Quantisation and localisation dynamical observables

Melissa Tacy

University of Wollongong, Analysis Seminar

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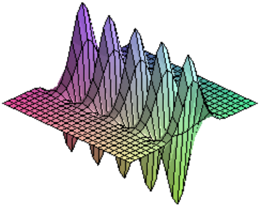
Classical and Quantum Mechanics

Quantum Mechanics

Dynamics described by Schrödinger equation

$$\frac{\hbar}{i}\frac{\partial}{\partial t}\Psi(t,x) = \widehat{H}\Psi(t,x)$$

$\|\Psi\|_{L^2}$ interpreted probabilistically

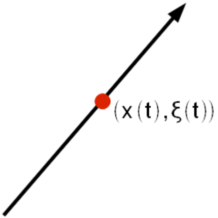


Classical Mechanics

Dynamics described by phase space flow.

$$\dot{x}(t) = \nabla_\xi H(x,\xi)$$

$$\dot{\xi}(t) = -\nabla_x H(x,\xi)$$



Correspondence Principle

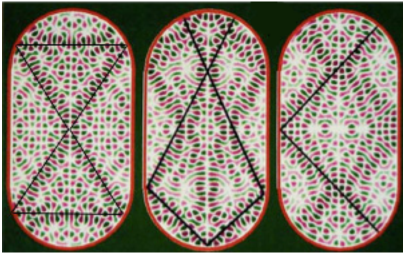
"The predictions of quantum mechanics and classical mechanics agree for large systems"

- A large system has relatively high energy so we interpret this as a statement about high energy systems.
- One way that quantum and classical mechanics can agree is for quantum states to concentrate on classical trajectories.
- The strongest form of this concentration is often referred to as a "scar"

Classical Dynamics

Suppose $H(x, \xi)$ is the Hamiltonian of a system (that is it defines the energy). The classical flow associated with $H(x, \xi)$ is

$$\begin{cases} \dot{x}(t) = \partial_\xi H(x, \xi) \\ \dot{\xi}(t) = -\partial_x H(x, \xi) \end{cases}$$



Intuition is that we should see concentration of quantum states near stable orbits

Dynamical Observables

Measurable quantities are called observables. They are given by a symbol $q(x, \xi)$ which evolves under the equation

$$\dot{q}_t(x, \xi) = \{q_t(x, \xi), H(x, \xi)\} = \sum_j \frac{\partial H}{\partial \xi_j} \frac{\partial q}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial q}{\partial \xi_j}$$

Note that $H(x, \xi)$ is constant in time (that is energy is conserved).

Some Important Observables

$$\begin{aligned} \dot{x}_i &= \{x_i, H(x, \xi)\} && \text{Velocity} \\ \ddot{x}_i &= \{\dot{x}_i, H(x, \xi)\} && \text{Acceleration} \\ \dddot{x}_i &= \{\ddot{x}_i, H(x, \xi)\} && \text{Jerk} \end{aligned}$$

Stationary States

Important set of solutions to Schrödinger equation

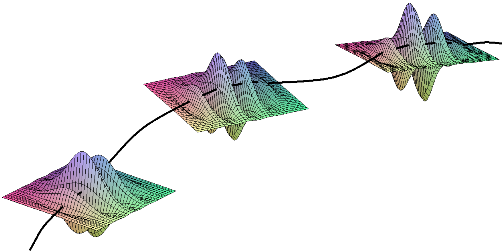
$$\Psi(t,x) = e^{\frac{i}{\hbar}t\lambda^2}u(x)$$

$$\widehat{H}u = \lambda^2 u$$

- λ^2 is interpreted as energy E
- The L^2 mass $\|u\|_{L^2(X)}$ gives the probability of particle being in the set X.
- We want to understand concentrations of the eigenfunction (stationary state) u and how they relate to dynamics.

Intuition - Wave Packets

Heuristically think of eigenfunction as being made of of wave packets tracking the classical flow.



