Maximal functions for Hilbert transforms along variable parabolas on $\mathbb{R}^2$

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Drawing upon works of Andreas...

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Introduction

- For $u \in (0, \infty)$, let $\mathcal{H}^u$ be the Hilbert transform along the direction $(1, u)$:

$$\mathcal{H}^u f(x) = \text{p.v.} \int_{\mathbb{R}} f(x_1 - t, x_2 - ut) \frac{dt}{t}, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

- For a subset $U \subset (0, \infty)$ we consider the maximal operator

$$\mathcal{H}_U f(x) = \sup_{u \in U} |\mathcal{H}^u f(x)|.$$

- Karagulyan showed that $\|\mathcal{H}_U\|_{L^2 \to L^{2,\infty}} \geq c\sqrt{\log \# U}$, and Łaba, Marinelli and Pramanik established the same lower bound for $\|\mathcal{H}_U\|_{p \to p}$ for all $1 < p < \infty$.
- In particular, $\mathcal{H}_U$ is unbounded on any L^p if U is infinite.

$$\mathcal{H}^u f(x) = \text{p.v.} \int_{\mathbb{R}} f(x_1 - t, x_2 - ut) \frac{dt}{t}, \quad \mathcal{H}_U f(x) = \sup_{u \in U} |\mathcal{H}^u f(x)|$$

- It was known (from Rademacher-Menshov) that

  $$\|\mathcal{H}_U\|_{2\to 2} \lesssim \log \# U;$$

  see Christ-Duoandikoetxea-Rubio de Francia who first stated this (c.f. also Cordoba, and Demeter).
  (We will abuse notation and write $\log(t)$ for $\log(2+t)$.)
- The above $L^2 \to L^2$ bound is sharp in general.
- Demeter and Di Plinio proved that $\|\mathcal{H}_U\|_{p\to p} \lesssim \log \# U$ for $p \in (2, \infty)$; also improved bounds for lacunary / Vargas U.
- See also Di Plinio and Parissis, who proved for $1 < p < \infty$, $\|\mathcal{H}_U\|_{p\to p} \lesssim \sqrt{\log \# U}$ for general lacunary U.
- The mapping properties of $\mathcal{H}^{u(x)} f(x)$ when $u(x)$ varies with x in a certain regular way (e.g. Lipschitz / depends only on x_1) is also very interesting (see Lacey, Li, Bateman, Thiele, Stein, Street), but we shall not discuss that today.

- Today we study the Hilbert transform H^u along the parabola parametrized by (t, ut^2), $t \in \mathbb{R}$:

$$H^u f(x) = \text{p.v.} \int_{\mathbb{R}} f(x_1 - t, x_2 - ut^2) \frac{dt}{t}, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

- It is bounded on L^p for $1 < p < \infty$, uniformly for $u \in (0, \infty)$ (H^u is a conjugation of H^1 by a dilation in the x_2 variable).
- For a subset $U \subset (0, \infty)$ we consider the maximal operator

$$H_U f(x) = \sup_{u \in U} |H^u f(x)|.$$

(See also Guo, Hickman, Lie, Roos for the study of $H^{u(x)}$ where $u(x)$ depends only on x_1, and Di Plinio, Guo, Thiele, Zorin-Kranich for the case where $u(x)$ is Lipschitz in x.)
- Let $N(U)$ be the number of dyadic intervals $[2^n, 2^{n+1})$ that U intersects (here $n \in \mathbb{Z}$).

Main Theorem

Theorem (Guo, Roos, Seeger, Y)

Let $p \in (2, \infty)$. Then H_U is bounded on $L^p(\mathbb{R}^2)$, if and only if $N(U) < +\infty$; furthermore,

$$\|H_U\|_{p \to p} \leq C_p \sqrt{\log N(U)}.$$

- In particular, H_U can be bounded on L^p for $p \in (2, \infty)$, even if U is infinite and contains an interval, contrary to $\mathcal{H}_U$!
- With some work, Karagulyan's counter-example can be adapted to show that the above bound is sharp; indeed

$$\|H_U\|_{p \to p} \geq c_p \sqrt{\log N(U)} \quad \text{for all } 1 < p < \infty.$$

- The assumption $p \in (2, \infty)$ allows for the use of local smoothing estimates for certain Fourier integral operators.
- The assumption $N(U) < \infty$ allows for the use of an inequality of Chang, Wilson and Wolff about martingales.
- Below we sketch the proof of the Theorem.

Step 1: Decomposition of the multiplier

- First let m be the Fourier multiplier of the Hilbert transform along the parabola (t, t^2):

$$m(\xi, \eta) = \int_{\mathbb{R}} e^{-2\pi i (t\xi + t^2\eta)} \frac{dt}{t}.$$

- The multiplier of H^u is then $m(\xi, u\eta)$.
- Decompose $1/t$ into sums of dilates of a suitable smooth odd function ψ supported on $[1/2, 2]$. Then $m(\xi, \eta)$ becomes

$$m(\xi, \eta) = \sum_{j \in \mathbb{Z}} m_j(\xi, \eta) \quad \text{where}$$

$$m_0(\xi, \eta) = \int_{\mathbb{R}} e^{-2\pi i (t\xi + t^2\eta)} \psi(t) dt, \quad m_j(\xi, \eta) = m_0(2^{-j} \circ (\xi, \eta))$$

and $2^{-j} \circ (\xi, \eta)$ is the non-isotropic dilation $(2^{-j}\xi, 2^{-2j}\eta)$.
- The multiplier of H^u is then $\sum_{j \in \mathbb{Z}} m_j(\xi, u\eta)$.

