Evolution equations and vector-valued L^p spaces

Strichartz estimates and symmetric diffusion semigroups

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Abstract

The results of this thesis are motivated by the investigation of abstract Cauchy problems. Our primary contribution is encapsulated in two new theorems.

The first main theorem is a generalisation of a result of E. M. Stein [65]. In particular, we show that every symmetric diffusion semigroup acting on a complex-valued Lebesgue space has a tensor product extension to a UMD-valued Lebesgue space that can be continued analytically to sectors of the complex plane. Moreover, this analytic continuation exhibits pointwise convergence almost everywhere. Both conclusions hold provided that the UMD space satisfies a geometric condition that is weak enough to include many classical spaces. The theorem is proved by showing that every symmetric diffusion semigroup is dominated by a positive symmetric diffusion semigoup. This allows us to obtain (a) the existence of the semigroup's tensor extension, (b) a vector-valued version of the Hopf–Dunford–Schwartz ergodic theorem and (c) an holomorphic functional calculus for the extension's generator. The ergodic theorem is used to prove a vector-valued version of a maximal theorem by Stein [65], which, when combined with the functional calculus, proves the pointwise convergence theorem.

The second part of the thesis proves the existence of abstract Strichartz estimates for any evolution family of operators that satisfies an abstract energy and dispersive estimate. Some of these Strichartz estimates were already announced, without proof, by M. Keel and T. Tao [42]. Those estimates which are not included in their result are new, and are an abstract extension of inhomogeneous estimates recently obtained by D. Foschi [24]. When applied to

physical problems, our abstract estimates give new inhomogeneous Strichartz estimates for the wave equation, extend the range of inhomogeneous estimates obtained by M. Nakamura and T. Ozawa [53] for a class of Klein–Gordon equations, and recover the inhomogeneous estimates for the Schrödinger equation obtained independently by Foschi [23] and M. Vilela [75]. These abstract estimates are applicable to a range of other problems, such as the Schrödinger equation with a certain class of potentials.

Contents

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Chapter 1

Introduction

Many initial value problems (such as the inhomogeneous Schrödinger, wave, heat and Klein–Gordon equations) can be written in abstract form as

$$\begin{cases} u'(t) + Lu(t) = F(t) & \forall t \geq 0 \\ u(0) = f \end{cases} \tag{1.1}$$

where L is a closed linear operator on a Banach space $\mathcal{B}$, u and F are $\mathcal{B}$-valued functions on $[0, \infty)$ and $f \in \mathrm{Dom}(L)$. A function u which satisfies (1.1) is called a solution to the problem, the point f in $\mathcal{B}$ is called the initial data, and F is called the forcing (or source) term of the equation. If $F = 0$ then (1.1) is called an homogeneous or *abstract Cauchy problem*; otherwise, it is referred to as an inhomogeneous Cauchy problem.

In this thesis we explore what can be said about solutions to such problems using techniques of functional analysis. Our journey takes us in two directions. First, we consider how the system has evolved locally, pointwise. More specifically, suppose that for each nonnegative time t, the solution $u(t)$ to the homogeneous problem lies in a Lebesgue function space. If we write u as a function $(t, x) \mapsto u(t, x)$ of time and spatial variables, then when can we say that

$$\lim_{\tau \to 0^+} u(t + \tau, x) = u(t, x)$$

for almost every point x? This question may be translated to asking which one-parameter semigroups acting on the Lebesgue space L^p exhibit pointwise convergence almost everywhere. E. M. Stein [65] showed that this question may be answered in the affirmative in the case of symmetric diffusion semigroups which act on scalar-valued L^p spaces. In this thesis we generalise Stein's result to show that this is also true of tensor extensions of symmetric diffusion semigroups which act on an important class of vector-valued L^p spaces. This generalisation is stated in Theorem 1.3.5.

