A smooth subadditive homogeneous norm on a homogeneous group

by

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Abstract. We prove that on every homogeneous group there exists a smooth, subadditive and homogeneous norm.

Introduction. Around 1970 E. M. Stein introduced the notion of a homogeneous group. Such a group $G$ admits a homogeneous norm $\| \cdot \|$, which for a $\gamma \geq 1$ satisfies
\[ \|xy\| \leq \gamma (\|x\| + \|y\|) \quad \text{for all } x, y \in G. \]
The group equipped with $\| \cdot \|$ and the Haar (Lebesgue) measure is a space of homogeneous type in the sense of [1]. A number of estimates become easier if $\gamma = 1$, i.e. if the homogeneous norm is subadditive, so that it gives rise to a left-invariant metric. It is known that for some homogeneous groups such a norm exists, e.g. for Heisenberg groups and the like [2]. Also for stratified groups the optimal control metric is homogeneous.
The aim of this note is to show that a homogeneous and subadditive norm exists for every homogeneous group and in fact the construction is quite simple. More information about such norms is supplied by Theorem 2.
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A smooth subadditive homogeneous norm on a homogeneous group.
A family of dilations on a nilpotent Lie algebra $G$ is a one-parameter group \[ \{ \delta_t \}_{t>0} \quad \delta_t \circ \delta_s = \delta_{ts} \] of automorphisms of $G$ determined by
\[ \delta_t e_j = t^{d_j} e_j, \]
where $e_1, \ldots, e_n$ is a linear basis for $G$, the $d_j$ are real numbers and $d_n \geq \ldots \geq d_1 \geq 1$. If we put $\langle x_1, \ldots, x_n \rangle = \sum x_i e_i$, then
\[ \delta_t \langle x_1, \ldots, x_n \rangle = \langle t^{d_1} x_1, \ldots, t^{d_n} x_n \rangle. \]

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If we regard $G$ as a Lie group with multiplication given by the Campbell–Hausdorff formula, then the dilations $\delta_t$ are also automorphisms of the group structure on $G$, and the nilpotent group $G$ equipped with these dilations is called a homogeneous group (cf. [3]).

We are going to show that on every homogeneous group $G$ there exists a subadditive and homogeneous norm, i.e. a function \( \| \cdot \| : G \to \mathbb{R}^+ \cup \{0\} \) such that

\[
\begin{align*}
(a) \quad \| xy \| &\leq \| x \| + \| y \|, \\
(b) \quad \| \delta_t x \| & = t \| x \|, \\
(c) \quad \| x \| = 0 \iff x = 0, \\
(d) \quad \| x \| & = \| x^{-1} \|, \\
(e) \quad \| \cdot \| & \text{ is continuous,} \\
(f) \quad \| \cdot \| & \text{ is smooth on } G - \{0\}.
\end{align*}
\]

The existence of \( \| \cdot \| \) which satisfies (a)-(e) is equivalent to the existence of a set \( A \subset G \) which satisfies the following conditions:

(a) \( A \) is open and \( \bar{A} \) is compact,
(b) \( A \) is convex, i.e. if \( x, y \in A \) and \( 1 \geq t \geq 0 \), then \( \delta_t x \cdot \delta_t^{-1} y \in A \),
(c) \( A \) is symmetric, i.e. if \( x \in A \), then \( x^{-1} \in A \).

In fact, given a set \( A \) satisfying (a)-(c), we put
\[\| x \| = \inf \{ t \mid \delta_t x, x \in A \} .\]

Now, if \( \| x \| < \varepsilon \) and \( \| y \| < \varepsilon \), then \( \delta_{1/\varepsilon} x, \delta_{1/\varepsilon} y \in A \) and by (b)
\[
\delta_{1/(\varepsilon + \varepsilon)} xy = \delta_{1/\varepsilon} x \cdot \delta_{1/\varepsilon} y = \delta_{1/(\varepsilon + \varepsilon)} \delta_{1/\varepsilon} x \cdot \delta_{1/\varepsilon} y, \quad y \in A ,
\]

so \( \| xy \| < \varepsilon + \varepsilon \). This proves (a). The rest is easy.

The converse is obtained by putting \( A = \{ x \in G \mid \| x \| < 1 \} \).

Moreover, we see that the condition

(e) (i) the boundary \( \partial A \) of \( A \) is a smooth manifold,
(ii) \( (d/dt) \delta_t x |_{t = 1} \neq T_x \partial A \) for every \( x \in \partial A \),

is equivalent to (f).

