Bilinear L^p estimates for quasimodes

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Bilinear estimates

Want to estimate the size of $u \cdot v$ where u has frequency $\sim \lambda$ and v has frequency $\sim \mu$

Why?

Estimates arise from non-linear PDE such as the non-linear Schrödinger equation (NLS)

$$i\partial_t v(t,x) + \Delta v(t,x) = \pm |v(t,x)|^\alpha v(t,x)$$

To control regularity of this equation need to know about the growth of the $|v|^\alpha v$ term.

Frequency decomposition

One fairly standard way to approach this problem is to decompose v into frequency bands

$$v = \sum_j v_j$$

$$v_j = \chi(\sqrt{\Delta} - \lambda_j)v$$

On $\mathbb{R}^n$ this is the same as restricting $|\xi| \approx \lambda$ on the frequency side. On a manifold we see $\chi(\sqrt{\Delta} - \lambda_j)$ is a smoothed spectral projector of $\sqrt{\Delta}$.

Need to understand interactions $v_i \cdot v_j$ particularly when $i \neq j$.

L^2 estimates and regularity

In 2005 Burq-Gérard-Tvetkov proved an L^2 estimate

$$\|uv\|_{L^2} \leq (\min(\mu, \lambda))^{1/4}$$

for $\chi(\sqrt{\Delta} - \lambda)u = u$, $\chi(\sqrt{\Delta} - \mu)v = v$.

- Important that the estimate only depends on the smallest frequency.
- They then use this estimate to obtain regularity properties for cubic NLS

$$i\partial_t v(t,x) + \Delta v(t,x) = \pm|v(t,x)|^2 v(t,x)$$

on Zoll manifolds.

L^p estimates in dimension 2

Theorem (Guo, Han, T)

Suppose u and v are approximate eigenfunctions of Δ with eigenvalues λ^2 and μ^2 respectively and $\lambda \leq \mu$. then

$$\|uv\|_{L^p} \lesssim F_{2,p}(\lambda,\mu)\,\|u\|_{L^2}\,\|v\|_{L^2}$$

$$F_{2,p}(\lambda,\mu)\begin{cases} \lambda^{\frac{1}{4}}\mu^{-\frac{1}{2p}+\frac{1}{4}} & \text{for } 2 \leq p \leq 3, \\ \lambda^{-\frac{3}{2p}+\frac{3}{4}}\mu^{-\frac{1}{2p}+\frac{1}{4}} & \text{for } 3 \leq p \leq 6, \\ \lambda^{\frac{1}{2}}\mu^{-\frac{2}{p}+\frac{1}{2}} & \text{for } 6 \leq p \leq \infty; \end{cases}$$

- These estimates are sharp in each of the three regimes
- Where $\mu \approx \lambda$ the three regimes collapse to two regimes namely $2 \leq p \leq 6$ and $6 \leq p \leq \infty$.

L^p estimates in dimension $n \geq 3$

Theorem (Guo, Han, T)

Suppose u and v are approximate eigenfunctions of Δ with eigenvalues λ^2 and μ^2 respectively and $\lambda \leq \mu$. then

$$\|uv\|_{L^p} \lesssim F_{n,p}(\lambda,\mu)\,\|u\|_{L^2}\,\|v\|_{L^2}$$

For $(n,p) \neq (3,2)$,

$$F_{n,p}(\lambda,\mu) = \begin{cases} \lambda^{\frac{3(n-1)}{4}-\frac{n+1}{2p}}\mu^{\frac{n-1}{4}-\frac{n-1}{2p}} & \text{for } 2 \leq p \leq \frac{2(n+1)}{n-1}, \\ \lambda^{\frac{n-1}{2}}\mu^{\frac{n-1}{2}-\frac{n}{p}} & \text{for } \frac{2(n+1)}{n-1} \leq p \leq \infty; \end{cases}$$

and $F_{3,2}(\lambda,\mu) = \lambda^{\frac{1}{2}}|\log(\lambda)|^{\frac{1}{2}}$.

- Apart from the $\log(\lambda)$ factor these estimates are sharp.
- Unlike the $n = 2$ case there are only two different regimes.