Second, given a particular inhomogeneous Cauchy problem (1.1), it is natural to ask the following question: Does (1.1) have a solution, and if so, is this solution unique and continuously dependent on the initial data f? The Cauchy problem is said to be well-posed if all parts of the above question have an affirmative answer. While there are various tools designed to answer this question, we focus on *Strichartz estimates* in particular. A Strichartz estimate for (1.1) is an a priori spacetime estimate to the solution of (1.1) depending on the norm of f and F. Classically, such estimates have the form

$$\|u\|_{L^p(\mathbb{R};L^s(\mathbb{R}^n))} \leq C \left(\|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\widetilde{p}}(\mathbb{R};L^{\widetilde{s}}(\mathbb{R}^n))} \right),$$

where n is the spatial dimension, p and $\widetilde{p}$ are the time exponents, and s and $\widetilde{s}$ are the spatial exponents. The challenge is to find a wide range of norms for which the spacetime estimate holds. In this thesis we prove the existence of certain Strichartz estimates in a very general abstract setting. The two main results are Theorem 4.2.2 (due to M. Keel and T. Tao [42, Theorem 10.1]) and Theorem 5.1.2 (which is a new result). The abstract Strichartz estimates of these theorems will then be applied to concrete problems, including inhomogeneous wave, Klein–Gordon and Schrödinger equations.

The structure of the thesis is as follows. Chapter 1 is devoted to elucidating these results within a broader mathematical and historical context. The goal of Chapter 2 is to prove our extension of Stein's pointwise convergence theorem for symmetric diffusion semigroups. In Chapter 3 we give a collection of mathematical tools which will be used to prove the results of the last two chapters. We thereby hope that the thesis, and in particular our exposition of the theory related to Strichartz estimates, will be mostly self-contained. Chapter 4 gives a proof of the abstract Strichartz estimates given by Keel and Tao [42, Theorem 10.1]. While Keel and Tao prove this theorem for a specific case, they omit the proof of the general case. We hope that that our detailed proof will be a welcome addition to the literature. Using techniques from Chapter 4, we prove some new abstract inhomogeneous Strichartz estimates in Chapter 5, before ending with a smattering of applications.

Throughout the thesis, there is one unifying example whose prominence is deserved: the Gaussian semigroup $\{e^{-t\Delta} : t \geq 0\}$ on L^2. On the one hand, this semigroup is the prototypical example of a symmetric diffusion semigroup. On the other hand, its boundary group $\{e^{it\Delta} : t \in \mathbb{R}\}$ is the prototypical example of an evolution group to which we apply the abstract Strichartz estimates.

1.1 Semigroups and Cauchy problems

The class of one-parameter semigroups of operators is one of the fundamental mathematical objects that arise in the study of abstract Cauchy problems. The aim of this section is to sketch out some key ideas behind this connection and indicate some basic strategies for solving inhomogeneous Cauchy problems.

Definition 1.1.1. Suppose that $\mathcal{B}$ is a Banach space and that, for each nonnegative t, there is a bounded linear operator T_t acting on $\mathcal{B}$. We say that the family $\{T_t : t \geq 0\}$ is a *one-parameter semigroup on* $\mathcal{B}$ if

(i) $T_0 = I$, where I is the identity operator on $\mathcal{B}$, and

(ii) $T_{s+t} = T_s T_t$ whenever s and t are nonnegative.

If, in addition to the above two axioms, the $\mathcal{B}$-valued map $t \mapsto T_t f$ is continuous on $[0, \infty)$ for each $f \in \mathcal{B}$, then we say that $\{T_t : t \geq 0\}$ is a *strongly continuous (one-parameter) semigroup on* $\mathcal{B}$.

For illustrative purposes, we give two explicit examples of strongly continuous semigroups. Our first is the (right) translation semigroup $\{T_t : t \geq 0\}$, which acts on the space $C(\mathbb{R})$ of continuous functions, and is given by

$$(T_t f)(x) = f(x - t) \qquad \forall t \geq 0 \quad \forall f \in C(\mathbb{R}).$$

