# Two nonlinear wave equations with conformal invariance

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# The wave equations

- Joint work with Sagun Chanillo
- Wave Constant Mean Curvature (CMC) equation on $\mathbb{R}^2$

$$u\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^3$$

$$-\partial_t^2 u + \Delta u = 2u_x \wedge u_y$$

(system of equations)

- Wave Liouville equation on $\mathbb{S}^2$

$$u\colon \mathbb{R} \times \mathbb{S}^2 \to \mathbb{R}$$

$$\partial_t^2 u - \Delta_g u = \alpha \left( \frac{e^{2u}}{\fint_{\mathbb{S}^2} e^{2u}} - 1 \right)$$

($\alpha$ is a real parameter, $g$ standard metric on $\mathbb{S}^2$)

# The elliptic analogs

- (Case $|u_x| = |u_y| = 1,\ u_x \cdot u_y = 0$)
  Constant Mean Curvature (CMC) equation on $\mathbb{R}^2$:

  $$\Delta u = 2 u_x \wedge u_y$$

  The image of any such $u$ is a surface with mean curvature $H \equiv 1$ in $\mathbb{R}^3$
- (Case $\alpha = 1,\ \fint_{\mathbb{S}^2} e^{2u} = 1$) Liouville equation on $\mathbb{S}^2$:

  $$-\Delta_g u = e^{2u} - 1$$

  If $u$ is as such, then $e^{2u} g$ is another metric on $\mathbb{S}^2$, conformal to $g$, that has Gaussian curvature equal to 1 everywhere, and that has area equal to $4\pi$.

# The CMC equation

- The static equation:

$$\Delta u = 2u_x \wedge u_y$$

- Energy ($\dot{H}^1$) critical: if $u$ is a solution, then a dilation of $u$ preserving its $\dot{H}^1$ norm is also a solution.
- (Entire) $\dot{H}^1(\mathbb{R}^2)$ solutions classified by Brezis-Coron: All are of the form

$$u(z) = \pi\left(\frac{P(z)}{Q(z)}\right) + C$$

  where $P$, $Q$ are polynomials in $z$ and $\pi\colon \mathbb{C} \to \mathbb{S}^2 \subseteq \mathbb{R}^3$ is the stereographic projection, and $C$ is a constant vector in $\mathbb{R}^3$.
- For such $u$,

$$\|\nabla u\|_{L^2}^2 = 8\pi \max\{\deg P, \deg Q\};$$

  kinetic energy quantized.

- Now let $W(z)$ be a ground state solution to the static CMC equation; in other words, $W(z)$ is a non-constant solution of the form
$$W(z) = \pi\left(\frac{P(z)}{Q(z)}\right) + C$$
where $\max\{\deg P, \deg Q\} = 1$ and $\|\nabla W\|_{L^2}^2 = 8\pi$.
- Sobolev inequality: for all functions $v \in \dot{H}^1(\mathbb{R}^2)$ taking values in $\mathbb{R}^3$, we have
$$\left|\int_{\mathbb{R}^2} v \cdot (v_x \wedge v_y) dxdy\right|^{1/3} \leq C\|\nabla v\|_{L^2}.$$
(Compensation compactness/Wente's inequality)
- Brezis-Coron (also Caldiroli-Musina): The $W$ we have above are minimizers of this inequality.
- $W$ is also a stationary solution to the wave CMC with initial data $u(0) = W$ and $\partial_t u(0) = 0$.

# The wave CMC equation

- The wave CMC equation again:

$$-\partial_t^2 u + \Delta u = 2u_x \wedge u_y$$

- Conserved energy (for 'nice' solutions):

$$E(u(t)) := \int_{\mathbb{R}^2} \frac{1}{2}(|\partial_t u|^2 + |\nabla u|^2) + \frac{2}{3} u \cdot (u_x \wedge u_y) dxdy$$

  is preserved along the flow of wave CMC (hence depends only on initial data). Sometimes we write

$$E(u_0, u_1) := \int_{\mathbb{R}^2} \frac{1}{2}(|u_1|^2 + |\nabla u_0|^2) + \frac{2}{3} u_0 \cdot ((u_0)_x \wedge (u_0)_y) dxdy$$

- The non-linearity of previous Sobolev inequality arises in this conserved energy.