Re-cast as semiclassical problem

Proves useful to use semiclassical calculus. This calculus is similar to the ψDO calculus. Given a symbol $p(x,\xi)$ define operator

$$p(x,hD)u = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}\langle x-y,\xi\rangle} p(x,\xi)u(y)d\xi dy$$

Scaling means $|\xi| \approx 1$ corresponds to frequency $\approx 1/h$.
For constant coefficient $p(\xi)$ this gives a scaled version of the Fourier transform method of solving a PDE

$$\xi_i \to hD_{x_i}$$

$$|\xi|^2 \to h^2\Delta$$

Conversion of eigenfunction problem

Suppose

$$(\Delta - \lambda^2)u = 0$$

Divide through by λ^2 and setting $\lambda = \frac{1}{h}$ we have

$$(h^2\Delta - 1)u = 0$$

That is the eigenfunction equation is associated with the semiclassical symbol $p(x, \xi) = |\xi|^2 - 1$. Now we study approximate eigenfunctions (quasimodes)

$$(h^2\Delta - 1)u = O_{L^2}(h)$$

$$(\sigma^2\Delta - 1)v = O_{L^2}(\sigma)$$

$$\lambda = \frac{1}{h} \quad \mu = \frac{1}{\sigma}$$

Seek estimates

$$\|uv\|_{L^p} \leq G_{n,p}(h, \sigma) \|u\|_{L^2} \|v\|_{L^2}$$

Gameplan

1. Reduce the problem to one localised near characteristic set
2. Factorise symbol to form $\xi_i - a(x, \xi')$
3. Introduce evolution operators $U_i(t,s)$ and use Duhammel's principle to write

$$u = U_1(t,0)u(0,x) + \frac{1}{h}\int_0^t U_1(t,s)[hf]ds$$

$$v = U_2(t,0)v(0,x) + \frac{1}{\sigma}\int_0^t U_2(t,s)[\sigma g]ds$$

   where hf and σg are the quasimode errors.
4. Analyse $U_1(t,s_1)U_2(t,s_2)U_1^\star(\tau,s_1)U_2^\star(\tau,s_2)$ as a Strichartz estimate

Characteristic set

Using semiclassical calculus had the advantage of easy cut offs. Suppose $\chi(x,\xi)$ has small support then

$$p(x,hD)\chi(x,hD)u = \chi(x,hD)p(x,hD)u + O_{L^2}(h)$$

so $\chi(x,hD)u$ is still an $O_{L^2}(h)$ quasimode.

Suppose $|p(x,\xi)| > c > 0$ the support of χ then $p^{-1}(x,hD)$ exists (its principal symbol is $1/p(x,\xi)$) and since

$$p(x,hD)\chi(x,hD)u = O_{L^2}(h)$$

$$\chi(x,hD)u = O_{L^2}(h)$$

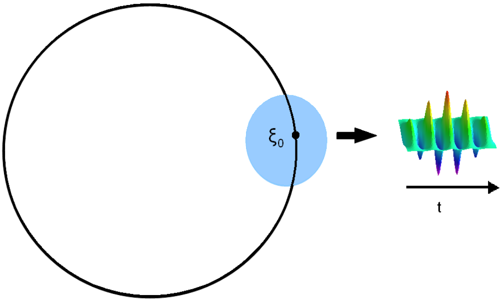
So we may assume we are localised near points (x_0,ξ_0) where $p(x_0,\xi_0) = 0$.

Laplacian example

Laplacian quasimodes obey

$$p(x, hD)u = O_{L^2}(h)$$

for $p(x,\xi) = |\xi|_g^2 - 1$. So can localise near the sphere $|\xi| = 1$



Key Points

- The characteristic set is a hypersurface.
- The characteristic set is curved.
- Localising near a point ξ_0 produces a wave-packet of frequency ξ_0.

