The Incompressible Navier Stokes Flow in Two Dimensions with Prescribed Vorticity

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To Dick Wheeden in friendship and appreciation

Abstract We study the incompressible two dimensional Navier–Stokes equation with initial vorticity in the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^2)$. This complements our earlier work for the case when the initial vorticity is in the inhomogeneous Sobolev space $W^{1,1}(\mathbb{R}^2)$.

The two-dimensional incompressible Navier–Stokes equation

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = \nu\Delta\mathbf{v} - \nabla p, \\ \nabla\cdot\mathbf{v} = 0, \end{cases} \tag{1}$$

models an incompressible flow of a fluid whose velocity and mechanical pressure at position $x \in \mathbb{R}^2$ and time $t \in \mathbb{R}$ are represented by the vector $\mathbf{v}(x,t) \in \mathbb{R}^2$ and the scalar $p(x,t) \in \mathbb{R}$; here ν is the kinematic viscosity coefficient. Note we have divided the Navier–Stokes equation by the constant density of the fluid ρ and thus ν in our equation is the dynamic viscosity coefficient divided by the density, assumed constant. Throughout this paper, ∇ will refer only to the spatial derivatives (i.e. derivative in the x variables). We also sometimes use the notation ∂_x to denote a derivative in the x variables when we have no need to be specific which space variable we are differentiating in.

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The vorticity of the Navier–Stokes flow is a scalar in the two-dimensional case, defined by

$$\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$$

where we wrote $\mathbf{v} = (v_1, v_2)$. It propagates according to the convection-diffusion equation

$$\omega_t - \nu \Delta \omega = -\nabla \cdot (\mathbf{v}\omega),$$

which one obtains from (1) by taking the curl of both sides. Formally the velocity $\mathbf{v}$ in the Navier–Stokes equation can be expressed in terms of the vorticity through the Biot–Savart relation

$$\mathbf{v} = (-\Delta)^{-1}(\partial_{x_2}\omega, -\partial_{x_1}\omega). \tag{2}$$

This follows formally by differentiating $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$, and using that $\nabla \cdot \mathbf{v} = 0$.

Our theorems concern the solution of the vorticity equation when the initial vorticity ω_0 is in the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^2)$. Here $\dot{W}^{1,1}(\mathbb{R}^2)$ is the completion of $C_c^\infty(\mathbb{R}^2)$ under the norm $\|u\|_{\dot{W}^{1,1}(\mathbb{R}^2)} := \|\nabla u\|_{L^1(\mathbb{R}^2)}$. The theorems are as follows:

Theorem 1 *Consider the two-dimensional vorticity equation*

$$\omega_t - \nu \Delta \omega = -\nabla \cdot (\mathbf{v}\omega), \tag{3}$$

where $\mathbf{v}$ *is defined through the Biot–Savart relation (2). Suppose we are given an initial vorticity* $\omega_0 \in \dot{W}^{1,1}(\mathbb{R}^2)$ *at time* $t = 0$. *If*

$$\|\nabla \omega_0(x)\|_{L^1(\mathbb{R}^2)} \le A_0,$$

then there exists a unique solution to the integral formulation of this vorticity equation up to time $t_0 = C\nu/A_0^2$, *such that*

$$\sup_{t \le t_0} \|\nabla \omega(x, t)\|_{L^1(\mathbb{R}^2)} \le 2A_0. \tag{4}$$

Moreover, the solution ω *depends continuously on the initial data* ω_0, *in the sense that if* $\omega_0^{(i)}$ *converges to* ω_0 *in* $\dot{W}^{1,1}(\mathbb{R}^2)$ *as* $i \to \infty$, *then the sequence of solutions* $\omega^{(i)}(x,t)$ *to (3) with initial data* $\omega_0^{(i)}$ *converges to* $\omega(x,t)$ *in* $L^\infty([0, t_0), \dot{W}^{1,1}(\mathbb{R}^2))$ *as* $i \to \infty$.