- Packets are localised in frequency and space
- Concentration in a region is related to time packets spend there
- Heuristic breaks down in time due to dispersion

Semiclassical Techniques

It is convenient to work in the semiclassical framework. Define a semiclassical pseudodifferential operator $p(x, hD)$ as

$$p(x, hD)u = Op_h u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} p(x,\xi)u(y)d\xi dy$$

Set $h = 1$ to get the standard pseudodifferential calculus

$$\xi_i \to hD_{x_i}$$

$$|\xi|^2 \to h^2\Delta$$

Very important identity

$$[p(x, hD), q(x, hD)] = ihOp_h(\{p(x,\xi), q(x,\xi)\}) + O(h^2)$$

The principal symbol of the commutator is given by the Poisson bracket

Eigenfunctions and Quasimodes

Suppose u is a Laplacian eigenfunction we can convert to semiclassical framework

$$(\Delta - \lambda^2)u = 0 \to (h^2\Delta - 1)u = 0$$

where $h = \lambda^{-1}$. So instead of eigenfunctions we study solutions to

$$p(x, hD)u = 0$$

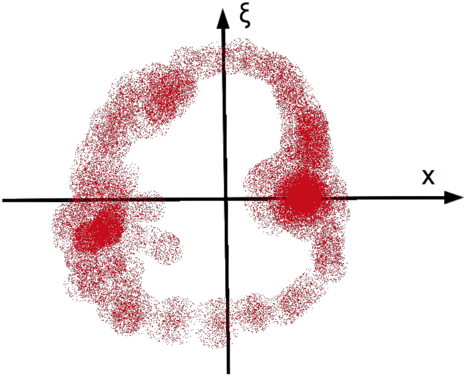
or quasimodes

$$p(x, hD)u = O_{L^2}(h^\beta)$$

We can look at a number of different β, it is common to look at $\beta = 1$ (or order h) quasimodes.

Localisation to regions of phase space

Think of u as having a "fuzzy" phase space picture



Want to measure how much of the "mass" lives where $q(x, \xi)$ is localised to some value. For instance where $x \approx 1$.

Measure this by applying a semiclassical pseudo with localised symbol.

Effect of localisation on Quasimodes

Suppose $\chi(x,\xi)$ is compactly supported in $T^{\star}M$. We then localise u to the support of $\chi(x,\xi)$ by considering

$$\chi(x,hD)u$$

How good a quasimode is this?

$$\begin{aligned} p(x,hD)\chi(x,hD)u = \chi(x,hD)p(x,hD)u \\ + hOp_h(\{p(x,\xi),\chi(x,\xi)\})u + O_{L^2}(h^2 \|u\|_{L^2}) \end{aligned}$$

So if u is an order h quasimode $\chi(x,hD)u$ is also one. This is one of the reasons why we tend to work with these quasimodes.

h-Dependent Localisation

Often want to study functions localised to a region the shrinks as $h \to 0$. Such as $\chi(h^{-\alpha}x_i)u$ (the function localised to the hypersurface $x_i = 0$).

- In this case we do not preserve order h quasimodes.
- We still have

$$p(x, hD)\chi(h^{-\alpha}x_i)u = \chi(x, hD)p(x, hD)u \\ + hOp_h(\{p(x, \xi), \chi(h^{-\alpha}x_i)\})u + O_{L^2}(h^{2-2\alpha}\|u\|_{L^2})$$

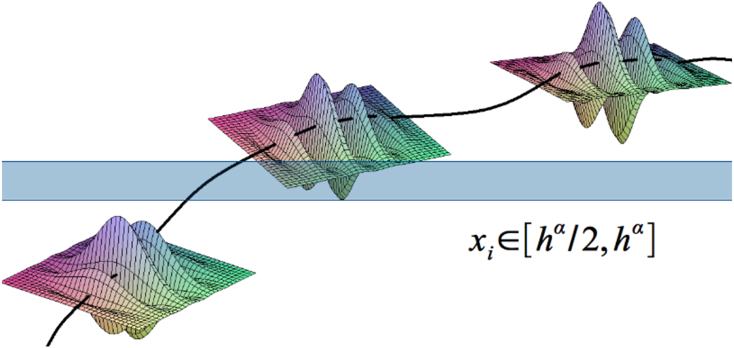
- Mapping norm of this depends on

$$\{p(x, \xi), \chi(h^{-\alpha}x_i)\} = h^{-\alpha}\chi'(h^{-\alpha}x_1)\{p(x, \xi), x_i\}$$