Symbol factorisation

We study a class of symbols that have similar properties to the Laplacian

1. **Non-Degeneracy**: For any (x_0, ξ_0) where $p(x_0, \xi_0) = 0$, we have $\nabla_\xi p(x_0, \xi_0) \neq 0$
2. **Curvature**: The hypersurface $\{\xi \mid p(x_0, \xi) = 0\}$ has positive definite 2nd fundamental form

Non-degeneracy means that there is some ξ_i such that

$$\partial_{\xi_i} p(x_0, \xi_0) \neq 0$$

Factorise $p(x, \xi)$ as

$$p(x, \xi) = e(x, \xi)(\xi_i - a(x, \xi'))$$

$$|e(x, \xi)| > c > 0$$

Evolution operators

Since $e(x,\xi)$ is bounded away from 0 we can invert this so if

$$p(x,hD)u = O_{L^2}(h)$$

we also have

$$(hD_{x_i} - a(x, hD_{x'}))u = O_{L^2}(h)$$

by setting $x_i = t$ we can convert this to the evolution equation

$$(hD_t - a(t, x', hD_{x'}))u = O_{L^2}(h)$$

We have two functions u and v both quasimodes need to compare directions from factorisation.

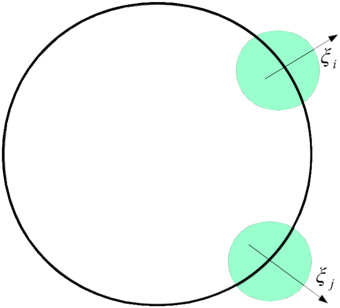
Principal direction of oscillation

Will specialise to $n = 2$ other dimensions are similar

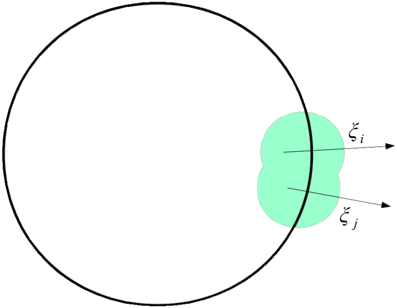
Flat Model

$$(|\xi|^2 - 1)$$

Localised in two different directions



Localised in the same direction



Case 1: Localised in the same direction $\xi_i = \xi_j$

We set $\xi_i = \xi_1$ and $x_1 = t$ then

$$(hD_t - a_1(t, x', hD_{x'}))u = O_{L^2}(h)$$

$$(\sigma D_t - a_2(t, x', \sigma D_{x'}))v = O_{L^2}(\sigma)$$

Case 2: Localised in different directions $\xi_i \neq \xi_j$

We set $\xi_i = \xi_1$, $\xi_j = \xi_2$, $(x_1, x_2) = (t_1, t_2) = t$

$$(hD_{t_1} - a_1(t, x', hD_{t_2}, hD_{x'}))u = O_{L^2}(h)$$

$$(hD_{t_2} - a_2(t, x', hD_{t_1}, hD_{x'}))v = O_{L^2}(\sigma)$$

Reconstruction via Duhammel's Principle

Suppose we have a solution operator $U(t,s)$

$$\begin{cases} (hD_t - a(t,x',hD_{x'}))U(t,s) = 0 \\ U(s) = \mathrm{Id} \end{cases}$$

and

$$(hD_t - a(t,x',hD_{x'}))u = hf(t,x)$$

then by Duhammel's principle

$$u = U(t,0)u(0,x') + \frac{1}{h}\int_0^t U(t,s)[hf(s,x)]ds$$

Key point is to write a parametrix solution for $U(t,s)$
Then study the $L^{p'} \to L^p$ norm of $U(t,s)U^\star(\tau,s)$.

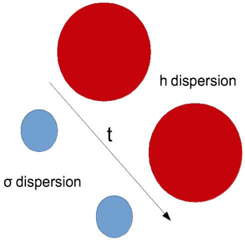
Intuition

Singularities Propagate

Need to consider three scales dependent on σ and h

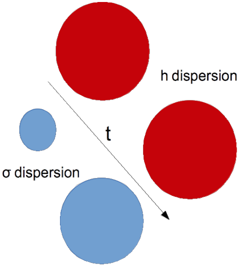
Small time

$$|t-\tau| \leq \sigma$$



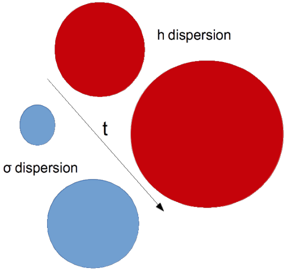
Mid time

$$\sigma \leq |t-\tau| \leq h$$



Long time

$$h \leq |t-s|$$



Constructing examples

The flat model is very useful.
That is study localised functions on $\mathbb{R}^2$ that obey