## Theorem (Failure of global existence)

*Suppose* $u\colon [0,T) \times \mathbb{R}^2 \to \mathbb{R}^3$ *is a smooth solution to wave CMC with initial data* $u(0) = u_0$, $u_t(0) = u_1$, *and that* $u$ *has compact support at each time slice* $t$. *Suppose also that*

$$E(u_0, u_1) < E(W, 0) \quad \textit{and} \quad \|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}.$$

*Then* $T$ *is finite; in fact* $\|u(t)\|_{L^2(\mathbb{R}^2)}$ *cannot remain finite for an infinite amount of time.*

- Analog of the finite time breakdown result of Kenig-Merle for the energy critical semi-linear focusing wave equation

$$\partial_t^2 u - \Delta u = |u|^{4/(N-2)} u \quad \text{on } \mathbb{R} \times \mathbb{R}^N,\ N \geq 3.$$

# Proof of Theorem

- For the moment $u$ will be a map from $\mathbb{R}^2$ into $\mathbb{R}^3$ independent of time.
- Stationary energy:

$$\mathcal{E}(u) := \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \frac{2}{3} u \cdot (u_x \wedge u_y).$$

- Recall the Sobolev inequality for such $u$:

$$\left| \int_{\mathbb{R}^2} u \cdot (u_x \wedge u_y) \right| \leq C \|\nabla u\|_{L^2}^3.$$

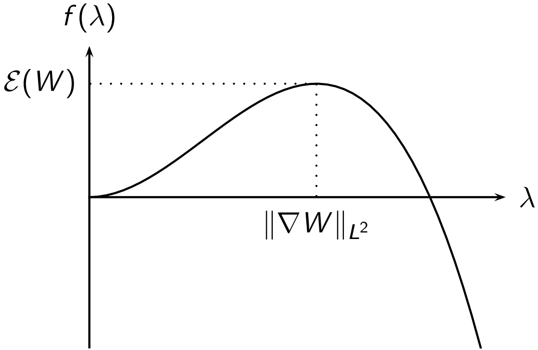
- Let $C$ be the best constant of this inequality and

$$f(\lambda) = \frac{1}{2}\lambda^2 - \frac{2}{3} C \lambda^3 \quad \text{for } \lambda > 0.$$

Then for all $u$ as above, by this Sobolev inequality,

$$f(\|\nabla u\|_{L^2}) \leq \mathcal{E}(u).$$

- Graph of $f$:




- Any point of the form $(\|\nabla u\|_{L^2}, \mathcal{E}(u))$ always lie above the graph of $f$.

- Now recall the conditions of our theorem: Suppose $u\colon [0,T) \times \mathbb{R}^2 \to \mathbb{R}^3$ is a smooth solution to wave CMC with initial data $u_0$, $u_1$. Suppose also that

$$E(u_0, u_1) < E(W, 0) \quad \text{and} \quad \|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}.$$

- Then by conservation of energy,

$$\mathcal{E}(u(t)) < \mathcal{E}(W) \quad \text{for all } t \in [0, T).$$

- By continuity in $t$ and the above picture, we have the following observation:

$$\|\nabla u(t,x)\|_{L^2(dx)} > \|\nabla W\|_{L^2(dx)}$$

  for all $t \in [0, T)$.
- We want to prove that $T$ must be finite.

- Let
$$y(t) = \|u(t,x)\|^2_{L^2(dx)}$$
for $t \in [0, T)$. Then
$$y'(t) = \int_{\mathbb{R}^2} 2u \cdot u_t$$
and
$$y''(t) = \int_{\mathbb{R}^2} 2|u_t|^2 - 2|\nabla u|^2 - 4u \cdot (u_x \wedge u_y).$$
- Suppose now $E(u_0, u_1) \leq E(W, 0) - \varepsilon$ for some $\varepsilon > 0$. By conservation of energy, we have at any time $t \in [0, T)$,
$$\int_{\mathbb{R}^2} -4u \cdot (u_x \wedge u_y) \geq \int_{\mathbb{R}^2} 3(|u_t|^2 + |\nabla u|^2) - 6E(W, 0) + 6\varepsilon.$$

- With a little more work, this implies

$$y''(t) \geq \int_{\mathbb{R}^2} (5|u_t|^2 + |\nabla u|^2 - |\nabla W|^2) dx + 6\varepsilon$$

(just use $\Delta W = 2W_x \wedge W_y$).