- Appear to loose a whole factor of $h^{-\alpha}$ but this is not the full story.

Back to Intuition

How long can packets remain in a region?



Depends on

$$\dot{q}(x,\xi) = \{q(x,\xi), p(x,\xi)\}$$

Theorem (T. 2015)

Suppose u is a $O(h)$ quasimode of $p(x, hD)$ and $\alpha \leq 1/2$. Let $\boldsymbol{\chi_q^\alpha} = Op(\chi_q^\alpha)$ where

$$\chi_q^\alpha(x,\xi) = \chi(h^{-\alpha}|q(x,\xi)|)$$

$$\chi : \mathbb{R}^+ \to \mathbb{R}, Supp(\chi) \subset [1/2, 1]$$

Then

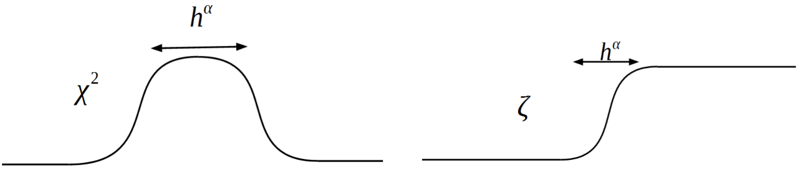
$$\left\| \dot{q}(x, hD)\boldsymbol{\chi_q^\alpha} u \right\|_{L^2} \lesssim h^{\alpha/2} \left\| u \right\|_{L^2}$$

- Can localised around other points than zero
- If $|\dot{q}(x,\xi)| \approx 1$ this implies

$$\left\| \boldsymbol{\chi_q^\alpha} u \right\|_{L^2} \lesssim h^{\alpha/2} \left\| u \right\|_{L^2}$$

Proof Sketch

Use commutation identity. Let ζ be defined so that $\zeta' = \chi^2$.



Consider $[p(x, hD), \zeta_{\boldsymbol{q}}^{\boldsymbol{\alpha}}]$ its symbol is given by

$$ih\{p(x,\xi), \zeta(h^{-\alpha}|q(x,\xi)|)\} + O(h^{2-2\alpha})$$

$$= ih^{1-\alpha}\{p(x,\xi), q(x,\xi)\}\zeta'(h^{-\alpha}|q(x,\xi)|) + O(h^{2-2\alpha})$$

$$= ih^{1-\alpha}\dot{q}(x,\xi)\chi^2(h^{-\alpha}|q(x,\xi)|) + O(h^{2-2\alpha})$$

Up to $O(h^{1-\alpha})$ error

$$\langle \dot{q}(x,hD)\boldsymbol{\chi}_{\boldsymbol{q}}^{\boldsymbol{\alpha}} u, \dot{q}(x,hD)\boldsymbol{\chi}_{\boldsymbol{q}}^{\boldsymbol{\alpha}} u\rangle = \langle \dot{q}(x,hD)u, \dot{q}(x,hD)(\boldsymbol{\chi}_{\boldsymbol{q}}^{\boldsymbol{\alpha}})^2 u\rangle$$