- By stationary phase, $m_0(\xi, \eta)$ can in turn be decomposed as

$$m_0(\xi, \eta) = \phi_0(\xi, \eta) + a_0(\xi, \eta) e^{-i\frac{\xi^2}{4\eta}}$$

where ϕ_0 is a Schwarz function vanishing at the origin, and a_0 is smooth and supported on $\{(\xi, \eta) \colon |\xi| \simeq |\eta| \geq 1\}$, with

$$|\nabla^k a_0(\xi, \eta)| \lesssim (|\xi| + |\eta|)^{-\frac{1}{2}-k} \quad \text{for } k \in \mathbb{N}.$$

- For $\ell \in \mathbb{Z}$ we write $\phi_{0,\ell}$ and $a_{0,\ell}$ for a smooth localization of ϕ_0 and a_0 to the annulus $\{|\xi| + |\eta| \simeq 2^\ell\}$, so that

$$\phi_0(\xi, \eta) = \sum_{\ell \in \mathbb{Z}} \phi_{0,\ell}(\xi, \eta), \quad a_0(\xi, \eta) = \sum_{\ell \geq 0} a_{0,\ell}(\xi, \eta).$$

(Note that $a_{0,\ell} = 0$ if $\ell < 0$ by the support condition on a_0.)

- Recall that m is the sum of (non-isotropic) dilates of m_0, and

$$m_0(\xi, \eta) = \sum_{\ell \in \mathbb{Z}} \phi_{0,\ell}(\xi, \eta) + \sum_{\ell \geq 0} a_{0,\ell}(\xi, \eta) e^{-i\frac{\xi^2}{4\eta}}.$$

- So writing

$$\phi_{j,\ell}(\xi,\eta) := \phi_{0,\ell}(2^{-j}\circ(\xi,\eta)), \quad a_{j,\ell}(\xi,\eta) := a_{0,\ell}(2^{-j}\circ(\xi,\eta)),$$

then

$$m(\xi,\eta) = \sum_{j\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}}\phi_{j,\ell}(\xi,\eta) + \sum_{j\in\mathbb{Z}}\sum_{\ell\geq 0} a_{j,\ell}(\xi,\eta)e^{-i\frac{\xi^2}{4\eta}}.$$

- We also let $T^u_{j,\ell}$ and $S^u_{j,\ell}$ be given by multipliers

$$\phi_{j,\ell}(\xi,u\eta) \quad \text{and} \quad a_{j,\ell}(\xi,u\eta)e^{i\frac{\xi^2}{4u\eta}}$$

respectively, so that

$$H^u = \sum_{\ell\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} T^u_{j,\ell} + \sum_{\ell\geq 0}\sum_{j\in\mathbb{Z}} S^u_{j,\ell}.$$

$$H^u = \sum_{\ell\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} T^u_{j,\ell} + \sum_{\ell\geq 0} \sum_{j\in\mathbb{Z}} S^u_{j,\ell}.$$

- We will prove the following two key estimates:
  1. For any $\ell \in \mathbb{Z}$,

$$\Big\| \sup_{u\in U} \Big| \sum_{j\in\mathbb{Z}} T^u_{j,\ell} f \Big| \Big\|_p \lesssim 2^{-|\ell|} \sqrt{\log N(U)} \|f\|_p \quad \text{for } 1 < p < \infty.$$

  2. There exists $\varepsilon = \varepsilon(p) > 0$ such that for any $\ell \geq 0$,

$$\Big\| \sup_{u>0} \Big| \sum_{j\in\mathbb{Z}} S^u_{j,\ell} f \Big| \Big\|_p \lesssim 2^{-\ell\varepsilon} \|f\|_p \quad \text{for } 2 < p < \infty.$$

- Together we bound

$$\begin{aligned} \| \sup_{u\in U} |H^u f| \|_p &\lesssim \sum_{\ell\in\mathbb{Z}} \Big\| \sup_{u\in U} \Big| \sum_{j\in\mathbb{Z}} T^u_{j,\ell} f \Big| \Big\|_p + \sum_{\ell\geq 0} \Big\| \sup_{u>0} \Big| \sum_{j\in\mathbb{Z}} S^u_{j,\ell} f \Big| \Big\|_p \\ &\lesssim \sqrt{\log N(U)} \|f\|_p \end{aligned}$$

for $2 < p < \infty$, and obtain our main theorem.

Step 2: Proof of the first key estimate

- Recall the multiplier of $T^u_{j,\ell}$ is $\phi_{j,\ell}(\xi, u\eta) = \phi_{0,\ell}(2^{-j} \circ (\xi, u\eta))$, and we want to prove that

$$\Big\| \sup_{u \in U} \Big| \sum_{j \in \mathbb{Z}} T^u_{j,\ell} f \Big| \Big\|_p \lesssim 2^{-|\ell|} \sqrt{\log N(U)} \|f\|_p \quad \text{for } 1 < p < \infty.$$

- The key is to prove it for $\ell = 0$.
- Indeed, $\{2^{|\ell|}\phi_{0,\ell}(2^\ell \xi, 2^\ell \eta) \colon \ell \in \mathbb{Z}\}$ form a bounded collection of C^{10} functions with compact support on the unit annulus. Applying the following argument to $2^{|\ell|}\phi_{0,\ell}(2^\ell \xi, 2^\ell \eta)$ in place of $\phi_{0,0}$ and performing an isotropic rescaling in (ξ, η) will give the desired conclusion for all $\ell \in \mathbb{Z}$.
- So from now on, let $T^u = \sum_{j \in \mathbb{Z}} T^u_{j,0}$ and prove that

$$\Big\| \sup_{u \in U} |T^u f| \Big\|_p \lesssim \sqrt{\log N(U)} \|f\|_p \quad \text{for } 1 < p < \infty;$$

in fact we only need to prove boundedness into weak L^p.

