# Applications of semiclassical analysis in PDE

Melissa Tacy

Department Colloquium, Macquarie University

21 October 2016

# What is Semiclassical Analysis?

Term for a collection of techniques the evolved from the Fourier Transform method for solving PDE and specially adapted to deal elegantly with a parameter.

**Fourier Transform Method** Suppose that for $\alpha = (\alpha_1, \ldots, \alpha_n)$

$$L = \sum_\alpha c_\alpha D^\alpha \quad D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

for example

$$-\Delta = \sum_{i=1}^n D_{x_i}^2$$

Then

$$\widehat{L[u]} = \frac{1}{(2\pi)^{n/2}} \int e^{-i\langle x,\xi\rangle} L[u](x)dx = \sum_\alpha c_\alpha \xi^\alpha \hat{u}(\xi)$$

So we can solve an algebraic equation on the Fourier side then invert to solve the differential equation $L[u] = 0$.

# What Kind of Problems

Semiclassical analysis is well suited to studying solutions to PDEs that involve a parameter.
Canonical examples are Laplacian eigenfunctions

$$-\Delta u = \lambda^2 u$$

What happens as $\lambda \to \infty$.

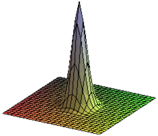
- How does $u$ behave, can it be concentrated in any region?
- What is the behaviour of $u$ near local features such as lower dimensional sets?
- How to products of different eigenfunctions behave?
- What is the connections between dynamics and geometry and the growth properties of eigenfunctions?

## Measuring Concentration

Understanding how the $L^p$ norm of $u$ grows helps us to understand the local features of a solution.

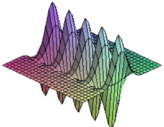
$$\left( \int |u(x)|^p dx \right)^{1/p} \leq ?$$

Point



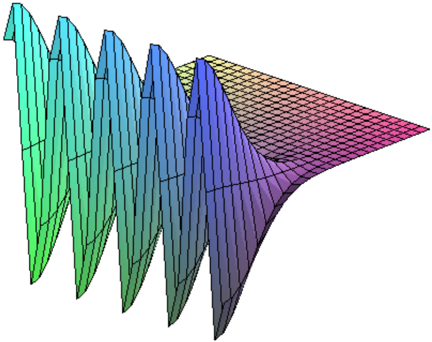
- High $L^\infty$ norm
- Sharp change in $L^p$ norm when $p < \infty$

Tube



- Lower $L^\infty$ norm
- Change in $L^p$ norm more gentle

# Submanifold Estimates



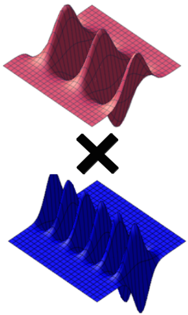
- Can take cross sections of eigenfunctions and study their $L^p$ norm
- Because cross sections are lower dimensional we expect growth for $L^p$, $p$ close to 2.

# Bilinear estimates

Suppose

$$-\Delta u = \lambda^2 u \qquad -\Delta v = \mu^2 v$$

How does $\|uv\|_{L^p}$ behave?



Applications to nonlinear PDE

$$i\partial_t v(t,x) + \Delta v(t,x) = \pm |v(t,x)|^\alpha v(t,x)$$

Interaction between frequency bands determines regularity properties.

# Laplacian Eigenfunctions

Want to solve

$$(-\Delta - \lambda^2)u = 0$$

so on the Fourier side we solve

$$(|\xi|^2 - \lambda^2)\hat{u} = 0$$

That is $\hat{u}$ is supported on $|\xi| = \lambda$. Semiclassical analysis builds the parameter into the operator. Let $h = \lambda^{-1}$ and introduce

$$\mathcal{F}_h[u] = \frac{1}{(2\pi h)^{n/2}} \int e^{-\frac{i}{h}\langle x.\xi\rangle} u(y)dy$$

we call this the semiclassical Fourier transform. Think of it as a rescaling

$$|\xi| = \lambda = h^{-1} \Rightarrow |\xi| = 1$$

# Properties of $\mathcal{F}_h$

Retain many of the standard Fourier transform properties

$$\xi_i \to hD_{x_i}$$

$$\sum_\alpha c_\alpha \xi^\alpha \to \sum_\alpha c_\alpha h^{|\alpha|} D^\alpha$$

The prefactor $(2\pi h)^{-n/2}$ is chosen so that this Fourier transform is still an isometry of $L^2$

$$\|\mathcal{F}_h[u]\|_{L^2} = \|u\|_{L^2}$$

With this scaling Laplacian eigenfunctions obey

$$(|\xi|^2 - 1)\mathcal{F}_h[u] = 0$$

# Semiclassical Psedodifferential Calculus

Can go step further and generalise to allow us to deal with non-constant coefficient equations. Let $p(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Define

$$p(x, hD)u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} p(x,\xi)u(y)dyd\xi$$

and call $p(x, hD)$ the left (or standard) quantisation of the symbol $p(x,\xi)$. If $p(x,\xi) = p(\xi)$ is independent of $x$ then

$$p(x, hD) = p(hD) = \mathcal{F}_h^{-1} p(\xi)\mathcal{F}_h$$

so if $p(x,\xi) = |\xi|^2 - 1$ then $p(x, hD) = -h^2\Delta - 1$.

