Strichartz Estimates for Non-Unitary Energy Bounds and Eigenfunction Estimates

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Abstract

The abstract Strichartz estimates of Keel and Tao [2] prove $L_t^p L_x^r$ estimates for operators $U(t)$ under a unitary energy bound $\|U(t)f\|_{L^2} \lesssim \|f\|_{L^2}$ and a dispersive estimate $\|U(s)U^\star(t)f\|_{L^\infty} \lesssim |t-s|^{-\sigma_\infty} \|f\|_{L^1}$. This paper presents an extended form of the abstract Strichartz estimates that assumes a L^2 bound similar to the dispersive estimate, that is $\|U(s)U^\star(t)f\|_{L^2} \lesssim |t-s|^{-\sigma_2} \|f\|_{L^2}$. These Strichartz estimates are useful for proving $L_t^p L_x^r$ bounds for evolution operators restricted to hypersurfaces. Using a trick derived from the implicit function theorem, questions about the L^p size of an approximate eigenfunction of a differential operator restricted to a hypersurface can be reduced to the study of just such a restricted evolution operator.

Introduction

Keel and Tao's [2] 1998 paper on Strichartz estimates united these estimates for evolution equations such as the wave and Schrodinger equations. This was achieved by putting Strichartz estimates in an abstract setting. Within this setting of a family of operators $U(t) : H \to L^2(X)$ (H a Hilbert space) Keel and Tao reduced proving the $\|U(t)f\|_{L_t^p L_x^r}$ estimates to proving the bilinear estimate

$$\left| \iint \langle (W(s))^\star F(s), (W(t))^\star G(t) \rangle \, ds dt \right| \lesssim \|F\|_{L_s^{p'} L_x^{r'}} \|G\|_{L_t^{p'} L_x^{r'}}. \quad (1)$$

One of the assumptions to obtain the abstract Strichartz estimates is that $U(t)$ has a uniform L^2 bound. This is not necessarily the case for an evolution operator restricted to a hypersurface (or any other submanifold). This paper extends the abstract Strichartz estimates by allowing the L^2 norm of $U(s)U^\star(t)$ to have a similar form to the usual Strichartz L^∞ dispersive estimate

$$\|U(s)U^\star(t)f\|_{L^\infty} \lesssim |t-s|^{-\sigma_\infty} \|f\|_{L^1}.$$

Included in these estimates is a (small) semiclassical parameter h. When performing asymptotic analysis of eigenfunctions this parameter appears as λ^{-1} for λ a large eigenvalue.

As an application of these methods, we consider an approximate solution to the differential equation $Pu = 0$ and ask how large can u be when restricted to a hypersurface, H. For example, an interesting class of operators are those such that

$$Pu = -\lambda^{-2}\Delta + \lambda^{-2}V(x) - 1.$$

Here a u such that $Pu = 0$ is an eigenfunction of $-\Delta + V(x)$ with eigenvalue λ^2. In this case we define the semiclassical parameter as $h = \lambda^{-1}$ and allow approximate eigenfunctions in the sense of $Pu = O_{L^2}(h)$. Under some technical assumptions on P and the family of approximate eigenfunctions $u(h)$, we obtain

Theorem 1. *Let $u(h)$ be a family of semiclassically microlocalised L^2 normalised functions such that $Pu = O_{L^2}(h)$. Further suppose that the symbol $p(x,\xi)$ of P satisfies the following non-degeneracy conditions when $p(x,\xi) = 0$*

$$p(x_0,\xi_0) = 0 \Rightarrow \partial_\xi p(x_0,\xi_0) \neq 0; \quad (2)$$

the second fundamental form on $\{\xi \mid p(x_0,\xi) = 0\}$ is positive definite; (3)

then

$$\|u\|_{L^p(H)} \lesssim h^{-\delta(p)}$$

$$\delta(p) = \begin{cases} \frac{n-1}{2} - \frac{n-1}{p}, & \frac{2n}{n-1} \leq p \leq \infty \\ \frac{n-1}{4} - \frac{n-2}{2p}, & 2 \leq p \leq \frac{2n}{n-1} \end{cases}.$$

This result generalises Burq, Gérard and Tzetkov's [1] estimates for Laplacian eigenfunctions restricted to hypersurfaces by moving it into the more general semiclassical framework similar to Koch, Tataru and Zworski's [3] semiclassical L^p estimates over the whole manifold.