- We use an inequality for martingales due to Chang, Wilson and Wolff (see Grafakos-Honzik-Seeger, Demeter(-Di Plinio) for some earlier applications of Chang-Wilson-Wolff, in the study of maximal functions for families of singular integrals).
- We need only the inequality for the standard dyadic martingale on $\mathbb{R}$, so let's focus on that case.
- Let g be an $L^p \cap L^\infty$ function on $\mathbb{R}$ for some finite p.
- For $k \in \mathbb{Z}$, let $\mathbb{E}_k g(x)$ be the average of g on the (essentially unique) dyadic interval containing x.
- Let $\mathbb{D}_k$ be the martingale difference $\mathbb{E}_k - \mathbb{E}_{k-1}$.
- Let $\mathbb{M}$ be the martingale maximal function

$$\mathbb{M}g(x) = \sup_{k \in \mathbb{Z}} |\mathbb{E}_k g(x)|,$$

and let $\mathbb{S}$ be the square function

$$\mathbb{S}g(x) = \left(\sum_{k \in \mathbb{Z}} |\mathbb{D}_k g(x)|^2 \right)^{1/2}.$$

- The Chang-Wilson-Wolff inequality says basically that if $|g(x)|$ is big, then most of the time $\mathbb{S}g(x)$ is also big (which would be obvious if we replaced the square function by an ℓ^1 sum).
- More precisely, the Chang-Wilson-Wolff inequality states that there exist universal constants c_1, c_2 such that for any $\varepsilon \in (0, 1/2)$ and any $\lambda > 0$, we have

$$|\{x\colon |g(x)| > 4\lambda \text{ and } \mathbb{S}g(x) < \varepsilon\lambda\}| \leq c_2 e^{-\frac{c_1}{\varepsilon^2}} |\{x\colon \mathbb{M}g(x) > \lambda\}|.$$

  (It is rare that $|g|$ is large but $\mathbb{S}g$ is extremely small.)
- We will apply this inequality for functions on $\mathbb{R}^2$, in the x_2 variable only; by abuse of notation, we still denote the corresponding operators on $\mathbb{R}^2$ by $\mathbb{E}$, $\mathbb{D}$, $\mathbb{M}$ and $\mathbb{S}$.
- First observe an elementary fact: for any function F, if $u \in [2^n, 2^{n+1})$ then

$$|F(u)| \leq |F(2^n)| + \int_1^2 |\partial_s F(2^n s)| ds.$$

- Given $U \subset (0, \infty)$, let $\mathcal{N}(U)$ be the set of all integers n so that $[2^n, 2^{n+1})$ that intersects U ('the set of relevant n's for U'). Then $N(U) = \#\mathcal{N}(U)$, and

$$\sup_{u \in U} |F(u)| \leq \sup_{n \in \mathcal{N}(U)} \left(|F(2^n)| + \int_1^2 |\partial_s [F(2^n s)]| ds \right).$$

- As a result,

$$\sup_{u \in U} |T^u f| \leq \sup_{n \in \mathcal{N}(U)} T_n f$$

where

$$T_n f := |T^{2^n} f| + \int_1^2 \left| \partial_s T^{2^n s} f \right| ds.$$

- Note that the multiplier of T^{2^n} is $\sum_{j \in \mathbb{Z}} \phi_{0,0}(2^{-j} \circ (\xi, 2^n \eta))$, which is the conjugation of a non-isotropic singular integral with a dilation in the variable x_2. So is $\partial_s T^{2^n s}$ for each s.
- Hence T_n is bounded on L^p for $1 < p < \infty$, uniformly in n.
- We want to prove that $\sup_{n \in \mathcal{N}(U)} T_n f$ is in weak L^p if $f \in L^p$; we use the martingale inequality of Chang-Wilson-Wolff.

Application of the Chang-Wilson-Wolff inequality

- So now let $\varepsilon \in (0, 1/2)$ to be chosen.
- If $\sup_{n \in \mathcal{N}(U)} T_n f(x) > 4\lambda$, then either

$$\sup_{n \in \mathcal{N}(U)} \mathbb{S} T_n f(x) \geq \varepsilon\lambda,$$

or there exists $n \in \mathcal{N}(U)$ such that

$$T_n f(x) > 4\lambda \quad \text{and} \quad \mathbb{S} T_n f(x) < \varepsilon\lambda.$$

- It follows that

$$\begin{aligned} &|\{x \colon \sup_{u \in U} |T^u f(x)| > 4\lambda\}| \\ \leq &\sum_{n \in \mathcal{N}(U)} |\{x \colon T_n f(x) > 4\lambda \text{ and } \mathbb{S} T_n f(x) < \varepsilon\lambda\}| \\ &+ |\{x \colon \sup_{n \in \mathcal{N}(U)} \mathbb{S} T_n f(x) \geq \varepsilon\lambda\}|. \end{aligned}$$

$$\sum_{n \in \mathcal{N}(U)} |\{T_n f > 4\lambda \text{ and } \mathbb{S} T_n f < \varepsilon\lambda\}| + \Big|\{\sup_{n \in \mathcal{N}(U)} \mathbb{S} T_n f \geq \varepsilon\lambda\}\Big|.$$