Insert the commutation identity for to get

$$= h^{\alpha-1}\langle \dot{q}(x,hD)u, [p(x,hD), \boldsymbol{\zeta}_{\boldsymbol{q}}^{\boldsymbol{\alpha}}]u\rangle$$

$$= h^{\alpha-1}\left(\langle p^\star(x,hD)\dot{q}(x,hD)u, \boldsymbol{\zeta}_{\boldsymbol{q}}^{\boldsymbol{\alpha}} u\rangle + \langle \dot{q}(x,hD)u, \boldsymbol{\zeta}_{\boldsymbol{q}}^{\boldsymbol{\alpha}} p(x,hD)u\rangle\right)$$

and use the fact that $p(x,hD)u = O_{L^2}(h)$ to get

$$\left|\langle \dot{q}(x,hD)\boldsymbol{\chi}_{\boldsymbol{q}}^{\boldsymbol{\alpha}} u, \dot{q}(x,hD)\boldsymbol{\chi}_{\boldsymbol{q}}^{\boldsymbol{\alpha}} u\rangle\right| \lesssim h^{\alpha} \left\|u\right\|_{L^2}^2$$

So

$$\left\|\dot{q}(x,hD)\boldsymbol{\chi}_{\boldsymbol{q}}^{\boldsymbol{\alpha}}(x,hD)u\right\|_{L^2} \lesssim h^{\alpha/2}\left\|u\right\|_{L^2}$$

Application to quasimode error

- Immediately tells us that this kind cut off is not as bad as we originally thought.
- Major error is given by

$$\left\| p(x,hD)\boldsymbol{\chi}^{\boldsymbol{\alpha}}_{\boldsymbol{q}}(x,hD)u \right\|_{L^2} \lesssim h^{1-\alpha} \left\| \dot{q}(x,hD)\chi'^{\alpha}_{q}(x,hD)u \right\|_{L^2}$$

  So we can say that the quasimode error is in fact $h^{1-\alpha/2}$ rather than $h^{1-\alpha}$ for general cut offs on scale h^{α}

- Even better for inner products. Same arguments yield

$$\left| \langle u, p(x,hD)\boldsymbol{\chi}^{\boldsymbol{\alpha}}_{\boldsymbol{q}}(x,hD)u \rangle \right| \lesssim h \left\| u \right\|^2_{L^2}$$

Localisation under the $h^{1/2}$ scale

- Commutation arguments break down due to the uncertainty principle.
- Need to be careful with definition of $\chi_{\boldsymbol{q}}^{\boldsymbol{\alpha}}$.

Toy Model

Suppose $p(x,\xi) = \xi_1$ and $q(x,\xi) = x_1$. That is

$$hD_{x_1}u = O_{L^2}(h)$$

Since $q(x,\xi)$ is ξ independent can define $\chi_{\boldsymbol{q}}^{\boldsymbol{\alpha}}$ for any α. In fact can even talk about restriction to $x_1 = 0$

Restriction of Quasimodes of hD_{x_1}

Let $\theta(x_1)$ be the Heaviside function. Consider

$$hD_{x_1}\theta(x_1)u = \theta(x_1)hD_{x_1}u + h\delta(x_1)u$$

So

$$h\langle u, u\rangle_{x'} = h\langle u, \delta(x_1)u\rangle = \langle u, \theta(x_1)hD_{x_1}u\rangle + \langle hD_{x_1}u, \theta(x_1)u\rangle$$

That is

$$\|u\|_{L^2_{x'}} \lesssim \|u\|_{L^2}$$

The best case, there is no concentration on the hypersurface at all (Could be up to $h^{-1/2}$ concentration).
How do we treat more general $p(x, hD)$?

Restriction of Quasimodes of general $p(x, hD)$

- Construct an operator W_h such that $hD_{x_1} W_h = W_h p(x, hD) + O(h^\infty)$
- This can be done as a semiclassical parametrix

$$W_h u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x', \xi' \rangle + \phi(x_1, y, \xi)} b(x_1, y, \xi) u(y) d\xi dy$$

$$\phi_{x_1}(x_1, y, \xi) + p(y, \nabla_y \phi) = 0 \quad \phi(0, y, \xi) = -\langle y, \xi \rangle$$

- This W_h also has the property that $W_h u\big|_{x_1=0} = u\big|_{x_1=0}$
- Let $v = W_h u$, now v is a quasimode of hD_{x_1} and we can use the toy model.
- Need to estimate the mapping norms of W_h.

