Variational norm estimates for some oscillatory integrals related to Carleson's operator

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Outline

- Variational norm estimates
- The operators of Carleson and Stein-Wainger
- Our main theorem
- Connections to local smoothing and square function estimates for the linear Schrödinger equation

Pointwise convergence and Variational norms

- In analysis one is often interested in questions about pointwise (almost everywhere) convergence.
- For instance, let $A_\lambda f(x)$ be the average of a function f over the ball $B(x,\lambda)$ centered at x and of radius λ:

$$A_\lambda f(x) := \frac{1}{|B(x,\lambda)|}\int_{B(x,\lambda)} f(y)dy.$$

The Lebesgue differentiation theorem states that for every (locally) L^1 function f on $\mathbb{R}^n$, we have

$$\lim_{\lambda\to 0^+} A_\lambda f(x) = f(x)$$

for almost every $x \in \mathbb{R}^n$.

- This is clearly true if f were continuous with compact support on $\mathbb{R}^n$, and the set of all such functions is dense in L^1.

- It then remains to establish a weak-type bound for a relevant maximal function.
- In this case the relevant maximal function is given by Hardy and Littlewood:

$$Mf(x) = \sup_{\lambda>0} |A_\lambda f(x)|,$$

and the relevant weak-type bound for M is

$$\|Mf\|_{L^{1,\infty}} \leq C\|f\|_{L^1},$$

where for a measurable function F on $\mathbb{R}^n$, its weak L^1 norm is defined by

$$\|F\|_{L^{1,\infty}} := \sup_{\alpha>0} \alpha|\{x \in \mathbb{R}^n \colon |F(x)| > \alpha\}|$$

is a slightly smaller quantity than $\|F\|_{L^1}$.

- In this talk we are interested in variational norm estimates, which are typically stronger estimates than bounds for maximal operators.
- If Λ is a subset of $\mathbb{R}$ and $\lambda \mapsto a_\lambda$ is a function defined on Λ, then for $r \in (0,\infty)$, the r-th variational norm of this function is
$$\|a_\lambda\|_{V^r(\lambda\in\Lambda)} := \sup_{N\in\mathbb{N}} \sup_{\substack{\lambda_1,\dots,\lambda_N\in\Lambda \\ \lambda_1<\dots<\lambda_N}} \left(\sum_{j=1}^{N-1} |a_{\lambda_{j+1}} - a_{\lambda_j}|^r \right)^{1/r}.$$
(Just take an arbitrary strictly increasing sequence
$$\lambda_1 < \lambda_2 < \cdots < \lambda_N$$
in Λ, compute the successive differences $a_{\lambda_{j+1}} - a_{\lambda_j}$, take ℓ^r norm over j, and take supremum over all choices of strictly increasing sequences.)

- In the context of Lebesgue differentiation, instead of a bound for the maximal function $f(x) \mapsto Mf(x) = \sup_{\lambda>0} |A_\lambda f(x)|$, one could consider the mapping property of

$$f(x) \mapsto \|A_\lambda f(x)\|_{V^r(\lambda>0)}.$$

- Indeed, one has the following theorem (drawing from earlier work of Lépingle, Bourgain, Pisier and Xu):

Theorem (Jones, Kaufman, Rosenblatt, Wierdl)

When $r \in (2, \infty)$, the map $f(x) \mapsto \|A_\lambda f(x)\|_{V^r(\lambda>0)}$ maps L^1 to $L^{1,\infty}$, and is bounded on L^p for $1 < p < \infty$.

- The theorem implies that M maps L^1 to $L^{1,\infty}$ and is bounded on L^p for $1 < p < \infty$, since

$$Mf(x) \leq A_1 f(x) + \|A_\lambda f(x)\|_{V^r(\lambda>0)}$$

for any r.