- The first term can be bounded using Chang-Wilson-Wolff, by

$$c_2 e^{-\frac{c_1}{\varepsilon^2}} \sum_{n \in \mathcal{N}(U)} |\{\mathbb{M} T_n f > \lambda\}| \lesssim e^{-\frac{c_1}{\varepsilon^2}} N(U) \frac{1}{\lambda^p} \|f\|_p^p;$$

  this is bounded by $\frac{1}{\lambda^p}\|f\|_p^p$, if we take $\varepsilon = [\log N(U)]^{-1/2}$.
- The second term will be bounded by

$$\frac{1}{\varepsilon^p \lambda^p} \Big\| \sup_{n \in \mathbb{Z}} \mathbb{S} T_n f \Big\|_p^p \lesssim \sqrt{\log N(U)}^p \frac{1}{\lambda^p} \|f\|_p^p$$

  as desired, if we can show

$$\Big\| \sup_{n \in \mathbb{Z}} \mathbb{S} T_n f \Big\|_p \lesssim \|f\|_p.$$

$$\|\sup_{n\in\mathbb{Z}} \mathbb{S}T_n f\|_p \lesssim \|f\|_p, \quad \text{where} \quad \mathbb{S}T_n f \lesssim \left(\sum_{k\in\mathbb{Z}} |\mathbb{D}_k T_n f|^2\right)^{1/2}.$$

- Recall that $\mathbb{D}_k$ are the dyadic differences in the x_2 variable.
- Let's pretend that $\mathbb{D}_k \simeq P_k^{(2)}$, the Littlewood-Paley frequency localization to 2^k in the x_2 variable (this is a small lie). Then

$$\mathbb{S}T_n f \leq \left(\sum_{k\in\mathbb{Z}} |P_k^{(2)} T^{2^n} f|^2\right)^{1/2} + \int_1^2 \left(\sum_{k\in\mathbb{Z}} |P_k^{(2)} \partial_s T^{2^n s} f|^2\right)^{1/2} ds.$$

- We claim

$$|P_k^{(2)} T^{2^n} f| \lesssim M^{(1)} T^{(1)} P_k^{(2)} f + M^{(1)} M^{(2)} P_k^{(2)} f$$

where $T^{(1)}$ is a singular integral in the x_1 variable, and $M^{(i)}$ are the maximal functions in the x_i variable, $i = 1, 2$.

- Indeed,

$$T^{2^n} f = \sum_{j\in\mathbb{Z}} \mathcal{F}^{-1}\left(\phi_{0,0}(2^{-j}\xi, 2^{-2j}2^n\eta)\widehat{f}\right)$$

where $\phi_{0,0}$ is a unit bump function on the unit annulus, so

$$P_k^{(2)} T^{2^n} f = \sum_{j:\, 2j-n\le k} \mathcal{F}^{-1}\left(\phi_{0,0}(2^{-j}\xi, 2^{-2j}2^n\eta)\widehat{P_k^{(2)} f}\right)$$

$$= \sum_{j:\, 2j-n\le k} \mathcal{F}^{-1}\left(\phi_{0,0}(2^{-j}\xi, 0)\widehat{P_k^{(2)} f}\right) + \text{error}$$

where the error is $\lesssim$ the strong maximal function of $P_k^{(2)} f$.

- The main term above is effectively a maximal truncation of a singular integral in the x_1 variable. So by Cotlar's lemma,

$$|P_k^{(2)} T^{2^n} f| \lesssim M^{(1)} T^{(1)} P_k^{(2)} f + M^{(1)} M^{(2)} P_k^{(2)} f$$

where $T^{(1)}$ is a singular integral in the x_1 variable, and $M^{(i)}$ are the maximal functions in the x_i variable, $i = 1, 2$.

$$|P_k^{(2)} T^{2^n} f| \lesssim M^{(1)} T^{(1)} P_k^{(2)} f + M^{(1)} M^{(2)} P_k^{(2)} f$$

- Moreover, rather importantly, we have a similar bound for $P_k^{(2)} \partial_s T^{2^n s} f$ for every s. Thus

$$\mathbb{S} T_n f \leq \left(\sum_{k \in \mathbb{Z}} |P_k^{(2)} T^{2^n} f|^2 \right)^{1/2} + \int_1^2 \left(\sum_{k \in \mathbb{Z}} |P_k^{(2)} \partial_s T^{2^n s} f|^2 \right)^{1/2} ds$$
$$\leq \left(\sum_{k \in \mathbb{Z}} |M^{(1)} T^{(1)} P_k^{(2)} f + M^{(1)} M^{(2)} P_k^{(2)} f|^2 \right)^{1/2}$$

pointwisely independent of n, and at this point we see that

$$\big\| \sup_{n \in \mathbb{Z}} \mathbb{S} T_n f \big\|_p \lesssim \big\| \big(\sum_{k \in \mathbb{Z}} |P_k^{(2)} f|^2 \big)^{1/2} \big\|_p \lesssim \|f\|_p$$

for $1 < p < \infty$, which completes the proof of our first key estimate.

- We remark that the same proof establishes the following:

Corollary

(a) *(Grafakos, Honzik, Seeger) Let K be a 'nice' Calderón -Zygmund kernel on $\mathbb{R}$. For $u > 0$, let*

$$K^u(x) := u^{-1}K(u^{-1}x).$$

Then for $U \subset (0, \infty)$, we have

$$\Big\| \sup_{u \in U} |f * K_u| \Big\|_p \lesssim_{p,b} \sqrt{\log N(U)} \|f\|_p \quad \text{for } 1 < p < \infty.$$

(b) *Let K be a 'nice' Calderón-Zygmund kernel on $\mathbb{R}^2$ with respect to some dilation $x \mapsto (\lambda x_1, \lambda^b x_2)$ where $b > 0$. For $u > 0$, let*

$$K^u(x) := u^{-1/b}K(x_1, u^{-1}x_2).$$

Then for $U \subset (0, \infty)$, we have

$$\Big\| \sup_{u \in U} |f * K_u| \Big\|_p \lesssim_{p,b} \sqrt{\log N(U)} \|f\|_p \quad \text{for } 1 < p < \infty.$$