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1 Extended Strichartz Estimates

Working with the Keel Tao formalism, we have a family of operators $U(t)$ such that

$$U(t) : H \to L^2(X)$$

for some Hilbert space H and measure space X.

Theorem 2. *Let $U(t) : H \to L^2(X)$ obey the estimates*

- *For all $t, s \in \mathbb{R}$ and $f \in L^2(X)$*

$$\|U(t)U^\star(s)f\|_{L^2(X)} \lesssim h^{-\mu_2}(h + |t-s|)^{-\sigma_2} \|f\|_{L^2(X)} \quad (4)$$

- *For all $t, s, \in \mathbb{R}$ and $f \in L^1(X)$*

$$\|U(t)U^\star(s)f\|_{L^\infty(X)} \lesssim h^{-\mu_\infty}(h + |t-s|)^{-\sigma_\infty} \|f\|_{L^1(X)} \quad (5)$$

then

$$\left(\int \|U(t)f\|_{L^r}^p \, dt\right)^{1/p} \lesssim h^{-\left(\frac{\mu_\infty - \mu_2}{p(\sigma_\infty - \sigma_2)} + \frac{\sigma_2\mu_\infty - \sigma_\infty\mu_2}{2(\sigma_\infty - \sigma_2)}\right)} \quad (6)$$

for pairs of (p, r), $2 < p \leq \infty$, $2 \leq r \leq \infty$ such that

$$\frac{1}{p} + \frac{1}{r}(\sigma_\infty - \sigma_2) = \frac{\sigma_\infty}{2}. \quad (7)$$

Note that the estimate (6) simplifies considerably when $\mu_\infty = \sigma_\infty$ and $\mu_2 = \sigma_2$, to become

$$\left(\int \|U(t)f\|_{L^r}^p \, dt\right)^{1/p} \leq h^{-1/p}.$$

The proof of this follows directly from the normal abstract Strichartz estimates proof. Estimates (4) and (5) are converted into bilinear forms giving

$$|\langle U(s)^\star F(s), U(t)^\star G(t)\rangle| \lesssim h^{-\mu_2}(h + |t-s|)^{-\sigma_2} \|F(s)\|_{L^2} \|G(t)\|_{L^2} \quad (8)$$

and

$$|\langle U(s)^\star F(s), U(t)^\star G(t)\rangle| \lesssim h^{-\mu_1}(h + |t-s|)^{-\sigma_1} \|F(s)\|_{L^1} \|G(t)\|_{L^1}. \quad (9)$$

Interpolation between (8) and (9) yields an estimate of the form

$$|\langle U(s)^\star F(s), U(t)^\star G(t)\rangle| \lesssim h^{-\gamma}(h + |t-s|)^{-\gamma} \|F(s)\|_{L^{r'}} \|G(t)\|_{L^{r'}}. \quad (10)$$

The s and t integrations are estimated via the Hardy-Littlewood-Soblev inequality. Combining the Hardy-Littlewood-Soblev and interpolation numerologies gives the governing equation (7) and estimate (6).

2 Estimating Eigenfunction Size

We attempt to estimate the size of u restricted to some hypersurface H where u obeys $Pu = O_{L^2}(h)$ for some differential operator P. To turn this into an evolution equation problem, we use the first non-degeneracy assumption (2) which by the implicit function theorem implies that locally we can rewrite the symbol as

$$p(x, \xi) = e(x, \xi)(\xi_1 - a(x, \xi)).$$

This symbol has an associated evolution equation

$$(hD_{x_1} - a(x, hD_{x'}))\tilde{u} = O(h)$$

which gives rise to an evolution operator $U(t)$, where we let $x_1 = t$. The L^p norm of u over the hypersurface can now be recovered from the $L_t^p L_x^r$ Strichartz estimates on $U(t)u(0, x')$ where $p = r$.

The relevant energy (4) and dispersion (5) estimates are obtained by writing $U(t)$ as a Fourier Integral Operator and making use of phase oscillations. These estimates make use of the second non-degeneracy assumption (3) and the method of stationary phase. The estimates give $\mu_2 = \sigma_2$ and $\mu_\infty = \sigma_\infty$ and so $p = r$ when $p = 2n/(n-1)$. This approach can also be used to obtain L^p bounds for approximate eigenfunctions restricted to submanifolds of co-dimension greater than one.

References

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