- One can also deduce directly from the theorem (i.e. without having to pass to a dense subset where pointwise convergence occurs) that whenever $f \in L^1$, $A_\lambda f(x)$ converges pointwisely for a.e. x as $\lambda \to 0^+$.
- This is because then $\|A_\lambda f(x)\|_{V^r(\lambda>0)}$ is finite for a.e. x, and at every such x the sequence $\{A_{\lambda_i} f(x)\}_{i\in\mathbb{N}}$ is Cauchy whenever $\{\lambda_i\}_{i\in\mathbb{N}}$ is a sequence that strictly decreases to zero.
- This feature of not requiring the exhibition of a dense subset is sometimes useful for proving pointwise convergence results in ergodic theory (c.f. earlier work of Bourgain).
- Also, the theorem gives a more quantitative rate at which $A_\lambda f(x)$ converges as $\lambda \to 0^+$, and variational estimates are often useful in bounding discrete maximal operators.
- Various variational norm estimates have since been established for many classical operators in harmonic analysis.
- For instance, let's call $K(y)$ a Calderón-Zygmund kernel on $\mathbb{R}^n$, if $K(y) = \frac{\Omega(y)}{|y|^n}$ where Ω is homogeneous of degree 0, smooth on the unit sphere, and $\int_{\mathbb{S}^{n-1}} \Omega(y) d\sigma(y) = 0$.

- One has the following variational norm counterpart, of the L^p boundedness of the maximally truncated singular integral:

Theorem (Campbell, Jones, Reinhold, Wierdl)

Let $K(y)$ be a Calderón-Zygmund kernel on $\mathbb{R}^n$. For $\lambda > 0$, let

$$T_\lambda f(x) = \int_{|y|>\lambda} f(x-y)K(y)dy.$$

When $r \in (2, \infty)$, the map $f(x) \mapsto \|T_\lambda f(x)\|_{V^r(\lambda>0)}$ is bounded on L^p for $1 < p < \infty$.

- As a result, one recovers the classical result that

$$\left\| \sup_{\lambda>0} |T_\lambda f| \right\|_{L^p} \lesssim \|f\|_{L^p} \quad \text{for } 1 < p < \infty.$$

- See also Jones, Seeger, Wright for an endpoint result at $r = 2$.

- There is also for instance a variational norm counterpart of the L^p boundedness of the maximal spherical averages in $\mathbb{R}^n$:

Theorem (Jones, Seeger, Wright)

Let $n \geq 2$, and let $\mathcal{A}_\lambda f(x)$ be the average of f on the $(n-1)$-dimensional sphere of radius λ centered at x, i.e.

$$\mathcal{A}_\lambda f(x) = \fint_{\mathbb{S}^{n-1}} f(x+\lambda y) d\sigma(y).$$

When $r \in (2, \infty)$, the map $f(x) \mapsto \|\mathcal{A}_\lambda f(x)\|_{V^r(\lambda>0)}$

(i) *is bounded on L^p for $\frac{n}{n-1} < p < nr$, and*

(ii) *fails to be bounded on L^p if $p > nr$.*

- As a result, one recovers the result of Bourgain and Stein that

$$\left\| \sup_{\lambda>0} |\mathcal{A}_\lambda f| \right\|_{L^p} \lesssim \|f\|_{L^p} \quad \text{for } \frac{n}{n-1} < p < \infty.$$

Carleson's operator and Pointwise convergence of Fourier series

- We now turn to Carleson's operator, which is a maximal operator relevant for the study of the following question: given an L^2 function on the unit circle, does the partial sum of its Fourier series converge pointwise almost everywhere?
- The answer can be shown to be yes, by first transferring the problem to the real line, and then bounding the following maximal operator:

$$f(x) \mapsto \sup_{\lambda > 0} \left| \int_{-\lambda}^{\lambda} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right|.$$

- Modulo some trivial operators, one is led to study the Carleson operator $\mathcal{C}$, defined by

$$\mathcal{C}f(x) = \sup_{\lambda > 0} \left| \int_{\mathbb{R}} f(x-y) \frac{e^{2\pi i \lambda y}}{y} dy \right|$$

where the integral is understood in the principal value sense.

$$\mathcal{C}f(x) = \sup_{\lambda>0} \left| \int_{\mathbb{R}} f(x-y) \frac{e^{2\pi i \lambda y}}{y} dy \right|$$

- It is a classical result of Carleson and Hunt that $\mathcal{C}$ is bounded on L^p if $1 < p < \infty$.
- Subsequently Fefferman, and Lacey and Thiele, gave new and inspiring proofs of the same result.
- They decomposed f into sums of wave packets, which are localized in both the physical and the frequency space (to the extent allowed by the uncertainty principle); this technique has since found many applications in analyzing operators that are invariant under modulations.
- Motivated by the above results, Stein and Wainger considered a variant of Carleson's operator, where the linear phase function $y \mapsto \lambda y$ is replaced by a polynomial of higher degree (and $1/y$ is replaced by a Calderón-Zygmund kernel in higher dimensions).