# Why Quantisation?

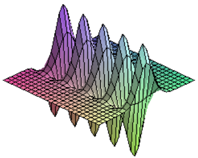
Deep links to the theory of quantum mechanics.

Quantum Mechanics

Dynamics described by Schrödinger equation

$$\frac{\hbar}{i}\frac{\partial}{\partial t}\Psi(t,x) = \widehat{H}\Psi(t,x)$$

$\|\Psi\|_{L^2}$ interpreted probabilistically

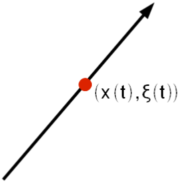


Classical Mechanics

Dynamics described by phase space flow.

$$\dot{x}(t) = \nabla_\xi H(x,\xi)$$

$$\dot{\xi}(t) = -\nabla_x H(x,\xi)$$



# Classical Dynamics

Suppose $H(x, \xi)$ is the Hamiltonian of a system (that is it defines the energy). The classical flow associated with $H(x, \xi)$ is

$$\begin{cases} \dot{x}(t) = \partial_\xi H(x, \xi) \\ \dot{\xi}(t) = -\partial_x H(x, \xi) \end{cases}$$

Measurable quantities are called observables. They are given by a symbol $q(x, \xi)$ which evolves under the equation

$$\dot{q}(x, \xi) = \{q(x, \xi), H(x, \xi)\} = \sum_j \frac{\partial H}{\partial \xi_j}\frac{\partial q}{\partial x_j} - \frac{\partial H}{\partial x_j}\frac{\partial q}{\partial \xi_j}$$

Note that $H(x, \xi)$ is constant in time (that is energy is conserved).

# Quantum Analogues

If $H(x, \xi)$ is the classical energy then the operator $H(x, hD)$ is the quantum energy operator. So if

$$H(x, hD)u = u$$

we are saying that $u$ is in some fixed energy state.
**Key Link**

$$[p(x, hD), q(x, hD)] = h\{p, q\}(x, hD) + O(h^2)$$

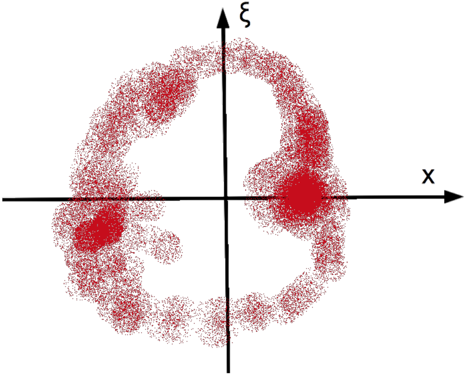
That is the principal symbol of a commutator (quantum mechanics) is given by the Poisson bracket (classical mechanics).

# Advantages of Semiclassical Analysis

- Can develop a calculus of operators of the $p(x, hD)$ including composition formulae and conditions for invertibility.
- By incorporating the parameter in the operator can efficiently treat cases at multiple scales.
- Relationship to quantum mechanics gives us a useful intuition. The correspondence principle states that for high energy systems classical and quantum mechanics must give the same result for any measurement.
- The semiclassical calculus is very flexible, can just as well consider approximate eigenfunctions (where $p(x, hD)u$ is small).

# Phase Portraits

Think of $u$ as having a "fuzzy" phase space picture




Semiclassical psuedodifferential operators act by multiplication on the phase space portrait.

Allow for one further generalisation. Instead of just acting by multiplcation on the phase space portrait what if we for instance wanted to change variables.

# Semiclassical Fourier Integral Operators

We can locally write a semiclassical Fourier integral operator, FIO

$$Tu = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(x,y,\xi)} b(x,y,\xi) u(y)$$

The phase function $\phi(x, y, \xi)$ defines the behaviour of the operator. Note that if

$$\phi(x, y, \xi) = \langle x - y, \xi \rangle$$

then we have a semiclassical psuedodifferential operator. If

$$\phi(x, y, \xi) = \langle x, \eta \rangle + \langle x - y, \xi \rangle$$

The operator represents a change of variables of the Fourier side $\xi \to \xi - \eta$.