Theorem (T. 2016)

Let $\nu(x,\xi) = \{p(x,\xi), x_1\}$ *and* $H = \{x \mid x_1 = 0\}$ *then*

$$\|\nu(x,hD)u\|_{L^2(H)} \lesssim \|u\|_{L^2} \tag{1}$$

$$\left\|\nu^{1/2}(x,hD)u\right\|_{L^2(H)} \lesssim \|u\|_{L^2} \tag{2}$$

where $\nu^{1/2}(x,hD)$ *is a suitable regularisation of* $\sqrt{\nu}$*.*

- The operator $\nu(x,hD)$ should be interpreted as the quantum version of normal velocity.
- This theorem says that even though a quasimode can concentrate, its normal velocity can't.
- (2) is much stronger where the symbol of $\nu(x,hD)$ is small.

Examples

These theorems hold for all smooth symbols $p(x, \xi)$. We will look at examples of the form $p(\xi)$ and construct solutions on the Fourier side.

$$\mathcal{F}_h[u] = \frac{1}{(2\pi h)^{n/2}} \int e^{-\frac{i}{h}\langle x,\xi\rangle} u(x)dx$$

Properties

$$\mathcal{F}_h[hD_{x_i}u] = \xi_i \mathcal{F}_h[u]$$

and

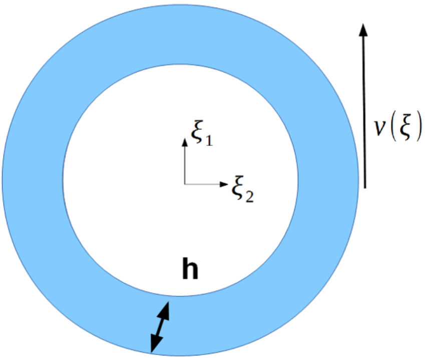
$$\|\mathcal{F}_h[u]\|_{L^2} = \|u\|_{L^2}$$

Will specialise to 2D case, higher dimensions are similar

Example 1

Let $p(x,\xi) = |\xi|^2 - 1$, then $\nu(x,\xi) = 2\xi_1$. This is the model flat Laplacian. On the Fourier side we need

$$(1 - |\xi|^2)\mathcal{F}_h u = O_{L^2}(h)$$



Behaviour of concentration depends on the size of $|\nu(x,\xi)|$.

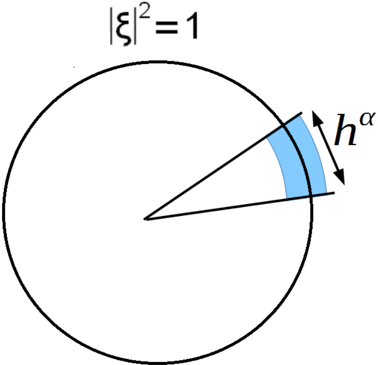
The extremes are when $|\nu(x,\xi)| \approx 1$ and when $|\nu(x,\xi)| < h^{1/2}$.

Let $0 \leq \alpha \leq 1/2$

$$\chi_\alpha^h(r,\omega) = \begin{cases} 1 & \text{if } |r-1| < h, |\omega - \omega_0| < h^\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then set

$$f_\alpha^h(\xi) = f_\alpha^h(r,\omega) = h^{-1/2-\alpha/2}\chi(r,\omega).$$



The function

$$T_\alpha^h = \mathcal{F}_h^{-1}[f_\alpha^h]$$

is an L^2 normalised, order h quasimode.