- In particular, Stein and Wainger showed that if $K(y)$ is a Calderón-Zygmund kernel on $\mathbb{R}^n$, and if y^α is a monomial on $\mathbb{R}^n$ of degree ≥ 2, then

$$f(x) \mapsto \sup_{\lambda>0} \left| \int_{\mathbb{R}} f(x-y)K(y)e^{2\pi i \lambda y^\alpha} dy \right|$$

is bounded on L^p for $1 < p < \infty$, using only relatively simple techniques in the study of oscillatory integrals.
- A natural question is to ask for variational norm variants of the above theorems of Carleson and Stein-Wainger.
- Oberlin, Tao, Thiele, Seeger and Wright proved that when $r > 2$, the map

$$f(x) \mapsto \left\| \int_{\mathbb{R}} f(x-y) \frac{e^{2\pi i \lambda y}}{y} dy \right\|_{V^r(\lambda>0)}$$

is bounded on L^p if $p \in (r', \infty)$, thereby strengthening Carleson's theorem. (Here $r' = \frac{r}{r-1}$; see also subsequent work of Do, Muscalu, Thiele for a substantial generalization.)

Main theorem

- In joint work with Shaoming Guo and Joris Roos, we strengthen the theorem of Stein and Wainger as follows.

Theorem (Guo, Roos, Yung)

Let $K(y)$ be a Calderón-Zygmund kernel on $\mathbb{R}^n$. Let $\alpha > 1$ be fixed. For $\lambda > 0$, let

$$\mathcal{H}_\lambda f(x) = \int_{\mathbb{R}^n} f(x-y) K(y) e^{2\pi i \lambda |y|^\alpha} dy.$$

When $r > 2$, the map $f(x) \mapsto \|\mathcal{H}_\lambda f(x)\|_{V^r(\lambda>0)}$

(i) *is bounded on L^p if $p \in ((nr)', \infty)$, and*

(ii) *fails to be bounded on L^p if $p < (nr)'$.*

- Indeed, (ii) follows by taking f to be dilations of a fixed Schwartz function whose Fourier transform is supported on the unit annulus.

- To prove (i), say when $K(y) = \frac{1}{y}$ on $\mathbb{R}$ and $\alpha = 2$, we need a square function estimate of Lee, Rogers and Seeger, and a local smoothing estimate of Rogers and Seeger.

Theorem (Lee, Rogers, Seeger)

Suppose $p \in (2, \infty)$. Then

$$\left\| \left\| e^{it\Delta} f(x) \right\|_{L^2(t \in [1,2])} \right\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

Theorem (Rogers, Seeger)

Suppose $p \in (4, \infty)$. If the Fourier transform $\widehat{f}(\xi)$ of f is supported on $\{|\xi| \lesssim A\}$ for some $A > 0$, then

$$\left\| \left\| e^{it\Delta} f(x) \right\|_{L^p(t \in [1,2])} \right\|_{L^p(\mathbb{R})} \lesssim A^{2\left[\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p}\right]} \|f\|_{L^p(\mathbb{R})}.$$

Corollary (Rogers, Seeger)

Suppose $p \in (2,\infty)$. Then there exists $\varepsilon(p) > 0$ such that the following holds: If the Fourier transform $\widehat{f}(\xi)$ of f is supported on $\{|\xi| \lesssim A\}$ for some $A > 0$, then

$$\left\| \left\| e^{it\Delta} f(x) \right\|_{L^p(t\in[1,2])} \right\|_{L^p(\mathbb{R})} \lesssim A^{2\left[\left(\frac{1}{2}-\frac{1}{p}\right)-\varepsilon(p)\right]} \|f\|_{L^p(\mathbb{R})}.$$

- The expression on the right hand side should be thought of as a $W^{s,p}$ norm of f, where $s =$ exponent of A.
- The above local smoothing estimate of Rogers and Seeger should be compared to the following fixed time estimate, which goes back to Miyachi, Fefferman and Stein:

$$\sup_{t\in[1,2]} \left\| e^{it\Delta} f(x) \right\|_{L^p(\mathbb{R})} \lesssim A^{2\left[\left(\frac{1}{2}-\frac{1}{p}\right)\right]} \|f\|_{L^p(\mathbb{R})}$$

for $p \in [2,\infty)$, if $\widehat{f}$ is supported on $\{|\xi| \lesssim A\}$.
[Local integration in $t \Rightarrow$ additional smoothing effect in L^p for $2 < p < \infty$! Requires less regularity of initial data.]

