The Marcinkiewicz-Zygmund theorem

Theorem 1. *Let T be a bounded linear operator from L^p to L^p for a certain $p \in [1, \infty]$. Then*

$$\left\| \left(\sum_{j=1}^{\infty} |Tf_j|^2 \right)^{1/2} \right\|_{L^p} \leq M \left\| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^p}$$

for the same p, where M is the operator norm of T from L^p to L^p.

Proof. It suffices to prove the theorem when only a finite number of f_j's, say f_1, $\dots$, f_N, are non-zero. The result then follows from the monotone convergence theorem by letting N go to infinity.

Let $\omega = (\omega_1, \dots, \omega_N)$ be a unit vector in $\mathbb{C}^N$. Consider the function $g = \sum_{j=1}^N \overline{\omega_j} f_j$. We then have $Tg = \sum_{j=1}^N \overline{\omega_j} Tf_j$ by the linearity of T. Since $\|Tg\|_{L^p} \leq M \|g\|_{L^p}$, it follows that

$$\int_{\mathbb{R}^n} \left| \sum_{j=1}^N \overline{\omega_j} Tf_j(x) \right|^p dx \leq M \int_{\mathbb{R}^n} \left| \sum_{j=1}^N \overline{\omega_j} f_j(x) \right|^p dx.$$

Integrating now over all unit vector ω in $\mathbb{C}^N$ and interchanging the order of integration, we get

$$\int_{\mathbb{R}^n} \int_{|\omega|=1} \left| \sum_{j=1}^N \overline{\omega_j} Tf_j(x) \right|^p d\omega dx \leq M \int_{\mathbb{R}^n} \int_{|\omega|=1} \left| \sum_{j=1}^N \overline{\omega_j} f_j(x) \right|^p d\omega dx. \quad (1)$$

However, for any vector $\mu = (\mu_1, \dots, \mu_N) \in \mathbb{C}^N$, the integral

$$\int_{|\omega|=1} \left| \sum_{j=1}^N \overline{\omega_j} \mu_j \right|^p d\omega$$

depends only on the length of μ; indeed by rotation invariance, the integral is equal to $C_N |\mu|^p$ where C_N is a dimensional constant and $|\mu| = \left(\sum_{j=1}^N |\mu_j|^2 \right)^{1/2}$. Hence from (1)

$$\int_{\mathbb{R}^n} C_N \left(\sum_{j=1}^N |Tf_j(x)|^2 \right)^{p/2} dx \leq M \int_{\mathbb{R}^n} C_N \left(\sum_{j=1}^N |f_j(x)|^2 \right)^{p/2} dx.$$

Dividing both sides by C_N, we get the desired inequality. □

Note how the linearity of T is used cruicially in the proof, together with the Hilbert space structure of $\mathbb{C}^N$. Note also that even $p = 1$ or ∞ is allowed in the theorem.

Alternative proof using Rademacher functions. Recall that the Rademacher functions are defined on $[0, 1)$ by

$$r_j(t) = \begin{cases} 1 & \text{if } t \in [k2^{-j}, (k+1)2^{-j}),\ k \text{ odd} \\ -1 & \text{if } t \in [k2^{-j}, (k+1)2^{-j}),\ k \text{ even} \end{cases} \qquad (j = 1, 2, \dots)$$

and are independent random variables on $[0, 1)$. They thus have the following property: for each $p \in [1, \infty)$, there exists C_p such that

$$C_p^{-1} \left(\sum_j |a_j|^2 \right)^{1/2} \leq \left\| \sum_j a_j r_j(t) \right\|_{L^p(dt)} \leq C_p \left(\sum_j |a_j|^2 \right)^{1/2}.$$

for any complex numbers a_j (See *Singular integrals*, Appendix D. Note that it suffices to prove the inequality for real a_j's, because the case for complex a_j's follow trivially from that.) Now consider the function $g(x,t) = \sum_j f_j(x) r_j(t)$. Then $Tg(x,t) = \sum_j Tf_j(x) r_j(t)$, and the boundedness of T gives for each t that

$$\int_{\mathbb{R}^n} \left| \sum_j Tf_j(x) r_j(t) \right|^p dx \leq M \int_{\mathbb{R}^n} \left| \sum_j f_j(x) r_j(t) \right|^p dx.$$

Integrating in t and interchanging the order of integral, we get

$$\int_{\mathbb{R}^n} \int_0^1 \left| \sum_j Tf_j(x) r_j(t) \right|^p dt dx \leq M \int_{\mathbb{R}^n} \int_0^1 \left| \sum_j f_j(x) r_j(t) \right|^p dt dx.$$