Theorem 2 *Let* $\omega_0 \in \dot{W}^{1,1}(\mathbb{R}^2)$, *and* ω *be the solution to the integral formulation of the vorticity equation (3) given by Theorem 1, with initial vorticity* ω_0. *Define a velocity vector* $\mathbf{v}$ *by the Biot–Savart relation (2). Then* $\mathbf{v}$ *is a distributional* *solution to the two-dimensional incompressible Navier–Stokes (1) up to time* $t_0 := C\nu\|\nabla\omega_0\|^{-2}_{L^1(\mathbb{R}^2)}$, *in the sense that*

$$\begin{cases} \int_0^{t_0}\int_{\mathbb{R}^2} [\mathbf{v}\cdot\partial_t\Phi + \nu\mathbf{v}\cdot\Delta\Phi + \mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\Phi]\, dxdt = -\int_{\mathbb{R}^2} \mathbf{v}(x,0)\cdot\Phi(x,0)dx \\ \int_0^{t_0}\int_{\mathbb{R}^2} \mathbf{v}\cdot\nabla\psi\, dxdt = 0 \end{cases}$$

holds for any $\psi \in C_c^\infty(\mathbb{R}^2\times[0,t_0),\mathbb{R})$, *and any* $\Phi \in C_c^\infty(\mathbb{R}^2\times[0,t_0),\mathbb{R}^2)$ *that satisfies* $\nabla\cdot\Phi = 0$ *for all* $t\in[0,t_0)$. *We also have*

$$\sup_{t\le t_0}\|\mathbf{v}(x,t)\|_{L^\infty(\mathbb{R}^2)} + \sup_{t\le t_0}\|\nabla\mathbf{v}(x,t)\|_{L^2(\mathbb{R}^2)} \le c\|\nabla\omega_0\|_{L^1(\mathbb{R}^2)}, \tag{5}$$

and the pressure $p(x,t) := (-\Delta)^{-1}\nabla\cdot((\mathbf{v}\cdot\nabla)\mathbf{v})$ *satisfies*

$$\sup_{t\le t_0}\|\nabla p(x,t)\|_{L^2(\mathbb{R}^2)} \le c\|\nabla\omega_0\|^2_{L^1(\mathbb{R}^2)}. \tag{6}$$

Note that in these theorems, we are only assuming that the initial vorticity ω_0 is in the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^2)$, contrary to [6] where we assumed the stronger assumption that the initial vorticity is in the inhomogeneous Sobolev space $W^{1,1}(\mathbb{R}^2)$. Giga et al. [8] and Kato [9] showed that the vorticity equation is globally well-posed under the hypothesis that the initial vorticity is a measure; see also an alternative approach in Ben-Artzi [1], and a stronger uniqueness result in Brezis [4]. We point out though that the scaling of their results is different from ours: we are assuming that the *gradient* of the initial vorticity is in L^1. This explains why the solution constructed by Kato satisfies the estimate

$$\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le Ct^{-\frac{1}{2}},\ t\to 0$$

(see Eq. (0.5) of [9]), whereas we can obtain bounds on $\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}$ and $\|\nabla\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)}$ that are uniform in t. Indeed it is known that they could not have done better, without further assumptions on the vorticity: the famous example of the *Lamb–Oseen vortex* for $\nu = 1$ [10] consists of an initial vorticity $\omega_0 = \alpha_0\delta_0$, a Dirac mass at the origin of $\mathbb{R}^2$ where α_0 is a constant (called the total circulation of the vortex). The corresponding solution ω to the vorticity equation (3) with this initial vorticity, and its corresponding velocity $\mathbf{v}$, are given by

$$\omega(x,t) = \frac{\alpha_0}{4\pi t}e^{-\frac{|x|^2}{4t}}, \qquad \mathbf{v}(x,t) = \frac{\alpha_0}{2\pi}\frac{(-x_2,x_1)}{|x|^2}\Big(1-e^{-\frac{|x|^2}{4t}}\Big).$$

We then have

$$\|\omega(\cdot,t)\|_{W^{1,1}(\mathbb{R}^2)} \sim \|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \sim \|\nabla\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \sim ct^{-\frac{1}{2}},\ t\to 0.$$

Hence the assumption that the initial vorticity is a measure cannot yield an estimate like in Theorems 1 or 2.