For the second example, consider an $n \times n$ matrix L with real entries. If T_t is defined by $T_t = e^{-tL}$ then $\{T_t : t \geq 0\}$ is a strongly continuous semigroup on $\mathbb{R}^n$. Among other, more important, examples are the Gaussian semigroup, given formally by

$$\{e^{-t\Delta} : t \geq 0\},$$

and the Poisson semigroup

$$\{e^{-t(-\Delta)^{1/2}} : t \geq 0\}.$$

If $\{T_t : t \geq 0\}$ is a strongly continuous semigroup on a Banach space $\mathcal{B}$ then we define the *generator* $-L$ of the semigroup by the formula

$$-Lf = \lim_{t \to 0^+} \frac{T_t f - f}{t} \qquad \forall f \in \mathrm{Dom}(L),$$

where the domain of L consists of all f in $\mathcal{B}$ for which the above limit exists. It is obvious that L is a linear operator. While L may be unbounded, one can show that L is closed and has dense domain in $\mathcal{B}$. Moreover,

$$T_t(\mathrm{Dom}(L)) \subseteq \mathrm{Dom}(L) \qquad \forall t \geq 0,$$

the map $t \mapsto T_t f$ is continuously differentiable on $[0, \infty)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t f = -L T_t f \qquad \forall f \in \mathrm{Dom}(L)$$

(see, for example, [18, Chapter 1]). Hence if $-L$ is the generator of a strongly continuous semigroup $\{T_t : t \in \mathbb{R}\}$, $f \in \mathrm{Dom}(L)$ and $F = 0$, then the homogeneous Cauchy problem

$$\begin{cases} u'(t) + Lu(t) = 0 & \qquad \forall t \geq 0 \\ u(0) = f \end{cases}$$

(that is, (1.1) without the forcing term) has a solution u given by

$$u(t) = T_t f \qquad \forall t \geq 0. \tag{1.2}$$

Moreover, since there is a one-to-one correspondence between strongly continuous semigroups and their generators, we deduce that the solution u given by (1.2) is unique.

With these considerations in mind, and taking inspiration from the case when L is a square matrix, we often write

$$T_t = e^{-tL},$$

where the right-hand side may not be more than a formal expression.

It is natural to ask, then, when a closed and densely defined operator is the generator of a one parameter semigroup. This problem has received much attention over the years. The following celebrated theorem, proved by K. Yosida [80] and independently by E. Hille and R. Phillips [33], gives a characterisation of generators in terms of spectral theory (see [58], [59] and [18]).

Theorem 1.1.2 (Hille–Phillips–Yosida theorem). *Suppose that $\omega \in \mathbb{R}$, $M > 0$ and L is a closed densely defined operator on a Banach space $\mathcal{B}$. Then the following statements are equivalent.*

(i) The operator $-L$ is the generator of a strongly continuous semigroup $\{T_t : t \geq 0\}$ satisfying

$$\|T_t\| \leq M e^{\omega t} \qquad \forall t \geq 0.$$

(ii) Every λ greater than ω lies in the resolvent set of $-L$ and

$$\left\|(\lambda I + L)^{-m}\right\| \leq M(\lambda - \omega)^{-m}$$

whenever m is a positive integer and $\lambda > \omega$.

It can be shown that many operators (such as $-\Delta$) associated with important Cauchy problems generate strongly continuous semigroups. In fact, although the theory for solving homogeneous Cauchy problems extends far beyond what we have alluded to here, these problems are relatively well understood compared to those that are inhomogeneous (see, for example, the well-posedness results in [1]).

One way to approach the inhomogeneous Cauchy problem is to decompose (1.1) into the homogeneous part with nonzero initial data

$$v'(t) + Lv(t) = 0, \qquad v(0) = f, \qquad t \geq 0,$$

and the inhomogeneous part with zero initial data

$$w'(t) + Lw(t) = F(t), \qquad w(0) = 0, \qquad t \geq 0.$$

By linearity it is easy to see that the solution u to (1.1) is equal to $v + w$. Formally, the solution to the homogeneous and inhomogeneous parts are

$$v(t) = e^{-tL} f$$

and

$$w(t) = \int_0^t e^{-(t-s)L} F(s) \, \mathrm{d}s$$

respectively. Hence the formal solution u of (1.1) is given by

$$u(t) = e^{-tL} f + \int_0^t e^{-(t-s)L} F(s) \, \mathrm{d}s.$$

The heuristics sketched here are known as *Duhamel's principle.*

One can give a rigorous treatment of the above ideas by imposing suitable conditions on L, f, F and the solution space (see, for example, [8, Chapter 4]). In this thesis, most of the generators we work with are selfadjoint on L^2 and spectral theory will, in some appropriate sense, justify the formalism above.