$$(h^2\Delta - 1)u = O_{L^2}(h)$$

$$(\sigma^2\Delta - 1)v = O_{L^2}(\sigma)$$

In the flat case $p(x,\xi) = p(\xi) = |\xi|^2 - 1$ so is constant coefficient.
Can use Fourier transform method to construct quasimodes.
Semiclassical Fourier transform

$$\mathcal{F}_h u = \frac{1}{2\pi h}\int e^{-\frac{i}{h}\langle x,\xi\rangle} u(x)dx$$

preserves L^2 norms,

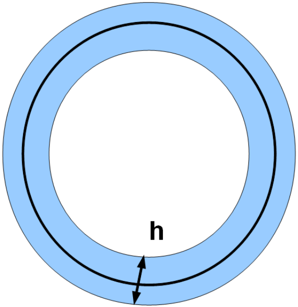
$$\|\mathcal{F}_h u\|_{L^2} = \|u\|_{L^2}$$

Solving on the Fourier side

On the Fourier side we must solve

$$(|\xi|^2 - 1)\hat{u} = O_{L^2}(h)$$

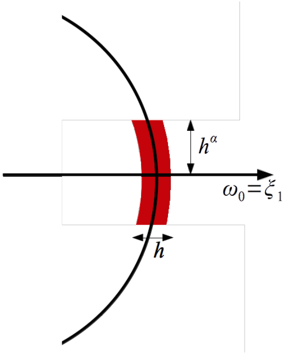
$\hat{u}$ supported anywhere in the annular region.



General Principles

- The more spread out $\hat{u}$ is; the more concentrated (at a point) u can be
- Therefore expect to saturate high p estimates be filling out full annulus and low p estimates by concentrating near a point.

Families of tubular concentrations



Let

$$\chi_\alpha(r,\omega) = \begin{cases} 1 & \text{if } |r-1| < h, |\omega - \omega_0| < h^\alpha \\ 0 & \text{otherwise} \end{cases}$$

$$f_\alpha^h(r,\omega) = h^{-\frac{1}{2}-\frac{\alpha}{2}}\chi_\alpha(r,\omega)$$

Now f_α^h is an L^2 normalised $O_{L^2}(h)$ quaismode of $|\xi|^2 - 1$. Let

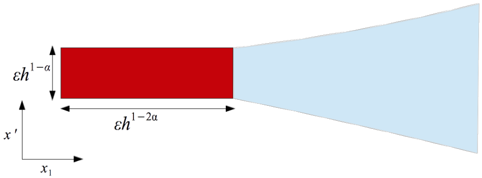
$$T_\alpha^h(x) = \mathcal{F}_h^{-1}[f_\alpha^h]$$

Size of T_α^h

$$T_\alpha^h = \frac{h^{-\frac{1}{2}-\frac{\alpha}{2}-1} e^{\frac{i}{h}x_1}}{2\pi} \int_{\mathbb{R}^2} e^{\frac{i}{h}(x_1(\xi_1-1)+x_2\cdot\xi_2)} \chi_\alpha(\xi) d\xi$$

So if $|x_1| < \epsilon h^{1-2\alpha}$ and $|x_2| < \epsilon h^{1-\alpha}$ the factor $e^{\frac{i}{h}(x_1(\xi_1-1)+x_2\xi_2)}$ doesn't oscillate much so

$$|T_\alpha^h| > ch^{-\frac{1}{2}-\frac{\alpha}{2}}$$



Products of tubes: High p

We construct examples by considering products of tubes T_α^σ and T_β^h.