- But we observed that at any time $t \in [0, T)$,

$$\|\nabla u(t,x)\|_{L^2(dx)} > \|\nabla W\|_{L^2(dx)}.$$

- Hence

$$y''(t) \geq 5\|u_t(t,x)\|^2_{L^2(dx)} + 6\varepsilon,$$

which implies

$$\frac{y''(t)}{y'(t)} \geq \frac{5}{4} \frac{y'(t)}{y(t)}$$

for sufficiently large $t$.

- Solving the equation, $y(t)$ becomes infinite in finite time, and therefore $T$ cannot be infinite.

# Local well-posedness of the wave CMC in low regularities

- Consider an initial value problem for the wave CMC:

$$-\partial_t^2 u + \Delta u = 2u_x \wedge u_y, \quad u(0) = u_0, \quad \partial_t u(0) = u_1.$$

  Given initial data $u_0 \in H^s$, $u_1 \in H^{s-1}$, does this initial value problem admits a unique solution in $C_t^0 H_x^s \cap C_t^1 H_x^{s-1}$?
- When $s > 2$ the answer is yes by standard iteration and Sobolev inequalities. We are interested going below $s > 2$.
- Difficulty: non-linearity contains first order derivatives, and that we are in low (2+1) dimensions
- Fortunately, the non-linearity has a certain *null structure*.
- Best hope: since the wave CMC is energy critical, we hope to go as low as $s = 1$.
- c.f. well-posedness of wave maps in $2 + 1$ dimensions in $H^s$; there one can actually go as low as $s = 1$.

# Null structures

- Given two functions $u$ and $v$ on the Minkowski space $\mathbb{R}^{1,n}$, define the null forms

$$Q_{00}(u,v) = -\partial_t u \partial_t v + \nabla_x u \cdot \nabla_x v$$
$$Q_{ij}(u,v) = \partial_{x_i} u \, \partial_{x_j} v - \partial_{x_j} u \, \partial_{x_i} v$$
$$Q_{0i}(u,v) = \partial_t u \, \partial_{x_i} v - \partial_{x_i} u \, \partial_t v$$

  Here $i, j$ ranges over $1, \ldots, n$.
- $Q_{00}$ arises in the nonlinearity of the wave map equation, while $Q_{ij}$ arises in the nonlinearity of the wave CMC.
- The null forms are better nonlinearities than things like $\partial_t u \, \partial_t v$. They 'damp' the interactions of the waves along the light cone, which are usually the hardest to control.

# The wave Sobolev spaces

- To proceed further, we need some Sobolev spaces adapted to the study of wave equations.
- A function $u(t,x)$ is said to be in $H^{s,b}$, if

$$\int_{\mathbb{R}^{1+2}} (1+|\xi|^2)^s (1+||\tau|-|\xi||^2)^b |\tilde{u}(\tau,\xi)|^2 d\tau d\xi < \infty.$$

  Here $\tilde{u}(\tau,\xi)$ is the space-time Fourier transform of $u(t,x)$.
- We say $u \in \mathcal{H}^{s,b}$, if $u \in H^{s,b}$ and $\partial_t u \in H^{s-1,b}$.
- Note that these are $L^2$ Sobolev spaces; $s$ and $b$ refer to differentiability in two different directions.

# The iteration scheme

- To solve the initial value problem for the wave CMC, we fix initial data $u_0$ and $u_1$. Given a function $u(t,x)$, let $S(u)$ be the solution of the following linear initial value problem:

$$-\partial_t^2 v + \Delta v = 2u_x \wedge u_y, \quad v(0) = u_0, \quad \partial_t v(0) = u_1.$$

- We want to show that $S$ has a fixed point, so we want $S$ to be a contraction mapping in some suitable function space.
- When the initial data has low regularity, say are in $H^s \times H^{s-1}$, the correct space to use is $\mathcal{H}^{s,b}$, for some $b$ slightly $> 1/2$.
- In fact, need the localization of $\mathcal{H}^{s,b}$ to the time interval $[0, T]$, which we denote $\mathcal{H}^{s,b}_T$.