From here, various approaches are possible. For example, suppose that $1 < p < \infty$ and that L satisfies the a priori estimate

$$\int_0^\infty \|u'(t)\|_{\mathcal{B}}^p \, \mathrm{d}t + \int_0^\infty \|Lu(t)\|_{\mathcal{B}}^p \, \mathrm{d}t \leq C_p \int_0^\infty \|F(t)\|_{\mathcal{B}}^p \, \mathrm{d}t \qquad \forall F \in L^p([0,\infty);\mathcal{B}),$$

where u is a solution to (1.1) with $f = 0$. Such an operator L is said to have *maximal L^p-regularity on* $[0,\infty)$. In this case, solving the inhomogeneous abstract Cauchy problem (1.1) with nonzero initial data f may be reduced to solving the corresponding homogeneous problem. In particular, this reduction enables one to determine whether an inhomogeneous problem is well-posed. Even though we do not pursue maximal L^p-regularity in the thesis, much of the mathematics that we explore in Chapter 2 has strong connections to the development of this method. Readers who wish to explore these ideas further are encouraged to read the excellent exposition [47] of P. Kunstmann and L. Weis.

A different approach to the same kind of problem involves finding an a priori spacetime estimate, known as a *Strichartz estimate*, to the solution of (1.1). Strichartz estimates facilitate the use of fixed point theorems and approximation methods, so that solutions to the inhomogeneous Cauchy problem can essentially be found from solutions to the homogeneous problem. The application of these estimates to inhomogeneous problems will be illustrated in Sections 4.1 and 5.9, justifying the search for Strichartz estimates, which forms a major component of the thesis.

Before giving a proper introduction to Strichartz estimates, we attend to some notation. Suppose that (X,μ) is a positive σ-finite measure space. If $1 \leq p < \infty$, denote by $L^p(X;\mathcal{B})$ the Bochner space of all $\mathcal{B}$-measurable functions F on X satisfying

$$\|F\|_p := \Big(\int_X \|F(x)\|_{\mathcal{B}}^p \, \mathrm{d}\mu(x) \Big)^{1/p} < \infty.$$

Denote by $L^\infty(X;\mathcal{B})$ the Bochner space of all $\mathcal{B}$-valued measurable functions which are μ-essentially bounded. (As is customary, we will not distinguish between equivalence classes of functions and members of each equivalence class.) The space of functions in $L^\infty(X;\mathcal{B})$ with compact support will be denoted by $L_0^\infty(X;\mathcal{B})$. In the case when $\mathcal{B}$ is the set $\mathbb{C}$ of complex numbers, $L^p(X;\mathcal{B})$ corresponds to the space of complex-valued pth power Lebesgue integrable functions on X, and we denote $L^p(X;\mathbb{C})$ simply by $L^p(X)$. If p is any Lebesgue–Bochner exponent in $[1,\infty]$, then denote by p' its conjugate exponent, given by

$$\frac{1}{p}+\frac{1}{p'}=1.$$

For $f\in L^p(X)$, we write $f\geq 0$ if $f(x)\geq 0$ for almost every $x\in X$. Whenever this is the case, the function f is said to be *nonnegative.* The subset of all nonnegative functions of $L^p(X)$ is a Banach lattice and shall be denoted by $L^p_+(X)$.

Throughout the thesis, we shall also use the notation

$$\|f\|_{\mathcal{C}}\lesssim K(j)\,\|g\|_{\mathcal{B}}$$

to mean

$$\|f\|_{\mathcal{C}}\leq CK(j)\,\|g\|_{\mathcal{B}},$$

where C is a constant, possibly changing from line to line, depending only on the Banach spaces $\mathcal{B}$ and $\mathcal{C}$. In particular, C does not depend on the functions f or g or on the variable j. If we write $A\approx B$ then we mean that $A\lesssim B$ and $B\lesssim A$.