Step 2: Proof of the second key estimate

- Let us now turn to our second key estimate.
- Let $S^u_{j,\ell}$ be the operator with multiplier $a_{j,\ell}(\xi, u\eta)e^{i\frac{\xi^2}{4u\eta}}$.
  Here $a_{j,\ell}$ is a non-isotropic dilate of $a_{0,\ell}$, and $a_{0,\ell}$ is a symbol of order $-1/2$ supported on $\{|\xi| \simeq |\eta| \simeq 2^\ell\}$:

$$|\nabla^k a_{0,\ell}(\xi,\eta)| \lesssim (2^\ell)^{-\frac{1}{2}-k} \quad \text{for } k \in \mathbb{N}.$$

- We will now establish our second key estimate, namely the existence of some $\varepsilon = \varepsilon(p) > 0$ such that for every $\ell \geq 0$,

$$\Big\| \sup_{u>0} \Big| \sum_{j\in\mathbb{Z}} S^u_{j,\ell} f \Big| \Big\|_p \lesssim 2^{-\ell\varepsilon} \|f\|_p \quad \text{for } 2 < p < \infty.$$

  The fact that $a_{0,\ell}$ is supported on $\{|\xi| \simeq |\eta| \simeq 2^\ell\}$ (and not the whole annulus of radius 2^ℓ, as in the case of the first key estimate) will allow us to take supremum over all $u > 0$; we illustrate this in a toy model below.

A toy model

- Let $\varphi(\xi, \eta)$ be a Schwartz function supported on the sector
$$\{|\xi| \simeq |\eta| \geq 1\}.$$
- For $\ell \geq 0$, let φ_ℓ be the localization of φ to $\{|\xi| \simeq |\eta| \geq 2^\ell\}$.
- For $j \in \mathbb{Z}$, let $\varphi_{j,\ell}(\xi, \eta) := \varphi_\ell(2^{-j} \circ (\xi, \eta))$.
- For $u > 0$, let $\tilde{S}^u_{j,\ell}$ be the operator with multiplier $\varphi_{j,\ell}(\xi, u\eta)$.
- Then we will prove that for any $N \in \mathbb{N}$,
$$\Big\| \sup_{u>0} \Big| \sum_{j\in\mathbb{Z}} \tilde{S}^u_{j,\ell} f \Big| \Big\|_p \lesssim 2^{-\ell N} \|f\|_p \quad \text{for } 2 \leq p < \infty.$$
- In fact, it suffices to prove that
$$\Big\| \sup_{n\in\mathbb{Z}} \Big| \sum_{j\in\mathbb{Z}} \tilde{S}^{2^n}_{j,\ell} f \Big| \Big\|_p + \Big\| \sup_{n\in\mathbb{Z}} \int_1^2 \Big| \sum_{j\in\mathbb{Z}} \partial_s \tilde{S}^{2^n s}_{j,\ell} f \Big| ds \Big\|_p \lesssim 2^{-\ell N} \|f\|_p,$$
which will hold as long as we show
$$\int \sum_{n\in\mathbb{Z}} \Big| \sum_{j\in\mathbb{Z}} \tilde{S}^{2^n}_{j,\ell} f \Big|^p + \sup_{s\in[1,2)} \int \sum_{n\in\mathbb{Z}} \Big| \sum_{j\in\mathbb{Z}} \partial_s \tilde{S}^{2^n s}_{j,\ell} f \Big|^p \lesssim \Big(2^{-\ell N} \|f\|_p \Big)^p.$$

- First note that for $\ell \geq 0$, φ_ℓ is the localization of a Schwartz function φ to $\{|\xi| \simeq |\eta| \simeq 2^\ell\}$.
- So $\{2^{\ell N}\varphi_\ell(2^\ell\xi, 2^\ell\eta) \colon \ell \geq 0\}$ is a bounded collection of C^{10} functions with compact support on $\{|\xi| \simeq |\eta| \simeq 1\}$, and the key is to prove our claim when $\ell = 0$.
- Let's write $D_{a,b}f(x) := f(2^a x_1, 2^b x_2)$ for the anisotropic dilation of f. Then

$$\sum_{j\in\mathbb{Z}} \tilde{S}_{j,0}^{2^n} = D_{0,-n} \circ \sum_{j\in\mathbb{Z}} \tilde{S}_j \circ D_{0,n},$$

where $\tilde{S}_j := \tilde{S}_{j,0}^1$ is the operator with multiplier $\varphi_0(2^{-j} \circ (\xi, \eta))$. In particular, $\sum_{j\in\mathbb{Z}} \tilde{S}_j$ is a non-isotropic Calderón-Zygmund operator on $\mathbb{R}^2$, so

$$\int |\sum_{j\in\mathbb{Z}} \tilde{S}_{j,0}^{2^n} f|^p \lesssim \int |f|^p$$

uniformly for $n \in \mathbb{Z}$.