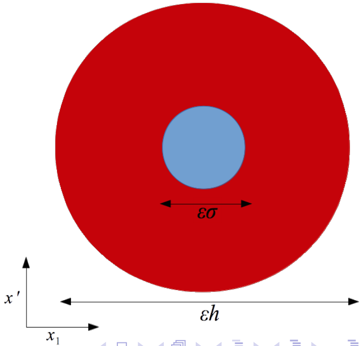
High p examples Want maximum concentration at a point, pick $\alpha = \beta = 0$.

For $|x| < \epsilon\sigma$

$$|T_0^\sigma| > c_1 \sigma^{-\frac{1}{2}} \quad \text{and} \quad |T_0^h| > c_2 h^{-\frac{1}{2}}$$

$$\left\| T_0^\sigma T_0^h \right\|_{L^p} > K \sigma^{-\frac{1}{2} + \frac{2}{p}} h^{-\frac{1}{2}}$$

Saturates for $p \geq 6$



Products of tubes: Mid-range p

Now we keep T_0^h concentrated but allow T_α^σ to spread a bit.

Select α_h so that

$$\sigma^{1-2\alpha_h} = h$$

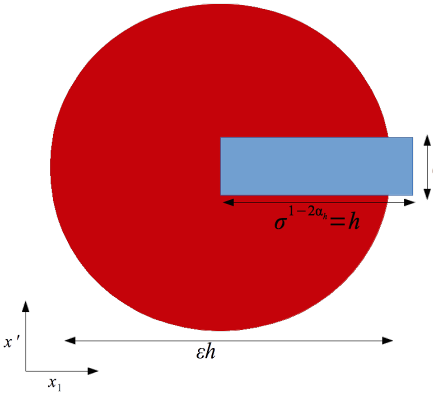
For $|x_1| < \epsilon h$ and $|x_2| < \epsilon \sigma^{1-\alpha}$

$$|T_0^h| > c_1 h^{-\frac{1}{2}} \quad \text{and} \quad |T_{\alpha_h}^\sigma| > \sigma^{-\frac{1}{2}+\frac{\alpha}{2}}$$

$$\left\| T_0^h T_{\alpha_h}^\sigma \right\| \geq K h^{-\frac{1}{2}+\frac{1}{p}} \sigma^{-\frac{1}{2}+\frac{\alpha}{2}+\frac{1}{p}-\frac{2\alpha}{p}}$$

$$K h^{-\frac{3}{4}+\frac{3}{2p}} \sigma^{-\frac{1}{4}+\frac{1}{2p}}$$

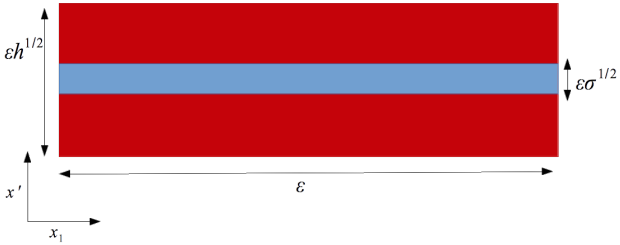
Saturates for $3 \leq p \leq 6$



Products of tubes: Low p

Allow maximum spread in both tubes $\alpha = \beta = 1/2$. For $|x_1| < \epsilon$ and $|x_2| < \epsilon\sigma^{1/2}$

$$|T^{\sigma}_{1/2}| > c_1 \sigma^{-1/4} \quad \text{and} \quad |T^{h}_{1/2}| > c_2 h^{-1/4}$$



$$\left\| T^{\sigma}_{1/2} T^{h}_{1/2} \right\|_{L^p} > K h^{-1/4} \sigma^{-1/4+\frac{1}{2p}}$$

Saturates for $2 \leq p \leq 3$.

Further Research

- We can go from the flat model to spherical harmonics easily enough so the estimates are sharp on manifolds. Can we obtain better results if we make some geometrical assumptions (such as negative curvature)?
- Can we extend Holder regularity results on Zoll manifolds to other manifolds? Burq-Gérard-Tvetkov went from the bilinear forms to L^2 estimates to get regularity estimates. Our L^p estimates go back to the bilinear form as part of the technical work. Could removing the middle step improve things?
- Could we use L^p estimates to look at regularity in other Sobolev spaces?