1.2 Strichartz estimates

Consider the inhomogeneous Schrödinger initial value problem

$$\begin{cases} iu'(t)+\Delta u(t)=F(t) & \forall t\geq 0\\ u(0)=f, \end{cases}\tag{1.3}$$

whose formal solution u is given by

via Duhamel's principle. In the seminal paper [67] published in 1977, R. Strichartz showed that if u is a solution to (1.3) in n spatial dimensions and $q=2(n+2)/n$, then

$$\|u\|_{L^q(\mathbb{R};L^q(\mathbb{R}^n))}\lesssim\|f\|_{L^2(\mathbb{R}^n)}+\|F\|_{L^{q'}(\mathbb{R};L^{q'}(\mathbb{R}^n))}\tag{1.4}$$

whenever $f \in L^2(\mathbb{R}^n)$ and $F \in L^{q'}(\mathbb{R}; L^{q'}(\mathbb{R}^n))$.

The above spacetime estimate was a corollary obtained from an investigation of Fourier transform restriction theorems, going back to the work of P. Tomas [71] and E. M. Stein [64]. In particular, Strichartz [67] posed the following question: If S is a subset of $\mathbb{R}^n$ and μ is a positive measure supported on S with temperate growth at infinity, then for which values of q is it true that the tempered distribution $f\mathrm{d}\mu$ has a Fourier transform in $L^q(\mathbb{R}^n)$ satisfying

$$\left\|\widehat{f\mathrm{d}\mu}\right\|_q \leq C_q \left(\int_{\mathbb{R}^n} |f|^2 \,\mathrm{d}\mu\right)^{1/2},$$

whenever $f \in L^2(\mathbb{R}^n, \mu)$? Strichartz gave a complete solution when S is a quadratic surface in $\mathbb{R}^n$. With the correct choice of quadratic surface S, the above estimate is equivalent to a spacetime estimate for the homogeneous part v of the solution u to (1.3), namely

$$\|v\|_{L^q(\mathbb{R};L^q(\mathbb{R}^n))} = \left\|e^{it\Delta} f\right\|_{L^q(\mathbb{R};L^q(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}. \tag{1.5}$$

Strichartz obtained the spacetime estimate

$$\|w\|_{L^q(\mathbb{R};L^q(\mathbb{R}^n))} = \left\|\int_0^t e^{i(t-s)\Delta} F(s)\,\mathrm{d}s\right\|_{L^q(\mathbb{R};L^q(\mathbb{R}^n))} \lesssim \|F\|_{L^{q'}(\mathbb{R};L^{q'}(\mathbb{R}^n))} \tag{1.6}$$

for the inhomogeneous part w by interpolating between

$$\left\|e^{it\Delta} h\right\|_{L^2(\mathbb{R}^n)} = \|h\|_{L^2(\mathbb{R}^n)} \tag{1.7}$$

and

$$\left\|e^{it\Delta} h\right\|_{L^\infty(\mathbb{R}^n)} \leq C|t|^{n/2} \|h\|_{L^1(\mathbb{R}^n)}, \tag{1.8}$$

and then applying fractional integration to the result. Since $u = v + w$, the estimate (1.4) is obtained. The key estimates (1.7) and (1.8) were easily deduced from Plancherel's theorem and an explicit integral representation of the Schrödinger operator $e^{it\Delta}$.