**Theorem 1.** For every homogeneous group \( G \) there exists a set \( A \) which satisfies (a)-(e), hence \( G \) admits a norm which satisfies (a)-(f).

**Proof.** If \( G \) is abelian we put \( A = \{ x = (x_1, \ldots, x_n) \mid \sum x_i^2 < 1 \} . \) To see that \( A \) satisfies (b) note that \( d_i > 1 \), so
\[
(\sum (t^{d_i} x_i + (1 - t)^{d_i} x_i)^2)^{1/2} = \left( \sum (t^{d_i} x_i)^2 \right)^{1/2} + \left( \sum (1 - t)^{d_i} x_i)^2 \right)^{1/2} \leq t \left( \sum x_i^2 \right)^{1/2} + (1 - t) \left( \sum x_i^2 \right)^{1/2} .
\]

(a), (b) and (e) are obvious.

We notice that if \( G \) is not abelian, then \( d_1 \geq 2 \) and \( e_x \) is in the center of \( G \), for \( \delta_t e_x = (\delta_t e_x, \delta_t e_x) = (e_x, e_x) \) and we assume that \( 1 \leq d_1 \leq \cdots \leq d_n \). By the Campbell–Hausdorff formula we have

\[
(x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_1 + y_1 + P_1(x_1, \ldots, x_n, y_1, \ldots, y_n), \ldots, x_n + y_n + P_n(x_1, \ldots, x_n, y_1, \ldots, y_n)),
\]

where the \( P_i \) are polynomials and since \( e_x \) is in the center of \( G \), \( [e_x, e_x] = 0 \) for \( 1 \leq i \leq n \), neither \( x_i \) nor \( y_i \) appears in any of the \( P_i \).

Now we proceed by induction on \( \dim G \). Let \( A' \) be a subset of the quotient group \( G' = G/\text{lin} \{ e_x \} = \{ \bar{x} = (x_1, \ldots, x_n) \mid x_i \in \mathbb{R} \} \) which satisfies (a)-(e) and \( \| \cdot \| \) the corresponding norm. There exists a constant \( C \) such that
\[
\| P_t (\delta_t x, \delta_t y) \| \leq 2Ct^t(1-t) \quad \text{for all } x, y \in A', \quad 0 \leq t \leq 1.
\]

Indeed, since \( P_t (x, 0) = P_t (0, y) = 0 \), we see that every monomial in \( P_t \) depends both on \( x \) and \( y \); hence, since \( A' \) is bounded, (a) holds for some \( C \). If \( x = (x_1, \ldots, x_n) \), then put \( \bar{x} = (x_1, \ldots, x_n) \). We prove that the set
\[
A = \{ x \in G \mid \bar{x} \in A' \} \quad \text{satisfies (a)-(e) too, where } \quad C = \text{the constant from (a), } f \in C^\omega (0, 1), \quad f' \leq 0, \quad f'' \leq 0, \quad f(k0) = 0, \quad f(0) = 1, \quad f(k0) = -\infty, \quad f(1) = 0 \quad \text{for } k = 1, 2, \ldots.
\]

**Remark.** With \( f = 0 \) the construction yields a set \( A \) which satisfies (a)-(e) but of course not (f).

**Proof of (a)-(f) for \( A \).** (a) and (f) are obvious. To show (b) notice that if \( x, y \in A \) and \( y \in \bar{A} \), then \( \delta_t x \cdot \delta_t^{-1} y = \delta_t \bar{x} \cdot \delta_t^{-1} \bar{y} \in A' \). So, it is sufficient to prove the following inequality:

\[
|t^{d_n} x_n + (1 - t)^{d_n} y_n + P_n(\delta_t \bar{x}, \delta_t^{-1} \bar{y})| < C + f((\delta_t \bar{x} \cdot \delta_t^{-1} \bar{y})
\]

But \( d_n \geq 2 \), \( 0 \leq t \leq 1 \), \( f' \leq 0 \), \( f'' \leq 0 \) and hence, by the definition of \( A \\
|t^{d_n} x_n + (1 - t)^{d_n} y_n + P_n(\delta_t \bar{x}, \delta_t^{-1} \bar{y})| < t^2 (C + f((\bar{x} \bar{y})) + (1 - t)^2 (C + f((\bar{x} \bar{y}))) + 2Ct(1-t)
\]

\[
\leq C + f(t(\| \bar{x} \| + (1 - t)\| \bar{y} \|)) < C + f(t(\| \bar{x} \| + (1 - t)\| \bar{y} \|)) < C + f((\delta_t \bar{x} \cdot \delta_t^{-1} \bar{y})).
\]