- Now we go back to part (i) of our theorem. Let's consider the case $K(y) = \frac{1}{y}$ on $\mathbb{R}$ and $\alpha = 2$. Then

$$\mathcal{H}_\lambda f(x) = \int_{\mathbb{R}} f(x-y) \frac{e^{2\pi i \lambda y^2}}{y} dy.$$

- To prove (i) of the theorem, we have to at least bound

$$\left\| \|\mathcal{H}_\lambda f(x)\|_{V^r(\lambda \in [1,2])} \right\|_{L^p(\mathbb{R})}$$

for $r \in (2, \infty)$ and $p \in (2, \infty)$.
- For $\lambda \in [1, 2]$, we decompose

$$\mathcal{H}_\lambda f(x) = \sum_{\ell \in \mathbb{Z}} \mathcal{H}_{\lambda,\ell} f(x)$$

where

$$\mathcal{H}_{\lambda,\ell} f(x) = \int_{|y| \simeq 2^\ell} f(x-y) \frac{e^{2\pi i \lambda y^2}}{y} dy.$$

- Then we hope to bound

$$\sum_{\ell \in \mathbb{Z}} \left\| \|\mathcal{H}_{\lambda,\ell} f(x)\|_{V^r(\lambda \in [1,2])} \right\|_{L^p(\mathbb{R})}.$$

- Note that $\mathcal{H}_{\lambda,\ell}$ is a multiplier operator with multiplier

$$\int_{|y|\simeq 2^\ell} \frac{e^{2\pi i(-\xi y+\lambda y^2)}}{y}\,dy.$$

- Since $\lambda \in [1,2]$, the phase function $y \mapsto -\xi y + \lambda y^2$ has a critical point in the domain of integration, namely $\{|y| \simeq 2^\ell\}$, if and only if

$$|\xi| \simeq 2^\ell,$$

  in which case the multiplier is approximately $2^{-\ell} e^{\frac{i(2\pi i\xi)^2}{8\pi\lambda}}$ by stationary phase.

- Hence if we denote by P_ℓ the Littlewood-Paley projection onto frequency $\simeq 2^\ell$, then

$$\mathcal{H}_{\lambda,\ell} f(x) \simeq 2^{-\ell} e^{i(8\pi\lambda)^{-1}\Delta} P_\ell f(x).$$

- So we are hoping to bound

$$\sum_{\ell\in\mathbb{Z}} 2^{-\ell} \left\| \left\| e^{it\Delta} P_\ell f(x) \right\|_{V^r(t\simeq 1)} \right\|_{L^p(\mathbb{R})} .$$

- We now invoke the following consequence of the Plancherel-Polya inequality:

Proposition

Let $F(t)$ be a function on $\mathbb{R}$ whose Fourier transform $\widehat{F}(\eta)$ is supported on $\{|\eta| \leq B\}$ for some $B > 0$. Then for $1 \leq r < \infty$,

$$\|F(t)\|_{V^r(t>0)} \lesssim B^{1/r} \|F\|_{L^r(\mathbb{R})}.$$

- Thus we may estimate the V^r norm in t, and reduce ourselves to bounding

$$\sum_{\ell\in\mathbb{Z}} 2^{-\ell} 2^{2\ell/r} \left\| \left\| e^{it\Delta} P_\ell f(x) \right\|_{L^r(t\simeq 1)} \right\|_{L^p(\mathbb{R})} .$$

$$\sum_{\ell \in \mathbb{Z}} 2^{-\ell} 2^{2\ell/r} \left\| \left\| e^{it\Delta} P_\ell f(x) \right\|_{L^r(t \simeq 1)} \right\|_{L^p(\mathbb{R})}.$$