We mention in passing a result in [7] where an estimate was established for systems of wave equations with divergence-free inhomogeneties.

In order to prove Theorem 1, we rely on a basic proposition that follows from the work of Bourgain and Brezis [2, 3].

Proposition 3 *If $\omega(\cdot, t) \in \dot{W}^{1,1}(\mathbb{R}^2)$ at a time t, then one can define a vector-valued function $\mathbf{v}(\cdot, t)$ via the Biot–Savart relation (2) at this time t, in which case we will have*

$$\|\mathbf{v}(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \mathbf{v}(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla\omega(\cdot, t)\|_{L^1(\mathbb{R}^2)}$$

at this time t. Here C is a constant independent of t and ω.

Proof of Proposition 3 For simplicity, let's fix the time t, and drop the dependence of ω and $\mathbf{v}$ on t in the notation. Note that $(-\partial_{x_2}\omega, \partial_{x_1}\omega)$ is a vector field in $\mathbb{R}^2$ with vanishing divergence. The desired conclusion then follows from (2) and the two-dimensional result of Bourgain and Brezis [3] (see also [11], [5] and [7]).

Since the proof of the two-dimensional result of Bourgain and Brezis [3] is actually quite simple, we adapt it here in our particular setting, for the convenience of the reader.

The main point here is that if $\omega \in C_c^\infty(\mathbb{R}^2)$, then $\mathbf{v} = (v_1, v_2) := (-\Delta)^{-1}(\partial_{x_2}\omega, -\partial_{x_1}\omega)$ satisfies

$$v_1(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \partial_2\omega(x-y)\log\frac{1}{|y|}dy = \frac{1}{2\pi}\int_{\mathbb{R}^2}\omega(x-y)\frac{-y_2}{|y|^2}dy$$

so

$$|v_1(x)| \leq \frac{1}{2\pi}\int_{\mathbb{R}^2}|\omega(x-y)|\frac{1}{|y|}dy \leq c\|\nabla\omega(x-\cdot)\|_{L^1(\mathbb{R}^2)}$$

the last inequality following from an application of Hardy's inequality (alternatively, one can see that the last inequality holds, by writing $\frac{1}{|y|}$ as $\nabla \cdot \frac{y}{|y|}$, and by integrating by parts). This shows

$$\|v_1\|_{L^\infty(\mathbb{R}^2)} \leq c\|\nabla\omega\|_{L^1(\mathbb{R}^2)}.$$

Similarly one shows $\|v_2\|_{L^\infty(\mathbb{R}^2)} \leq c\|\nabla\omega\|_{L^1(\mathbb{R}^2)}$, so

$$\|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)} \leq c\|\nabla\omega\|_{L^1(\mathbb{R}^2)}.$$

Finally,

$$\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)} \leq \|\nabla^2(-\Delta)^{-1}\omega\|_{L^2(\mathbb{R}^2)} \leq \|\omega\|_{L^2(\mathbb{R}^2)} \leq c\|\nabla\omega\|_{L^1(\mathbb{R}^2)},$$

the last inequality following from the Gagliardo–Nirenberg inequality. The above proves the desired conclusion of the proposition under the extra assumption that $\omega \in C_c^\infty(\mathbb{R}^2)$. Since such functions are dense in $\dot{W}^{1,1}(\mathbb{R}^2)$, a standard approximation argument shows that these estimates extend to the general case when $\omega \in \dot{W}^{1,1}(\mathbb{R}^2)$. Hence the full proposition follows. □

Proof of Theorem 1 In the sequel by a scaling we may assume without loss of generality that the viscosity coefficient $\nu = 1$.