(a)(i) is obvious. We first prove (a)(ii) for \( x = (x_1, \ldots, x_n) \) in \( \partial A \) such that \( \| x \| < C \). Then \( \bar{x} \in \partial A' \) and \( T_{\bar{x}} \partial A = T_{\bar{x}} A' \ominus \mathbb{R} e_x \). So if \( (d/dt) \delta_t x |_{t = 1} = T_{\bar{x}} \partial A \), then \( (d/dt) \delta_t x |_{t = 1} = (d/dt) \delta_t x |_{t = 1} = T_{\bar{x}} \partial A' \). But this contradicts the induction hypothesis. Now, we observe that the set \( \partial A \cap \{ x \in \mathbb{R}^n \mid \| x \| > C \} \) is the graph of the function \( g(x) = C + f((\| x \|)) \), \( A' \to \mathbb{R} \), and that if \( v = (v_1, \ldots, v_n) \in T_{\bar{x}}(\mathbb{R}^n) \cdot M \), where \( M \) is the graph of a function \( g: \mathbb{R} \to \mathbb{R} \), \( \bar{x} \in \mathbb{R}^n \), then \( v = (d/dt) g(\bar{x} \bar{y}) |_{t = 0} = d\bar{g}(\bar{x}) \). Hence if \( (d/dt) \delta_t x |_{t = 1} = T_{\bar{x}} \partial A \), where \( x = (\bar{x}, C + f((\| \bar{x} \|)) \), then by the definition of \( f \) (f' \leq 0),

\[
0 < d_n x = (d/dt) \delta_t x |_{t = 1} \cdot f((\| \bar{x} \|) + C) \leq (d/dt) f((\| \bar{x} \|) + (d/dt) f((\| \bar{x} \|) + f((\| \bar{x} \|) + \| \bar{x} \|') \leq 0.
\]
This contradiction proves (e)(ii) for \( \partial A \cap \{ x \in \mathbb{R}^n : x_n > C \} \). For \( \partial A \cap \{ x \in \mathbb{R}^n : x_n < -C \} \), (e)(ii) follows by symmetry.

Theorem 2 below exhibits a very simple "convex body", i.e., a set satisfying (a)-(e), which yields a homogeneous subadditive norm. The proof, however, is more complicated.

**Theorem 2.** Let \( G \) be a homogeneous group and \( x = (x_1, \ldots, x_n) \) homogeneous coordinates (\( \delta, x = (t^d x_1, \ldots, t^n x_n) \)). There exists \( \varepsilon > 0 \) such that for \( r < \varepsilon \) the set

\[
A = \{ x : \sum x_i^2 < r^2 \}
\]

satisfies the conditions (a)-(e). Consequently there is a homogeneous subadditive norm on \( G \)

\[
\| x \| = \inf \{ \| (\delta_1, x) \| < r \}
\]

such that the unit ball \( \{ x : \| x \| < 1 \} \) coincides with the Euclidean ball \( \{ x : \| x \| < r \} \) (\( \| x \| = (\sum x_i^2)^{1/2} \)).

Proof. We verify only the condition (\( \beta \)) because the others are satisfied trivially. Put

\[
V_1 = \{ e_i : d_i < 2 \}, \quad V_2 = \{ e_i : d_i \geq 2 \};
\]

then \( G = V_1 \oplus V_2 \) as a linear space. Define \( (x_1, x_2) = x_1 + x_2 \), where \( x_1 \in V_1 \), \( x_2 \in V_2 \). Since \( \delta \epsilon_1, \epsilon_2 = t^{d_1 + d_2} \epsilon_1 \), \( \epsilon_2 \) and \( d_4 \geq 1 \), it follows that \( \| (x, y) \| \in V_2 \) for all \( x, y \in G \); so for \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) we have

\[
x \cdot y = (x_1 + y_1, x_2 + y_2 + R(x, y)).
\]

Let \( R_1(x, y) = R((x_1, 0), (y_1, 0)) \) and \( R_2 = R - R_1 \). In virtue of the Campbell–Hausdorff formula there is a constant \( C_1 \) such that for all \( \| x \|, \| y \| < 1 \)

\[
\| R_1(x, y) \| \leq C_1 \| [x_1, y_1] \|.
\]