But by the property of Rademacher functions listed above, the left hand side is comparable to

$$\int_{\mathbb{R}^n} \left(\sum_j |Tf_j|^2 \right)^{p/2} dx$$

while the right hand side is comparable to

$$\int_{\mathbb{R}^n} \left(\sum_j |f_j|^2 \right)^{p/2} dx$$

(up to a constant C_p). Hence we obtain

$$\left\| \left(\sum_{j=1}^\infty |Tf_j|^2 \right)^{1/2} \right\|_{L^p} \leq M C_p^2 \left\| \left(\sum_{j=1}^\infty |f_j|^2 \right)^{1/2} \right\|_{L^p},$$

a slightly weaker inequality than was stated in the theorem. □

Yet another proof using other independent random variables. Suppose on a certain probability space (Ω, dt) we have independent Gaussian random variables $h_1(t), \dots, h_N(t)$ $(t \in \Omega)$. Say all of them have density $e^{-\pi\alpha^2}$. Then for any real numbers $a_1, \dots, a_N$, the random variable

$$\sum_{j=1}^N a_j h_j(t)$$

is again a Gaussian random variable with density

$$A^{-1} e^{-\pi\alpha^2/A^2}$$

where $A = \left(\sum_j a_j^2\right)^{1/2}$. Hence

$$\begin{aligned}\int_\Omega \left|\sum_{j=1}^N a_j h_j(t)\right|^p dt &= \int_0^1 \alpha^p A^{-1} e^{-\pi\alpha^2/A^2} d\alpha \\ &= A^p \int_0^1 \alpha^p e^{-\pi\alpha^2} d\alpha \\ &= C_p \left(\sum_{j=1}^N a_j^2\right)^{\frac{p}{2}}.\end{aligned}$$

Let now $T = T_1 + iT_2$, where T_1 and T_2 are the real and imaginary parts of T respectively. Write also $f_j = u_j + iv_j$ where u_j and v_j are the real and imaginary parts of f_j. Apply the above identity to $a_j = T_1 u_j(x)$, we get, upon integrating in x and invoking the boundedness and linearity of T, that

$$C_p \int \left(\sum_{j=1}^N |T_1 u_j(x)|^2\right)^{\frac{p}{2}} dx \leq M C_p \int \left(\sum_{j=1}^N |u_j(x)|^2\right)^{\frac{p}{2}} dx.$$

Cancelling C_p from both sides yields an inequality for $T_1 u_j$. Repeating with $a_j = T_1 v_j(x)$, $T_2 u_j(x)$ and $T_2 v_j(x)$, and combining the resulting inequalities, we get

$$\int \left(\sum_{j=1}^N |Tf_j(x)|^2\right)^{\frac{p}{2}} dx \leq M \int \left(\sum_{j=1}^N |f_j(x)|^2\right)^{\frac{p}{2}} dx.$$

□

Remark 1. *Here is an explicit construction of independent Gaussian random variables. View $[0,1]$ as a probability space with the ordinary Lebesgue measure playing the role of the probability measure. Let $h(s)$ be a Gaussian random variable on $[0,1]$, i.e. a real valued function on $[0,1]$ such that*

$$|\{s \colon h(s) < a\}| = \int_{-\infty}^a e^{-\pi\alpha^2} d\alpha$$

for all real values of a. Consider a product probability space $[0,1]^N$, equipped with the product probability measure (which of course again happens to be the ordinary Lebesgue measure). We shall write each $t \in [0,1]^N$ as $t = (t_1, \ldots, t_N)$. Let $h_1(t)$, …, $h_N(t)$ be random variables on $[0,1]^N$, defined by

$$h_j(t) = h(t_j)$$

for all j. Then they are independent Gaussian random variables; indeed they correspond to repeating a process independently N times. Let now $f(\alpha)$ be the *density of the random variable* $\sum_{j=1}^{N} a_j h_j(t)$. *Then*

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-2\pi i\xi\alpha} f(\alpha) d\alpha \\
=&\int_{[0,1]^N} e^{-2\pi i\xi \sum_j a_j h_j(t)} dt \\
=&\int_{[0,1]^N} \prod_j e^{-2\pi i\xi a_j h(t_j)} dt \\
=&\prod_j \int_0^1 e^{-2\pi i\xi a_j h(t_j)} dt_j
\end{aligned}$$

(this is a transparent way of seeing the independence of h_j *at work)*

$$\begin{aligned}
=&\prod_j \int_{-\infty}^{\infty} e^{-2\pi i\xi a_j \alpha} e^{-\pi\alpha^2} d\alpha \\
=&\prod_j e^{-\pi\xi^2 a_j^2} \int_{-\infty}^{\infty} e^{-\pi(\alpha+i\xi a_j)^2} d\alpha \\
=&e^{-\pi\xi^2 \sum_j a_j^2}.
\end{aligned}$$

By inverse Fourier transform now,

$$f(\alpha) = A^{-1} e^{-\pi\alpha^2/A^2}$$

where $A = \left(\sum_j a_j^2\right)^{1/2}$, *and we are back to the proof of the main theorem.*

(The above is basically a proof that if X *and* Y *are independent random variables and* f_X, f_Y *are their densities, then* f_{X+Y}, *the density of* $X+Y$, *satisfies*

$$\widehat{f_{X+Y}}(\xi) = \widehat{f_X}(\xi)\widehat{f_Y}(\xi),$$

i.e.

$$f_{X+Y} = f_X * f_Y.$$

Note the Gaussian random variable was chosen for convenience, because convolutions of Gaussians can be easily computed by multiplication on the Fourier side.)