# Energy estimates

- Reason for using $\mathcal{H}^{s,b}$: There is the following energy estimate (when $b > 1/2$; here $\Box = \partial_t^2 v - \Delta$):

$$\|v\|_{\mathcal{H}_T^{s,b}} \lesssim \|v(0)\|_{H^s} + \|\partial_t v(0)\|_{H^{s-1}} + \|\Box v\|_{H^{s-1,b-1}}.$$

- When applied to $v = S(u)$, we get

$$\|S(u)\|_{\mathcal{H}_T^{s,b}} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \|2u_x \wedge u_y\|_{H^{s-1,b-1}}, \quad b > 1/2.$$

- Good news: if $u_x \wedge u_y$ here were the null form $Q_{00}(u,u)$, then we can finish this off by estimating

$$\|Q_{00}(u,u)\|_{H^{s-1,b-1}} \lesssim T^{\varepsilon/2} \|u\|_{\mathcal{H}_T^{s,b}}$$

  for some small $\varepsilon > 0$ (when $b$ is only slightly $> 1/2$). If $T$ were sufficiently small, then $S$ is a contraction map.
- Bad news: $u_x \wedge u_y$ is NOT $Q_{00}(u,u)$; it is the null form $Q_{ij}$ instead, which behaves worse than $Q_{00}$ in 2+1 dimensions.

- At this point, it is instructive to compare the energy estimate in $\mathcal{H}^{s,b}$ ($b > 1/2$):

$$\|v\|_{\mathcal{H}^{s,b}} \lesssim \|v(0)\|_{H^s} + \|\partial_t v(0)\|_{H^{s-1}} + \|\Box v\|_{H^{s-1,b-1}}$$

to the energy estimate in $C_t^0 H_x^s \cap C_t^1 H_x^{s-1}$:

$$\|v\|_{C_t^0 H_x^s \cap C_t^1 H_x^{s-1}} \lesssim \|v(0)\|_{H^s} + \|\partial_t v(0)\|_{H^{s-1}} + \|\Box v\|_{L_t^1 H_x^{s-1}}.$$

- The latter is a gain in integrability in $t$: it says that to control $v$ in $L_t^\infty$, one only needs to control $\Box v$ in $L_t^1$. But one needs the same number of $t$ derivatives of $v$ on both sides.
- The former is better because one gains differentiability in $b$ there: to control $b$ derivatives of $v$, one only needs to control $b-1$ derivatives of $\Box v$.
- When the equation says that $\Box v$ is already one derivative of $v$, gain in differentiability is certainly better than gain in integrability.

# Open question

- It is still open whether the wave CMC is locally well-posed in $H^s$ for any $s \leq 2$.
- Because of our specific null structures ($Q_{ij}$ instead of $Q_{00}$), some additional difficulties may occur below $s > 5/4$ (contrary to wave maps)
- One may also need to iterate only in a subspace of $\mathcal{H}^{s,b}$, as in the case of Maxwell-Klein-Gordon, or Yang-Mills, equations.

# The wave Liouville equation

- The equation again:

$$u : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{R}$$

$$\partial_t^2 u - \Delta_g u = \alpha \left( \frac{e^{2u}}{\fint_{\mathbb{S}^2} e^{2u}} - 1 \right)$$

($\alpha$ is a real parameter, and we use standard metric $g$ on $\mathbb{S}^2$)

**Theorem (local well-posedness in $\dot{H}^1$)**

*Suppose $\alpha \in \mathbb{R}$. For any $u_0 \in \dot{H}^1(\mathbb{S}^2)$ and $u_1 \in L^2(\mathbb{S}^2)$ with $\int_{\mathbb{S}^2} u_1 = 0$, there exists $u$ in $C_t^0 \dot{H}_x^1 \cap C_t^1 L_x^2$ that solves the wave Liouville with initial data $u(0) = u_0$, $\partial_t u(0) = u_1$.*

(Proof by standard iteration; omitted)

- The facts that $\fint_{\mathbb{S}^2} u_1 = 0$ and $\fint_{\mathbb{S}^2} \left( \frac{e^{2u}}{\fint_{\mathbb{S}^2} e^{2u}} - 1 \right) = 0$ are useful in iterating using the Duhamel formula:

$$u(t,x) = \cos(t\sqrt{-\Delta_g})u_0 + \frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}} u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}} \left( \frac{e^{2u}}{\fint_{\mathbb{S}^2} e^{2u}} - 1 \right)(s)ds.$$

- The solution in this theorem is guaranteed to exist for time $T > 0$ with $T$ depending only on $\alpha$, $\|u_0\|_{\dot{H}^1}$ and $\|u_1\|_{L^2}$.
- Furthermore,

$$\int_{\mathbb{S}^2} u(t) = \int_{\mathbb{S}^2} u_0$$

for all $t \in [0, T]$.