- Recap: We know

$$\int |\sum_{j\in\mathbb{Z}} \tilde{S}^{2^n}_{j,0} f|^p \lesssim \int |f|^p$$

uniformly in n, and we want

$$\int \sum_{n\in\mathbb{Z}} |\sum_{j\in\mathbb{Z}} \tilde{S}^{2^n}_{j,0} f|^p \lesssim \int |f|^p$$

- But the multiplier for $\tilde{S}^{2^n}_{j,0}$ is $\varphi_0(2^{-j} \circ (\xi, 2^n\eta))$, which is non-zero only when $|\xi| \simeq 2^j$ and $|\eta| \simeq 2^{2j-n}$.
- It follows that

$$\sum_{j\in\mathbb{Z}} \tilde{S}^{2^n}_{j,0} f = \sum_{j\in\mathbb{Z}} \tilde{S}^{2^n}_{j,0} \left[\sum_{k\in\mathbb{Z}} P^{(1)}_k P^{(2)}_{2k-n} f \right],$$

where $P^{(i)}_j$ is Littlewood-Paley projection to frequency 2^j in the x_i variable, $i = 1, 2$; note that the frequency supports of $[\dots]$ above are disjoint as n varies.

$$\sum_{j\in\mathbb{Z}} \tilde{S}^{2^n}_{j,0} f = \sum_{j\in\mathbb{Z}} \tilde{S}^{2^n}_{j,0} \left[\sum_{k\in\mathbb{Z}} P^{(1)}_k P^{(2)}_{2k-n} f \right],$$

- From the boundedness of $\sum_{j\in\mathbb{Z}} \tilde{S}^{2^n}_{j,0}$ on L^p, we have

$$\int |\sum_{j\in\mathbb{Z}} \tilde{S}^{2^n}_{j,0} f|^p \lesssim \int |\sum_{j\in\mathbb{Z}} P^{(1)}_j P^{(2)}_{2j-n} f|^p \lesssim \int \Big(\sum_{j\in\mathbb{Z}} |P^{(1)}_j P^{(2)}_{2j-n} f|^2\Big)^{p/2}$$

  by reversed Littlewood-Paley inequality.
- We sum over $n \in \mathbb{Z}$; if $2 \le p < \infty$, we have

$$\int \sum_{n\in\mathbb{Z}} |\sum_{j\in\mathbb{Z}} \tilde{S}^{2^n}_{j,0} f|^p \lesssim \int \Big(\sum_{n\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |P^{(1)}_j P^{(2)}_{2j-n} f|^2\Big)^{p/2} \lesssim \|f\|^p_p,$$

  by Littlewood-Paley again. Similarly, one can prove

$$\sup_{s\in[1,2)} \int \sum_{n\in\mathbb{Z}} |\sum_{j\in\mathbb{Z}} \partial_s \tilde{S}^{2^n s}_{j,0} f|^p \lesssim \|f\|^p_p, \quad 2 \le p < \infty.$$

  This completes our analysis for the toy model case.

Return to the actual case

- We have to show the existence of some $\varepsilon = \varepsilon(p) > 0$, such that for every $\ell \geq 0$,

$$\Big\| \sup_{u>0} \Big| \sum_{j\in\mathbb{Z}} S^u_{j,\ell} f \Big| \Big\|_p \lesssim 2^{-\ell\varepsilon} \|f\|_p \quad \text{for } 2 < p < \infty.$$

  Here the multiplier of $S^u_{j,\ell}$ is $a_{j,\ell}(\xi, u\eta) e^{-i\frac{\xi^2}{4u\eta}}$, where $a_{j,\ell}$ is a non-isotropic dilate of $a_{0,\ell}$, and $a_{0,\ell}$ is a symbol of order $-1/2$ supported on $\{|\xi| \simeq |\eta| \simeq 2^\ell\}$.
- The difficulty is with the oscillation $e^{-i\frac{\xi^2}{4u\eta}}$; without it the argument for the toy model case shows that the above holds with $\varepsilon = 1/2$.
- So now fix $\ell \geq 0$. We will first show that the estimate holds when we take supremum only over a lacunary sequence of u: actually, we will see that

$$\int \sum_{n\in\mathbb{Z}} \Big| \sum_{j\in\mathbb{Z}} S^{2^n}_{j,\ell} f \Big|^p \lesssim \left(\ell 2^{-\ell/p} \|f\|_p\right)^p \quad \text{for } 2 \leq p < \infty.$$

- To do so, we just need to show that

$$\int |\sum_{j\in\mathbb{Z}} S_{j,\ell}^{2^n} f|^p \lesssim \left(\ell 2^{-\ell/p}\|f\|_p\right)^p \quad \text{for } 2 \le p < \infty,$$

uniformly over $n \in \mathbb{Z}$; using the disjointness of Fourier supports, we may then sum over n for free just as before.

- Motivated by our previous calculation, let's write

$$\sum_{j\in\mathbb{Z}} S_{j,\ell}^{2^n} = D_{\ell,\ell-n} \circ \sum_{j\in\mathbb{Z}} S_j \circ D_{-\ell,-\ell+n}$$

where $S_j = S_{j,\ell} := D_{-\ell,-\ell} \circ S_{j,\ell}^1 \circ D_{\ell,\ell}$; we need to show that

$$\|\sum_{j\in\mathbb{Z}} S_j f\|_p \lesssim \ell 2^{-\ell/p}\|f\|_p \quad \text{for } 2 \le p < \infty.$$

- But S_j is just the operator with multiplier $a_{j,\ell}(2^\ell\xi, 2^\ell\eta)e^{-i2^\ell\frac{\xi^2}{4\eta}}$.
- Let now $\sigma_0(\xi,\eta) = a_{0,\ell}(2^\ell\xi, 2^\ell\eta)e^{-i2^\ell\frac{\xi^2}{4\eta}}$.