Hence, by the inequality

\[
\| [x, y] \| \leq C_1 \| x \| \| y \| \| [x/y, y/y] \|,
\]

which is an easy consequence of the bilinearity and antisymmetry of \([ \cdot, \cdot \])\), we have for some constant \( C_1 \)

\[
(1) \quad \| R_1(x, y) \| \leq C_1 \| x_1 \| \| y_1 \| \| x_1/x_1 - y_1/y_1 \|.
\]

for all \( \| x \|, \| y \| < 1 \). Also by the Campbell–Hausdorff formula there is a constant \( C' \) such that for \( \| x \|, \| y \| < 1 \)

\[
(\ast) \quad \| R_2(x, y) \| \leq C'(\| x_1 \| \| y_2 \| + \| x_2 \| \| y_1 \| + \| x_2 \| \| y_2 \|).
\]

Let \( v = \delta_{x_2} + \delta_{y_1} - \delta_{x_1} - \delta_{y_1} \). By the definition \( d_i \geq 2 \) for \( e_i \in V_2 \), so in virtue of (\ast)

\[
\| v \| \leq t^2 \| x_2 \| + (1 - t^2) \| y_2 \| + C'(1 - t)(\| x_1 \| \| y_1 \| + \| x_2 \| \| y_1 \| + \| x_2 \| \| y_2 \|).
\]

Now, if we assume that \( C'(\| x_1 \| + \| x_2 \| + \| y_1 \|) \leq 1/2 \) and \( 0 < t \leq 1 \), then

\[
\| v \| \leq t^2 \| x_2 \| + (1 - t^2) \| y_2 \| + \frac{1}{2} t(1 - t)(\| x_2 \| + \| y_2 \|) \leq \| x_2 \| + \| y_2 \|
\]

and

\[
\| v \| \leq t^2 \| x_2 \| + (1 - t^2) \| y_2 \| + \frac{1}{2} t(1 - t)(\| x_2 \| + \| y_2 \|) = t \| x_2 \| + (1 - t)^2 \| y_2 \| + \frac{1}{2} t(1 - t)(\| x_2 \| + \| y_2 \|).
\]

Therefore

\[
\| v \| + \frac{1}{2} t(1 - t)(\| x_2 \| + \| y_2 \|) \leq t \| x_2 \| + (1 - t) \| y_2 \| \quad \text{and}
\]

\[
(2) \quad \| v \| + t(1 - t) \leq \| v \|^2 + t(1 - t) \| \| x_2 \| + \| y_2 \| \|
\]

\[
\leq \| v \|^2 + \frac{1}{2} t(1 - t)(\| x_2 \| + \| y_2 \|)^2 \leq \| v \|^2 + (1 - t) \| y_2 \|^2.
\]

Note that \( 2(t_1, t_2) \leq t(1 - t) \| v_1 \|^2 + 4 \| t_2 \|^2/(1 - t) \), where \( (x, y) = \sum x_i y_i \) is the scalar product. Hence

\[
(3) \quad \| v + R_1(\delta, x_1, \delta, y_1) \| \leq \| v \|^2(1 + (1 - t)) + \| R_1 \|^2 \left[ 1 + 4/(t(1 - t)) \right].
\]

Observe also that

\[
(4) \quad \| (x + y) \|^2 \leq \| x \| \| y \| \| x/y, y/y \|.
\]

Finally, by (1)-(4) we have

\[
\| \delta_i x_1 + \delta_{-1} y_1 \|^2 \leq \| \delta_i x_1 + \delta_{-1} y_1 \|^2 + \| v + R_1(\delta, x_1, \delta, y_1) \|^2
\]

\[
\leq \| \delta_i x_1 \| + \| \delta_{-1} y_1 \|^2 + \| v + R_1(\delta, x_1, \delta, y_1) \|^2
\]

\[
\leq \| \delta_i x_1 \| + \| \delta_{-1} y_1 \|^2 + \| v + R_1(\delta, x_1, \delta, y_1) \|^2
\]

\[
\| v + R_1(\delta, x_1, \delta, y_1) \|^2 \leq \| v \|^2 + (1 - t) \| y_2 \|^2 + (1 - t)y_2^2/4.
\]

\[
\| v + R_1(\delta, x_1, \delta, y_1) \|^2 \leq \| v \|^2(1 + (1 - t)) + \| R_1 \|^2 \left[ 1 + 4/(t(1 - t)) \right].
\]

However, if \( 5C_1^2 \| x_1 \| \| y_1 \| < 1 \), then the sum of the last two expressions will be nonpositive, so

\[
\| \delta_i x_1 + \delta_{-1} y_1 \|^2 \leq \| v + R_1(\delta, x_1, \delta, y_1) \|^2 \leq \| v \|^2 + (1 - t) \| y_2 \|^2.
\]

This proves Theorem 2.