Let K_t be the heat kernel on $\mathbb{R}^2$, i.e.

$$K_t(x) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}.$$

Rewriting (3) as an integral equation for ω using Duhamel's theorem, where ω_0 is the initial vorticity, we have,

$$\omega(x,t) = K_t \star \omega_0(x) + \int_0^t \partial_x K_{t-s} \star [\mathbf{v}\omega(x,s)]ds \tag{7}$$

where $\mathbf{v}$ is given by (2).

We apply a Banach fixed point argument to the operator T given by

$$T\omega(x,t) = K_t \star \omega_0(x) + \int_0^t \partial_x K_{t-s} \star [\mathbf{v}\omega(x,s)]ds, \tag{8}$$

where again $\mathbf{v}$ is given by (2). Let us set

$$E = \Big\{ g : \mathbb{R}^2 \times [0, t_0] \to \mathbb{R} \mid \sup_{0<t<t_0} \|\nabla g(x,t)\|_{L^1(\mathbb{R}^2)} \leq A \Big\}.$$

We will first show that T maps E into itself, for t_0 chosen as in the theorem.

Differentiating (8) in the space variable once, we get

$$(T\omega(x,t))_x = K_t \star (\omega_0)_x + \int_0^t \partial_x K_{t-s} \star \big(\mathbf{v}_x\omega\big)ds + \int_0^t \partial_x K_{t-s} \star \big(\mathbf{v}\omega_x\big)ds.$$

By Young's convolution inequality, we have

$$\|(T\omega(\cdot,t))_x\|_{L^1(\mathbb{R}^2)} \leq \|(\omega_0)_x\|_{L^1(\mathbb{R}^2)} + C\int_0^t (t-s)^{-1/2}(\|\mathbf{v}_x\omega\|_{L^1(\mathbb{R}^2)} + \|\mathbf{v}\omega_x\|_{L^1(\mathbb{R}^2)})ds.$$

Now we apply Proposition 3 to each of the terms in the integral on the right hand side. For the first term we have, by Cauchy-Schwarz followed by Gagliardo–Nirenberg, that

$$\|\mathbf{v}_x\omega\|_{L^1(\mathbb{R}^2)} \leq C\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)}\|\omega\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)}\|\omega_x\|_{L^1(\mathbb{R}^2)}.$$

We control $\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}$ with Proposition 3: this gives

$$\|\mathbf{v}_x \omega\|_{L^1(\mathbb{R}^2)} \le C\|\omega_x\|^2_{L^1(\mathbb{R}^2)}.$$

Similarly, by Proposition 3, for the second term, we have

$$\|\mathbf{v}\omega_x\|_{L^1(\mathbb{R}^2)} \le \|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)}\|\omega_x\|_{L^1(\mathbb{R}^2)} \le C\|\omega_x\|^2_{L^1(\mathbb{R}^2)}.$$

Altogether, we have,

$$\|(T\omega)_x\|_{L^1(\mathbb{R}^2)} \le \|\nabla\omega_0\|_{L^1(\mathbb{R}^2)} + C\int_0^t (t-s)^{-1/2}\|\nabla\omega\|^2_{L^1(\mathbb{R}^2)} ds.$$

Thus if $\|\nabla\omega_0\|_{L^1(\mathbb{R}^2)} \le A_0$, then since $\omega \in E$, we have

$$\sup_{0\le t\le t_0} \|\nabla(T\omega)(x,t)\|_{L^1(\mathbb{R}^2)} \le A_0 + Ct_0^{1/2}A^2.$$

By choosing A so that $A_0 = A/2$ and $t_0 = 1/(2CA)^2$, we see that if $\omega \in E$, then

$$\sup_{0\le t\le t_0} \|\nabla_x(T\omega)(x,t)\|_{L^1(\mathbb{R}^2)} \le A,$$

i.e. $T\omega \in E$. It remains to show that T is a contraction on E.