  Then the multiplier of $\sum_{j\in\mathbb{Z}} S_j$ is $\sum_{j\in\mathbb{Z}} \sigma_0(2^{-j}\circ(\xi,\eta))$.

$$\sigma_0(\xi,\eta) = a_{0,\ell}(2^\ell\xi, 2^\ell\eta)e^{-i2^\ell \frac{\xi^2}{4\eta}}, \quad \text{multiplier for } S_j \text{ is } \sigma_0(2^{-j}\circ(\xi,\eta))$$

- To bound $\sum_j S_j$ in L^p, one may observe that σ_0 is supported on $\{|\xi| \simeq |\eta| \simeq 1\}$, and that

$$|\sigma_0(\xi,\eta)| \lesssim 2^{-\frac{\ell}{2}}, \quad \text{and} \quad |\partial^\alpha \sigma_0(\xi,\eta)| \lesssim 2^{-\frac{\ell}{2}+\ell|\alpha|}.$$

- So for instance Hörmander-Mikhlin theorem gives that

$$\|\sum_{j\in\mathbb{Z}} S_j f\|_p \lesssim 2^{4\ell}\|f\|_p, \quad 1 < p < \infty.$$

  But this is NOT enough! We need an operator norm that decays as $\ell \to +\infty$.
- It turns out that one can also show that

$$\|S_0\|_{p\to p} \lesssim 2^{-\ell/p} \quad \text{for } 2 \le p < \infty,$$

  which allows one to apply the following theorem of Seeger about localized multipliers (c.f. also Carbery):

Summing dilations of a localized multiplier

Theorem (Carbery / Seeger)

Let $\sigma_0(\xi,\eta)$ be a smooth multiplier supported on an unit annulus in $\mathbb{R}^2$, and S_j be the operator with multiplier $\sigma_0(2^{-j}\circ(\xi,\eta))$. Suppose $1<p<\infty$. Let A, B be constants so that

$$\|S_0 f\|_p \lesssim A\|f\|_p \quad \textit{with} \quad |\partial^\alpha \sigma_0(\xi,\eta)| \lesssim B \quad \textit{for } |\alpha| \leq 4.$$

Then

$$\|\sum_{j\in\mathbb{Z}} S_j f\|_p \lesssim A\left[\log\left(2+\frac{B}{A}\right)\right]^{\left|\frac{1}{2}-\frac{1}{p}\right|}\|f\|_p.$$

- We saw $|\partial^\alpha \sigma_0(\xi,\eta)| \lesssim 2^{4\ell}$ for $|\alpha| \leq 4$, so if we can also prove

$$\|S_0\|_{p\to p} \lesssim 2^{-\ell/p} \quad \text{for } 2 \leq p < \infty,$$

then from the above theorem of Seeger, we have

$$\|\sum_{j\in\mathbb{Z}} S_j\|_{p\to p} \lesssim \ell 2^{-\ell/p} \quad \text{for } 2 \leq p < \infty.$$

- It remains to see that $\|S_0\|_{p\to p} \lesssim 2^{-\ell/p}$ for $2 \le p < \infty$.
- But the multiplier of S_0 is given by

$$\sigma_0(\xi,\eta) = a_{0,\ell}(2^\ell\xi, 2^\ell\eta)e^{-i2^\ell\frac{\xi^2}{4\eta}}.$$

- Let S^t be the operator

$$S^t f(x) = \int_{\mathbb{R}^2} \widehat{f}(\xi,\eta)a_{0,\ell}(\xi,\eta)e^{-it\frac{\xi^2}{4\eta}}e^{2\pi ix\cdot(\xi,\eta)}d\xi d\eta,$$

so that

$$D_{\ell,\ell}\circ S_0\circ D_{-\ell,-\ell} = S^1.$$

- The phase $\xi^2/(4\eta)$ in the multiplier of S^t is homogeneous of degree 1 and has rank 1 Hessian on the support of $a_{0,\ell}$.
- A fixed time estimate of Miyachi shows that for $t \simeq 1$,

$$\left\|S^t f\right\|_p \lesssim 2^{-\frac{\ell}{2}}(2^\ell)^{\left(\frac{1}{2}-\frac{1}{p}\right)}\|f\|_p \lesssim 2^{-\ell/p}\|f\|_p \quad \text{for } 2\le p<\infty,$$

so the same estimate holds for $S_0 f$ in place of $S^t f$, as desired.

- Recap: We wanted to prove the existence of some $\varepsilon = \varepsilon(p) > 0$, such that for every $\ell \geq 0$,

$$\Big\| \sup_{u>0} \big| \sum_{j\in\mathbb{Z}} S^u_{j,\ell} f \big| \Big\|_p \lesssim 2^{-\ell\varepsilon} \|f\|_p \quad \text{for } 2 < p < \infty.$$

- We saw this holds with $\varepsilon = \frac{1}{p} - 0$ if we replace $\sup_{u>0}$ by supremum over 2^n, $n \in \mathbb{Z}$.
- One may be tempted to try using

$$\sup_{u>0} \big| \sum_{j\in\mathbb{Z}} S^u_{j,\ell} f \big| \leq \sup_{n\in\mathbb{Z}} \big| \sum_{j\in\mathbb{Z}} S^{2^n}_{j,\ell} f \big| + \int_1^2 \sup_{n\in\mathbb{Z}} \big| \sum_{j\in\mathbb{Z}} \partial_s S^{2^n s}_{j,\ell} f \big| ds,$$

and see whether $\partial_s S^{2^n s}_{j,\ell}$ is as good as $S^{2^n}_{j,\ell}$ for $s \in [1,2)$.
- Unfortunately this is not the case now: $\partial_s S^{2^n s}_{j,\ell}$ is actually worse than $S^{2^n}_{j,\ell}$ by a factor of 2^ℓ, and 2^ℓ is worse than the gain of $2^{-\frac{\ell}{p}}$ we had for $\sup_{n\in\mathbb{Z}} |\sum_{j\in\mathbb{Z}} S^{2^n}_{j,\ell} f|$.