- Conservation of energy:

$$E(u(t)) = \fint_{\mathbb{S}^2} \left(|\partial_t u|^2 + |\nabla u|^2\right) - \alpha \log\left(\fint_{\mathbb{S}^2} e^{2(u-\bar{u})}\right)$$

  conserved along the flow, where $\bar{u} = \fint_{\mathbb{S}^2} u$. Using this and the Moser-Trudinger inequality, one can prove:

Theorem (global well-posedness in $\dot{H}^1$ when $\alpha < 1$)

*The solution $u(t,x)$ in the previous theorem exists for all time if $\alpha < 1$.*

- Recall Moser-Trudinger on $\mathbb{S}^2$: if $u$ is a function on $\mathbb{S}^2$ satisfying $\int_{\mathbb{S}^2} |\nabla u|^2 \leq 1$, then

$$\fint_{\mathbb{S}^2} e^{4\pi (u-\bar{u})^2} \leq C.$$

- This inequality is sharp in that one cannot replace $4\pi$ in the exponent by anything that is strictly bigger. Note that this inequality can also be stated as

$$\fint_{\mathbb{S}^2} \exp\left( \frac{(u-\bar{u})^2}{\fint_{\mathbb{S}^2} |\nabla u|^2} \right) \leq C,$$

if $\fint_{\mathbb{S}^2} |\nabla u|^2 < \infty$.

- What we will usually use is the following corollary of the above inequality, namely

$$\fint_{\mathbb{S}^2} e^{2(u-\bar{u})} \leq C \exp\left(\fint_{\mathbb{S}^2} |\nabla u|^2\right),$$

which holds because pointwise $2(u-\bar{u}) \leq \frac{(u-\bar{u})^2}{\fint_{\mathbb{S}^2} |\nabla u|^2} + \fint_{\mathbb{S}^2} |\nabla u|^2$. Equivalently, the above inequality can be stated

$$\log\left(\fint_{\mathbb{S}^2} e^{2(u-\bar{u})}\right) \leq 1 \cdot \fint_{\mathbb{S}^2} |\nabla u|^2 + \log C.$$

Note that the left-hand side above arises in the conserved energy.

- Onofri: $C$ can be taken to be zero here. But we will not need this refinement.

- Using Moser-Trudinger and conservation of energy:

$$E(u(t)) = \fint_{\mathbb{S}^2} \left(|\partial_t u|^2 + |\nabla u|^2\right) - \alpha \log\left(\fint_{\mathbb{S}^2} e^{2(u-\bar{u})}\right)$$

one can then control, as long as the solution exists, the quantity

$$\|\partial_t u(t)\|_{L^2(\mathbb{S}^2)} + \|\nabla u(t)\|_{L^2(\mathbb{S}^2)}$$

uniformly in $t$ when $\alpha$ is small, and this will prove global well-posedness when $\alpha < 1$.

- Moser also proved the following refinement of Moser-Trudinger: if $u$ is an *even* function on $\mathbb{S}^2$ satisfying $\int_{\mathbb{S}^2} |\nabla u|^2 \leq 1$, then

$$\fint_{\mathbb{S}^2} e^{8\pi(u-\bar{u})^2} \leq C.$$

It follows that for such functions,

$$\log\left(\fint_{\mathbb{S}^2} e^{2(u-\bar{u})}\right) \leq \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla u|^2 + \log C.$$

- Using this (and conservation of energy), one can prove

Theorem (global well-posedness in $\dot{H}^1$ for even data when $\alpha < 2$)

*If both initial data $u_0$ and $u_1$ are even functions on $\mathbb{S}^2$, then the solution $u(t,x)$ of the wave Liouville exists for all time if $\alpha < 2$.*

- We also mention the analogous results for wave Liouville systems:

$$\partial_t^2 u_i - \Delta_g u_i = \sum_{j=1}^{N} a_{ij} M_j \left( \frac{e^{2u_j}}{\fint_{\mathbb{S}^2} e^{2u_j}} - 1 \right), \quad i = 1, \dots, N \quad \text{on } \mathbb{S}^2.$$

  Here $(a_{ij})$ is a (constant) $N$ by $N$ symmetric matrix, and $(M_j)$ is a vector.
- Key is a Moser-Trudinger inequality for systems: c.f. work of Shafrir-Wolansky, Wang, Chanillo-Kiessling.