# Applications of Semiclassical FIOs

Semiclassical FIOs are a large class of operators. Will focus on two different techniques commonly used in the field

1. Using evolution equations to reconstruct eigenfunctions.
   - In this case the relevant FIO will be the propagator $e^{\frac{i}{h}p(x,hD)}$
   - Develop a small time parametrix representation
2. "Changing variables on phase space"
   - In this case we use a FIO to quantise a change of variables turns $p(x,\xi)$ into for instance $\xi_1$.
   - Then we only have to deal with the simple operator $hD_{x_1}$

# Evolutions equations and eigenfunctions

In our first PDE class we learn to separate solutions, look for

$$v(t,x) = f(t)u(x)$$

so

$$(hD_t - h^2\Delta)v(t,x) = 0$$

In this case have a solution

$$v(t,x) = e^{\frac{i}{h}t}u(x)$$

where

$$-\Delta u = h^{-2}u$$

We go the other way. Use evolution equations to solve for eigenfunctions.

# Duhamel method for describing eigenfunctions

Suppose

$$\|p(x,hD)u\|_{L^2} \leq Ch\|u\|_{L^2}$$

we say $u$ is an $O_{L^2}(h)$ quasimode of $p(x,hD)$. Then clearly

$$(hD_t + p(x,hD))u = hf(x)$$

where $\|f\|_{L^2} \leq C$. Consider $hf(x)$ as an inhomogeneity, Duhamel's principle givens us

$$u = U(t)u + \frac{1}{h}\int_0^t U(t-s)[hf(x)]ds$$

Therefore estimating the $L^p$ norms of the function $u$ is the same as analysing the $L^2 \to L^p$ mapping properties of $U(t)$.

# A parametrix for $U(t)$

We can write $U(t)$ as a semiclassical FIO and produce an explicit local representation of it as an oscillatory integral operator.
We use the PDE

$$\begin{cases} (hD_t + p(x, hD))U(t) = 0 \\ U(0) = \mathrm{Id} \end{cases}$$

to produce a parametrix solution

$$U(t)u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(t,x,y,\xi)} b(x, y, \xi) u(y)$$

$$\partial_t \phi(t, x, y, \xi) = p(x, \nabla_x \phi) \quad \phi(0, x, y, \xi) = \langle x - y, \xi \rangle$$

# Some results

**Theorem (Koch-Tataru-Zworkski, 2005)**

*Suppose $u$ is an $O_{L^2}(h)$ quasimode of a Laplace-like semiclassical pseudodifferential operator $p(x, hD)$, then*

$$\|u\|_{L^p(M)} \lesssim h^{-\delta(n,p)} \|u\|_{L^2}$$

$$\delta(n,p) = \begin{cases} \frac{n-1}{2} - \frac{n}{p} & \frac{2(n+1)}{n-1} \leq p \leq \infty \\ \frac{n-1}{4} - \frac{n-1}{2p} & 2 \leq p \leq \frac{2(n+1)}{n-1} \end{cases}$$

- The Laplace-like condition is necessary to obtain the correct dispersive estimates.
- There are sharp examples for all $p$.

### Theorem (T, 2010)

*Suppose u is an $O_{L^2}(h)$ quasimode of a Laplace-like semiclassical pseudodifferential operator $p(x, hD)$ and $H$ is a smooth embedded hypersurface, then*

$$\|u\|_{L^p(H)} \lesssim h^{-\tilde{\delta}(n,p)} \|u\|_{L^2}$$

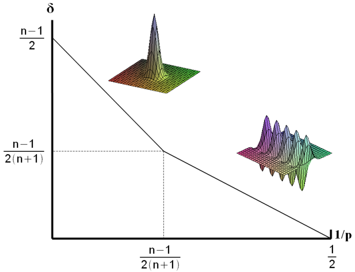
$$\tilde{\delta}(n,p) = \begin{cases} \frac{n-1}{2} - \frac{n-1}{p} & \frac{2n}{n-1} \leq p \leq \infty \\ \frac{n-1}{4} - \frac{n-2}{2p} & 2 \leq p \leq \frac{2n}{n-1} \end{cases}$$

### Theorem (Hassell-T, 2011)

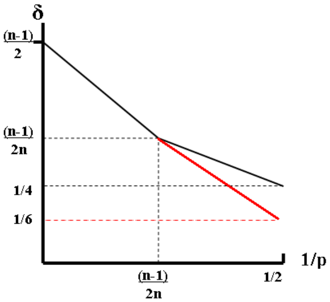
*If in addition the trajectories of classical flow given by $p(x, \xi)$ only glance $H$ then*

$$\|u\|_{L^2(H)} \lesssim h^{-\left(\frac{n-1}{3} - \frac{2n-3}{3p}\right)} \quad 2 \leq p \leq \frac{2n}{n-1}$$