- Let's fix ℓ momentarily and estimate the L^p norm over $\mathbb{R}$, with $p \in (2, \infty)$.
- The summand increases as r decreases to 2, and if $r = 2$ the summand is bounded by

$$2^{-\ell} 2^{2\ell/2} \|f\|_{L^p(\mathbb{R})} = \|f\|_{L^p(\mathbb{R})}$$

  using the square function estimate of Lee, Rogers and Seeger. This cannot be summed over ℓ!
- Fortunately, if r is not just in $(2, \infty)$, but $r = p$ as well, then we can use the local smoothing estimate of Rogers and Seeger, and bound the summand by

$$2^{-\ell} 2^{2\ell/p} [2^\ell]^{2(\frac{1}{2} - \frac{1}{p}) - \varepsilon(p)} \|f\|_{L^p(\mathbb{R})} = 2^{-\ell \varepsilon(p)} \|f\|_{L^p(\mathbb{R})}$$

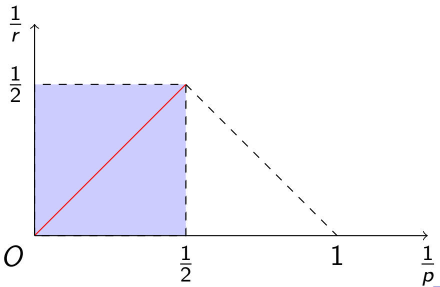
  where $\varepsilon(p) > 0$.

- To summarize, we have proved the following estimates:

$$2^{-\ell}2^{2\ell/r}\left\|\left\|e^{it\Delta}P_\ell f(x)\right\|_{L^r(t\simeq 1)}\right\|_{L^p(\mathbb{R})}$$

$$\lesssim\begin{cases}\|f\|_{L^p(\mathbb{R})} & \text{if } r\in[2,\infty),\ p\in(2,\infty)\\ 2^{-\ell\varepsilon(p)}\|f\|_{L^p(\mathbb{R})} & \text{if } r=p\in(2,\infty)\end{cases}$$

- Interpolating between the two, we get a favourable estimate for the left hand side for $r\in(2,\infty)$, $p\in(2,\infty)$, which we can then sum over $\ell\geq 1$.
- This is one of the key steps in the proof of (i) of the theorem.



Appendix: An estimate of Seeger for multipliers with localized bounds

Proposition (Seeger)

Let $m(\xi)$ be a Fourier multipliers on $\mathbb{R}^n$, compactly supported on $\{\xi\colon 1/2 \leq |\xi| \leq 2\}$, and satisfies

$$|\partial_\xi^\tau m(\xi)| \leq B \quad \text{for each } 0 \leq |\tau| \leq n+1$$

for some constant B. For $j \in \mathbb{Z}$, write T_j the multiplier operator with multiplier $m(2^{-j}\xi)$. Fix some $p \in (1,\infty)$. Assume that there exists some constant A such that

$$\sup_{j\in\mathbb{Z}} \| T_j f \|_{L^p(\mathbb{R}^n)} \leq A \| f \|_{L^p(\mathbb{R}^n)}.$$

Then

$$\left\| \| T_j f \|_{\ell^2(\mathbb{Z})} \right\|_{L^p(\mathbb{R}^n)} \lesssim A \left| \log\left(2 + \frac{B}{A}\right) \right|^{\left|\frac{1}{2} - \frac{1}{p}\right|} \| f \|_{L^p(\mathbb{R}^n)}.$$

A vector-valued version

Proposition

Let $I \subset \mathbb{R}$ be a compact interval. Let $\{m_u(\xi) \colon u \in I\}$ be a family of Fourier multipliers on $\mathbb{R}^n$, each of which is compactly supported on $\{\xi \colon 1/2 \leq |\xi| \leq 2\}$, and satisfies

$$\sup_{u \in I} |\partial_\xi^\tau m_u(\xi)| \leq B \quad \text{for each } 0 \leq |\tau| \leq n+1$$

for some constant B. For $u \in I$ and $j \in \mathbb{Z}$, write $T_{u,j}$ the multiplier operator with multiplier $m_u(2^{-j}\xi)$. Fix some $p \in [2, \infty)$. Assume that there exists some constant A such that

$$\sup_{j \in \mathbb{Z}} \left\| \|T_{u,j} f\|_{L^2(I)} \right\|_{L^s(\mathbb{R}^n)} \leq A \|f\|_{L^s(\mathbb{R}^n)}$$

for both $s = p$ and $s = 2$. Then

$$\left\| \| \|T_{u,j} f\|_{L^2(I)} \|_{\ell^2(\mathbb{Z})} \right\|_{L^p(\mathbb{R}^n)} \lesssim A \left| \log \left(2 + \frac{B}{A} \right) \right|^{\frac{1}{2} - \frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)}.$$