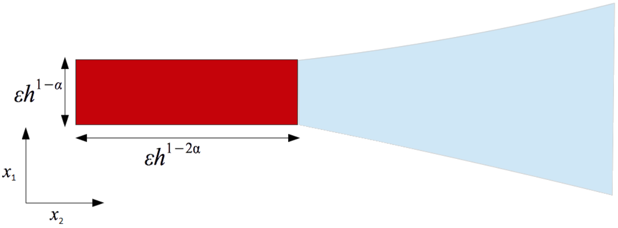
$$\nu(x, hD)\, T_\alpha^h(x) = \frac{1}{(2\pi h)} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} 2\xi_1 f_\alpha(\xi)\, d\xi.$$

We write

$$\nu(x, hD)\, T_\alpha^h(x)\big|_H = \frac{h^{-1/2-\alpha/2-1/2} e^{\frac{i}{h}x_2}}{2\pi} \int_{\mathbb{R}^n} e^{\frac{i}{h}x_2(\xi_2-1)} 2\xi_1 \chi_\alpha(\xi)\, d\xi.$$

Note that if $|x_2| < \epsilon h^{1-2\alpha}$ the factor $e^{\frac{i}{h}x_2(\xi_2-1)}$ does not oscillate so in this region

$$|\nu(x, hD)\, T_\alpha^h(x)| \approx c h^{-1/2+3\alpha/2}.$$



So we have a $h^{1-2\alpha}$ region where

$$|\nu(x, hD)T_\alpha^h| \approx ch^{-1/2+3\alpha/2}$$

$$\left\|\nu(x, hD)T_\alpha^h\right\|_{L^2} \approx h^{\alpha/2}$$

and

$$\left|\langle T_\alpha^h, \nu(x, hD)T_\alpha^h\rangle\right| \approx 1$$

This is a Laplacian example however the result is true for symbols very far from the Laplacian. For example

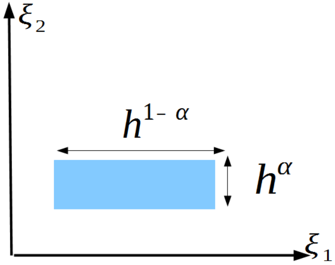
$$p(\xi) = \xi_2\xi_1$$

Does not obey curvature conditions on it's characteristic set $\{\xi \mid p(\xi) = 0\}$. Can have maximal $h^{-1/2}$ concentration on $x_1 = 0$.

Example 2

Let $p(x,\xi) = \xi_2\xi_1$, in this case $\nu(x,\xi) = \xi_2$. Must have

$$\xi_1\xi_2 f_\alpha = O_{L^2}(h).$$



Let $\zeta(r)$ supported in $[1/2, 1]$

$$\chi_\alpha(\xi) = \zeta(h^{-\alpha}|\xi_2|)\zeta(h^{-1+\alpha}|\xi_1|)$$

$$f_\alpha = h^{-\frac{1}{2}}\chi_\alpha(\xi)$$

and $v_\alpha = \mathcal{F}_h^{-1}[f_\alpha]$.

Localised where $|\nu(x,\xi)| \approx h^{\alpha}$

$$\nu(x,hD)v_\alpha|_H = \frac{h^{-\frac{1}{2}}}{(2\pi h)}\int e^{\frac{i}{h}x_2\cdot\xi_2}\xi_2\chi_\alpha(\xi)d\xi.$$

Now for $|x_2| < h^{1-\alpha}$ the $e^{\frac{i}{h}x_2\cdot\xi_2}$ factor does not significantly oscillate. So in this region

$$|\nu_\alpha(x,hD)v_\alpha| > h^{\alpha-\frac{1}{2}}$$

So

$$\|\nu_\alpha(x,hD)v_\alpha\|_{L^2(H)} > ch^{\alpha/2}.$$

and

$$|\langle v_\alpha, \nu_\alpha(x,hD)v_\alpha\rangle| \approx 1$$

Further work

- As we can localise quasimodes with less loss to the quasimode error we can use these results to study local (in h) properties of quasimodes
- Localisation beyond the $h^{1/2}$ scale for other symbols $q(x, \xi)$
- Need to define exactly what this means for general symbols (it is clear for restriction to hypersurfaces)