# Blow-up for (scalar) wave Liouville when $1 \leq \alpha < 2$

- If $u$ is a function defined on $\mathbb{S}^2$, define

$$CM(u) = \frac{\int_{\mathbb{S}^2} x e^{2u}}{\int_{\mathbb{S}^2} e^{2u}},$$

  where $x$ is the position vector in $\mathbb{R}^3$ (This is the center of mass of the measure $e^{2u} d\mathrm{vol}_g$).
- Thus $CM(u) \in \mathbb{R}^3$; in fact $|CM(u)| \leq 1$.
- Aubin proved the following improved Moser-Trudinger inequality on $\mathbb{S}^2$ (see also Chang-Yang and Han): if $|CM(u)| \leq 1 - \delta$ for some $\delta > 0$, then for any $\mu > 1/2$, there exists a constant $C = C(\mu, \delta)$ such that

$$\log \left( \fint_{\mathbb{S}^2} e^{2(u - \bar{u})} \right) \leq \mu \fint_{\mathbb{S}^2} |\nabla u|^2 + \log C.$$

- One should compare this with the improved Moser-Trudinger inequality for even functions, since when $u$ is even, $CM(u) = 0$.

We have the following blow-up criteria.

### Theorem (Blow up criteria for scalar wave Liouville)

*Let* $1 \leq \alpha < 2$. *Suppose a local in time solution* $u$ *to wave Liouville exists on a time interval* $[0, T_0)$ *for some* $T_0 < \infty$, *and fails to continue beyond* $T_0$. *Then there is a sequence of times* $t_i \to T_0^-$ *such that*

$$\lim_{i\to\infty} |CM(u, t_i)| = 1, \quad \lim_{i\to\infty} \int_{\mathbb{S}^2} e^{2u(t_i)} = \infty,$$

*and*

$$\lim_{i\to\infty} \|\nabla u(t_i)\|_{L^2} = \infty,$$

*where* $CM(u, t)$ *is the center of mass of* $u(t)$. *Furthermore, if* $\alpha = 1$, *then there is some point* $p \in \mathbb{S}^2$ *such that for any* $\varepsilon > 0$,

$$\lim_{i\to\infty} \frac{\int_{B(p,\varepsilon)} e^{2u(t_i)}}{\int_{\mathbb{S}^2} e^{2u(t_i)}} \geq 1 - \varepsilon.$$

*Here* $B(p, \varepsilon) =$ *geodesic ball on* $\mathbb{S}^2$ *centered at* $p$ *and of radius* $\varepsilon$.

# Proof of Blow-up criteria

- The fact that one can take $t_i \to T^-$ such that

$$\lim_{i\to\infty} |CM(u, t_i)| = 1$$

follows from Aubin's improvement of Moser-Trudinger when the center of mass stays away from the unit sphere; if $\limsup_{t\to T^-} |CM(u, t)| < 1$ then one can continue the solution past $T$.

- The fact that one can take $t_i \to T^-$ such that

$$\lim_{i\to\infty} \int_{\mathbb{S}^2} e^{2u(t_i)} = \infty$$

follows from conservation of energy:

$$\limsup_{t\to T^-} \left(\|\partial_t u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)$$
$$\lesssim E(u_0, u_1) + \alpha \log\left(\limsup_{t\to T^-} \fint_{\mathbb{S}^2} e^{2u(t)}\right) + 2\alpha|\bar{u}(0)|$$

If $\limsup_{t\to T^-} \int_{\mathbb{S}^2} e^{2u(t)} < \infty$ then one can continue the solution past $T$.

- Since
$$\lim_{i\to\infty}\int_{\mathbb{S}^2} e^{2u(t_i)} = \infty,$$
it follows from Moser-Trudinger that
$$\lim_{i\to\infty}\|\nabla u(t_i)\|_{L^2} = \infty.$$

- The last part of the theorem follows from the following concentration lemma of Chang-Yang:

## Lemma (Chang-Yang)

*Suppose* $v_i \in \dot{H}^1(\mathbb{S}^2)$ *is a sequence of functions with* $⨍_{\mathbb{S}^2} e^{2v_i} = 1$ *and* $\sup_i ⨍_{\mathbb{S}^2}(|\nabla v_i|^2 + 2v_i) = C < \infty$. *We have either*

$$\sup_i ⨍_{\mathbb{S}^2} |\nabla v_i|^2 = C' < \infty,$$

*or there exists a point* $p \in \mathbb{S}^2$ *and a subsequence of* $v_i$ *(which we still denote by* $v_i$*) such that for any* $\varepsilon > 0$, *we have*

$$\lim_{i \to \infty} \frac{1}{4\pi} \int_{B(p,\varepsilon)} e^{2v_i} \geq 1 - \varepsilon.$$

*Here* $B(p, \varepsilon)$ *is the geodesic ball on* $\mathbb{S}^2$ *centered at* $p$ *and of radius* $\varepsilon$.