Whole Manifold Estimates



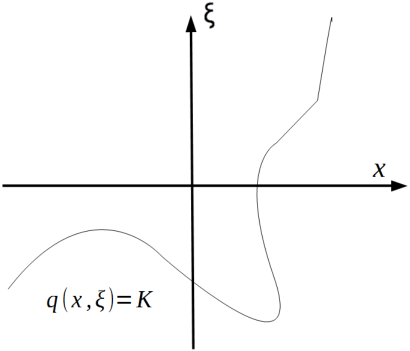

Hypersurface Estimates




- High $p$ estimates are saturated by point type examples
- Low $p$ estimates are saturated by tube type examples

## Localisation near level sets

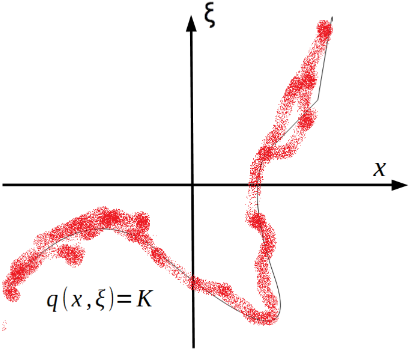
In classical mechanics we measure important quantities such as velocity and acceleration by evaluating an observable $q(x,\xi)$. Since behaviour is deterministic we can talk about cases when $q(x,\xi) = K$.




**Quantum Analogue** When is $u$ "localised" near $q(x,\xi) = K$?

# Localisation to regions of phase space

Go back to the "fuzzy" phase space picture




Want to measure how much of the "mass" lives where $q(x, \xi)$ is localised to $K$.

Measure this by applying a semiclassical pseudo with localised symbol.

## Theorem (T. 2015)

*Suppose $u$ is a $O(h)$ quasimode of $p(x, hD)$ and $\alpha \leq 1/2$. Let $\boldsymbol{\chi}_{\boldsymbol{q}}^{\boldsymbol{\alpha}} = Op(\chi_q^\alpha)$ where*

$$\chi_q^\alpha(x, \xi) = \chi(h^{-\alpha}|q(x, \xi)|)$$

$$\chi : \mathbb{R}^+ \to \mathbb{R}, Supp(\chi) \subset [1/2, 1]$$

*Then*

$$\left\| \dot{q}(x, hD)\boldsymbol{\chi}_{\boldsymbol{q}}^{\boldsymbol{\alpha}} u \right\|_{L^2} \lesssim h^{\alpha/2} \|u\|_{L^2}$$

- Can localised around other points than zero
- If $|\dot{q}(x, \xi)| \approx 1$ this implies

$$\left\| \boldsymbol{\chi}_{\boldsymbol{q}}^{\boldsymbol{\alpha}} u \right\|_{L^2} \lesssim h^{\alpha/2} \|u\|_{L^2}$$

- Proof only needs commutator relationship

$$[p(x, hD), q(x, hD)] = ihOp(\{p(x, \xi), q(x, \xi)\}) + O(h^2)$$

# Restricting to actual level sets

We consider the special cases were $q(x, \xi) = x_1$, want to restrict to $H = \{x \in M \mid x_1 = 0\}$

- Construct an operator $W_h$ such that

$$hD_{x_1} W_h = W_h p(x, hD) + O(h^\infty)$$

  think of this as a change of variables so that $p(x, hD)$ becomes the operator $hD_{x_1}$.
- Let $v = W_h u$. Then if $u$ is a $O_{L^2}(h)$ quasimode of $p(x, hD)$, $v$ is a $O_{L^2}(h)$ quasimode of $hD_{x_1}$.
- Approximate solutions to $hD_{x_1} v = 0$ are easy to treat.
- Use a local representation of $W_h$ in terms of a phase function to study its mapping properties.

**Theorem (T. 2016)**

*Let* $\nu(x,\xi) = \{p(x,\xi), x_1\}$ *and* $H = \{x \mid x_1 = 0\}$ *then*

$$\|\nu(x,hD)u\|_{L^2(H)} \lesssim \|u\|_{L^2} \quad (1)$$

$$\left\|\nu^{1/2}(x,hD)u\right\|_{L^2(H)} \lesssim \|u\|_{L^2} \quad (2)$$

*where* $\nu^{1/2}(x,hD)$ *is a suitable regularisation of* $\sqrt{\nu}$*.*

- The operator $\nu(x,hD)$ should be interpreted as the quantum version of normal velocity.
- This theorem says that even though a quasimode can concentrate, its normal velocity can't.
- (2) is much stronger where the symbol of $\nu(x,hD)$ is small.

# Current Directions

- Developing ways to deal with long time propagation particularly focussed on including geometry into the analysis.
- Restriction to general level sets
- Small scale behaviour (up to the minimum scale given by the uncertainty principle)
- Applications to nonlinear PDE