- Fortunately, to bound $\sup_{u>0} F(u)$, we only 'need' $1/p$ derivative of F in $L^p(du)$.
- More precisely, for $F(u) = \sum_{j\in\mathbb{Z}} S^u_{j,\ell} f$, we use

$$\sup_{u\in[2^n,2^{n+1})} |F(u)|^p$$
$$\leq |F(2^n)|^p + p\Big(\int_1^2 |F(2^n s)|^p ds\Big)^{\frac{1}{p'}}\Big(\int_1^2 |\partial_s F(2^n s)|^p ds\Big)^{\frac{1}{p}}$$

for every $n \in \mathbb{Z}$, and take supremum over n on both sides; we would be done if we can show that for every $2 < p < \infty$, there exists $\varepsilon(p) > 0$ such that

$$\left(\int_{\mathbb{R}^2}\int_1^2 \sum_{n\in\mathbb{Z}} \Big|\sum_{j\in\mathbb{Z}} S^{2^n s}_{j,\ell} f\Big|^p ds dx\right)^{1/p} \lesssim 2^{-\ell(\frac{1}{p}+\varepsilon(p))}\|f\|_p$$
$$\left(\int_{\mathbb{R}^2}\int_1^2 \sum_{n\in\mathbb{Z}} \Big|\sum_{j\in\mathbb{Z}} \partial_s S^{2^n s}_{j,\ell} f\Big|^p ds dx\right)^{1/p} \lesssim 2^{\ell} 2^{-\ell(\frac{1}{p}+\varepsilon(p))}\|f\|_p$$

$$\left(\int_1^2 \int_{\mathbb{R}^2} \sum_{n\in\mathbb{Z}} |\sum_{j\in\mathbb{Z}} S_{j,\ell}^{2^n s} f|^p dx\, ds\right)^{1/p} \lesssim 2^{-\ell(\frac{1}{p}+\varepsilon(p))}\|f\|_p$$

- Our previous methods will give these estimates if we are willing to drop the gain of $\varepsilon(p)$ on the right hand side; indeed we can replace $\int_1^2 ds$ by $\sup_{s\in[1,2)}$ and still get the estimate without $\varepsilon(p)$.
- But the integral over $s \in [1,2)$ is really what allows us to gain $2^{-\ell\varepsilon(p)}$ on the right hand side of these inequalities.
- Recall we had S^t whose multiplier is $a_{0,\ell}(\xi,\eta)e^{-it\frac{\xi^2}{4\eta}}$.
- A local smoothing estimate of Mockenhaupt, Seeger and Sogge shows that for $2 < p < \infty$, there exists $\varepsilon(p) > 0$ so that

$$\left(\int_1^2 \int_{\mathbb{R}^2} |S^t f|^p dx\, dt\right)^{1/p} \lesssim 2^{-\ell(\frac{1}{p}+\varepsilon(p))}\|f\|_p.$$

- This additional gain, together with a vector-valued variant of Seeger's theorem for localized multipliers (due to Jones, Seeger and Wright), give the desired estimates above.

A vector-valued version of Seeger's theorem

Theorem (Jones, Seeger, Wright)

Let $I \subset \mathbb{R}$ be a compact interval. Let $\{m_u(\xi) \colon u \in I\}$ be a family of Fourier multipliers on $\mathbb{R}^n$, each of which is compactly supported on $\{\xi \colon 1/2 \leq |\xi| \leq 2\}$, and satisfies

$$\sup_{u \in I} |\partial_\xi^\tau m_u(\xi)| \leq B \quad \text{for each } 0 \leq |\tau| \leq n+1$$

for some constant B. For $u \in I$ and $j \in \mathbb{Z}$, write $T_{u,j}$ the multiplier operator with multiplier $m_u(2^{-j} \circ \xi)$. Fix some $p \in [2, \infty)$. Assume that there exists some constant A such that

$$\Big\| \| T_{u,0} f \|_{L^2(I)} \Big\|_{L^s(\mathbb{R}^n)} \leq A \| f \|_{L^s(\mathbb{R}^n)}$$

for both $s = p$ and $s = 2$. Then

$$\Big\| \big\| \| T_{u,j} f \|_{L^2(I)} \big\|_{\ell^2(\mathbb{Z})} \Big\|_{L^p(\mathbb{R}^n)} \lesssim A \left[\log \left(2 + \frac{B}{A} \right) \right]^{\frac{1}{2} - \frac{1}{p}} \| f \|_{L^p(\mathbb{R}^n)}.$$

The case $1 < p \leq 2$

- Finally, we briefly discuss what happens to the boundedness of H_U when $1 < p \leq 2$. It is known, for instance, that if $U = [1, 2]$ then H_U is not bounded on L^p for $1 \leq p \leq 2$.
- For $r > 0$, let $U^r = (r^{-1}U) \cap [1, 2]$ and $N(U^r, \delta)$ be the minimum number of intervals of length δ required to cover U^r.
- Let

$$p(U) = 1 + \limsup_{\delta \to 0^+} \frac{\sup_{r>0} \log N(U^r, \delta)}{\log \delta^{-1}};$$

note that $1 \leq p(U) \leq 2$, and e.g.

$$p(U) = \begin{cases} 1 & \text{if } U \text{ is lacunary,} \\ 2 & \text{if } U \text{ contains an interval.} \end{cases}$$

Theorem (Guo, Roos, Seeger, Y)

(a) *H_U is unbounded on L^p if $p < p(U)$;*

(b) *If $p(U) < p \leq 2$, then H_U is bounded on L^p, if and only if $N(U) < \infty$.*

Happy birthday Andreas!