- Remember what we want to prove: let $\alpha = 1$ and $u$ solve the wave Liouville on $[0, T)$ that does not extend past $T$. Then there is some point $p \in \mathbb{S}^2$ such that for any $\varepsilon > 0$,

$$\lim_{i \to \infty} \frac{\int_{B(p,\varepsilon)} e^{2u(t_i)}}{\int_{\mathbb{S}^2} e^{2u(t_i)}} \geq 1 - \varepsilon.$$

- To do so, let $m_i = \fint_{\mathbb{S}^2} e^{2u(t_i)}$ and $v_i(x) = u(t_i, x) - \frac{1}{2} \log m_i$. One can apply Chang-Yang lemma on these $v_i$: it is easy to check that $\fint_{\mathbb{S}^2} e^{2v_i} = 1$, and

$$\begin{aligned}
\sup_i \fint_{\mathbb{S}^2} (|\nabla v_i|^2 + 2v_i) &= \fint_{\mathbb{S}^2} |\nabla u(t_i, x)|^2 - \log \left( \fint_{\mathbb{S}^2} e^{2(u(t_i) - \bar{u}(t_i))} \right) \\
&= E(u(t_i)) - \fint_{\mathbb{S}^2} |\partial_t u(t_i)|^2 \quad \text{since } \alpha = 1 \\
&\leq E(u(0))
\end{aligned}$$

independent of $i$.

- Hence we have two scenarios: Either

$$\fint_{\mathbb{S}^2} |\nabla v_i|^2 = C' < \infty,$$

or there exists a point $p \in \mathbb{S}^2$ and a subsequence of $v_i$ (which we still denote by $v_i$) such that for any $\varepsilon > 0$, we have

$$\lim_{i\to\infty} \frac{1}{4\pi} \int_{B(p,\varepsilon)} e^{2v_i} \geq 1 - \varepsilon.$$

The first alternative cannot happen since

$$\|\nabla v_i\|_{L^2} = \|\nabla u(t_i)\|_{L^2} \to \infty \quad \text{as } i \to \infty$$

by the third part of the blow-up criteria. Hence the second alternative holds, and this is the conclusion of the theorem.

# Constructing finite time blow-ups for wave Liouville

- Does one have global existence for the scalar wave Liouville in the critical case $\alpha = 1$? If not, can one exhibit a finite time blow up of the equation?
- Problem: when $\alpha = 1$, there is no initial data of negative energy, thanks to Onofri. So one cannot easily construct blow-up as we have done before.
- One could try an ODE blow-up, but it didn't work either.

- Let's borrow some analogy from the study of the energy critical focussing semilinear wave equation in $3+1$ dimensions (c.f. Duyckaerts-Kenig-Merle, 2011, 2012): the equation is

$$\partial_t^2 u - \Delta u = u^5 \quad \text{on } \mathbb{R} \times \mathbb{R}^3.$$

- Two kinds of blow-up: (Let $T$ be the blow-up time)

  Type I: $\sup_{t \in [0,T)} \|\nabla u(t)\|_{L^2_x} + \|\partial_t u(t)\|_{L^2_x} = \infty$

  Type II: otherwise

- Krieger-Schlag-Tataru constructed a type II blow up with

$$\sup_{t \in [0,T)} \|\nabla u(t)\|_{L^2} < \|\nabla W\|_{L^2} + \varepsilon$$

  where $W$ is the groundstate for this equation.

- Duyckaerts-Kenig-Merle showed a profile decomposition for type II radial blow-ups

- We would be happy to see even just a type I blow-up for the wave Liouville when $\alpha = 1$.
- Question: understand $\|\nabla u\|_{L^2}$ when the Onofri energy of $u$, namely

$$\mathcal{E}(u) := \fint_{\mathbb{S}^2} |\nabla u|^2 - \log\left(\fint_{\mathbb{S}^2} e^{2(u-\bar{u})}\right),$$

is controlled.