For this let $\omega_1(x,t), \omega_2(x,t) \in E$. We just need to observe that from Proposition 3, we get

$$\|\mathbf{v}_1 - \mathbf{v}_2\|_\infty + \|\nabla\mathbf{v}_1 - \nabla\mathbf{v}_2\|_2 \le C\|\nabla(\omega_1 - \omega_2)\|_{L^1(\mathbb{R}^2)}.$$

Thus repeating our earlier computations, we see that

$$\sup_{0\le t\le t_0} \|\nabla(T\omega_1 - T\omega_2)\|_{L^1(\mathbb{R}^2)} \le Ct_0^{1/2}A \sup_{0\le t\le t_0} \|\nabla(\omega_1 - \omega_2)\|_{L^1(\mathbb{R}^2)}.$$

By the choice of t_0, it is seen that T is a contraction. Thus using the Banach fixed point theorem on E, we obtain our operator T has a fixed point and so the integral equation (7) has a unique solution in E. The continuous dependence on initial data can be proved in an identical way, and we will not repeat the details here. □

Proof of Theorem 2 Let $\omega_0 \in \dot{W}^{1,1}(\mathbb{R}^2)$, and $\omega(x,t)$ be the unique solution to (7) given above. Let $\mathbf{v}(x,t)$ be defined by the Biot–Savart relation (2) as in Proposition 3. If $\omega_0^{(i)}$ is a sequence of functions in $C_c^\infty(\mathbb{R}^2)$ converging to ω_0 in $\dot{W}^{1,1}(\mathbb{R}^2)$, then the corresponding solution $\omega^{(i)}(x,t)$ to the vorticity equation (3) converges to $\omega(x,t)$ in $L^\infty([0,t_0), \dot{W}^{1,1}(\mathbb{R}^2))$. Thus the velocities $\mathbf{v}^{(i)} := (-\Delta)^{-1}(-\partial_{x_2}\omega^{(i)}, \partial_{x_1}\omega^{(i)})$ converges in $L^\infty([0,t_0); L^\infty(\mathbb{R}^2))$ to $\mathbf{v}$. But since $\omega_0^{(i)} \in C_c^\infty(\mathbb{R}^2)$, which are in particular in the inhomogeneous Sobolev space $W^{1,1}(\mathbb{R}^2)$, so we may apply

Theorem II of Kato [9] as in [6], and conclude that the $\mathbf{v}^{(i)}$ defined above solves the Navier–Stokes equation (1), at least in the distributional sense. We can now pass to limit as $i \to \infty$, using the convergence of $\mathbf{v}^{(i)}$ to $\mathbf{v}$ in $L^\infty([0,t_0), L^\infty(\mathbb{R}^2))$ we obtained above, and appealing to the dominated convergence theorem: this shows that $\mathbf{v}(x,t)$ is also a distributional solution to (1) up to time t_0, in the sense that

$$\begin{cases} \int_0^{t_0} \int_{\mathbb{R}^2} [\mathbf{v} \cdot \partial_t \Phi + \mathbf{v} \cdot \Delta \Phi + \mathbf{v} \cdot (\mathbf{v} \cdot \nabla)\Phi] \, dxdt = -\int_{\mathbb{R}^2} \mathbf{v}(x,0) \cdot \Phi(x,0) dx \\ \int_0^{t_0} \int_{\mathbb{R}^2} \mathbf{v} \cdot \nabla \psi \, dxdt = 0 \end{cases}$$

holds for any $\psi \in C_c^\infty(\mathbb{R}^2 \times [0,t_0), \mathbb{R})$, and any $\Phi \in C_c^\infty(\mathbb{R}^2 \times [0,t_0), \mathbb{R}^2)$ that satisfies $\nabla \cdot \Phi = 0$ for all $t \in [0,t_0)$. The estimate (5) then follows from Propositions 3 and (4). Lastly we observe that the estimate (6) follows, from the fact that the pressure $p(x,t)$ satisfies the equation

$$-\Delta p = \nabla \cdot ((\mathbf{v} \cdot \nabla)\mathbf{v}),$$

which is a consequence of taking the divergence of the Navier–Stokes equation. □

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