Similar estimates were obtained for the inhomogeneous Klein–Gordon equation, of which the inhomogeneous acoustic wave equation

$$\begin{cases} -u''(t) + \Delta u(t) = F(t) & \forall t \geq 0 \\ u(0) = f \\ u'(0) = g \end{cases} \tag{1.9}$$

is a particular example. Whenever $\rho \in \mathbb{R}$, let $\dot{H}^\rho(\mathbb{R}^n)$ denote the homogeneous Sobolev space $(-\Delta)^{-\rho/2} L^2(\mathbb{R}^n)$, with norm given by

$$\|f\|_{\dot{H}^\rho} = \left\|(-\Delta)^{\rho/2} f\right\|_{L^2(\mathbb{R}^n)}.$$

Strichartz showed that if u is a solution to (1.9) then

$$\|u\|_{L^q(\mathbb{R};L^q(\mathbb{R}^n))} \lesssim \|f\|_{\dot{H}^\rho(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\rho-1}(\mathbb{R}^n)} + \|F\|_{L^{q'}(\mathbb{R};L^{q'}(\mathbb{R}^n))} \tag{1.10}$$

whenever $q = 2(n+1)/(n+1-2\rho)$, $1 \leq \rho < (n+1)/2$, $f \in \dot{H}^\rho(\mathbb{R}^n)$, $g \in \dot{H}^{\rho-1}(\mathbb{R}^n)$ and $F \in L^{q'}(\mathbb{R}; L^{q'}(\mathbb{R}^n))$. His proof of the spacetime estimate was similar to that for the inhomogeneous Schrödinger equation. The restriction theorem was used to obtain the homogeneous estimate, while the inhomogeneous estimate was obtained via an interpolation argument. In fact, the spacetime estimate for the solution w to the inhomogeneous wave equation with zero initial data was already contained in Strichartz' earlier paper [66]. In view of related work [62] by I. Segal, who was another early pioneer, inhomogeneous spacetime estimates have a longer history than their homogeneous counterparts.

The spacetime estimates (1.4) and (1.10), and variations of these, are now universally known as *Strichartz estimates*. Strichartz estimates, such as (1.5), for the homogeneous Cauchy problem with initial data are known as *homogeneous Strichartz estimates*, while those, such as (1.6), for the inhomogeneous problem with zero initial data are known as *inhomogeneous Strichartz estimates*. The estimates (1.7) and (1.8) are examples of what is respectively known as an energy and dispersive estimate. For the wave equation, fundamental dispersive estimates are the result of work by P. Brenner [5] and H. Pecher [55].

Following the early developments of Strichartz, work by B. Marshall [49] on the Klein–Gordon equation suggested that the space and time exponents in the Strichartz estimate (1.10) need not be equal. While this possibility had been noted in an early paper [66] of Strichartz, it was H. Pecher [56] who obtained almost all possible Strichartz estimates of the form

$$\|v\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{\dot{H}^\rho(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\rho-1}(\mathbb{R}^n)}$$

for the homogeneous wave equation with nonzero initial data. Around this time, it became evident that the restriction theorems were unnecessary for obtaining Strichartz estimates for the homogeneous equations. In fact, it was shown by J. Ginibre and G. Velo [27] that the duality arguments used to establish such estimates were more efficiently applied in an abstract operator setting where the Fourier transform did not play an essential role. This point of view was fully exploited by K. Yajima [78] who provided a large set of Strichartz inequalities for the homogeneous Schrödinger equation.

Strichartz estimates for the wave equation were later generalised by replacing the Lebesgue space norms (for the functions of the spatial variables) with norms of more general spaces. After contributions by various authors (including [38], [25] and [48]), Ginibre and Velo [28] gave a unified presentation of the known generalised Strichartz estimates for the inhomogeneous wave equation.

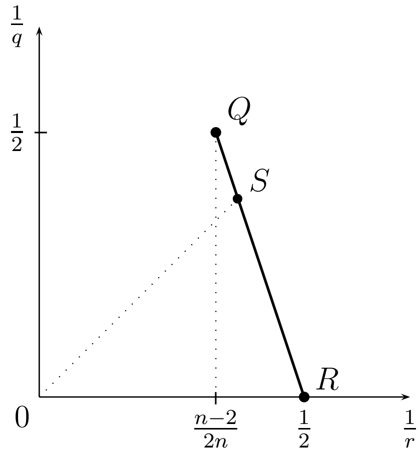
By this stage, the picture was almost complete for Strichartz-type estimates of the homogeneous parts of the wave and Schrödinger equations. Necessary and sufficient conditions for exponent pairs (q, r), such that the homogeneous Strichartz estimates

$$\|v\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$$

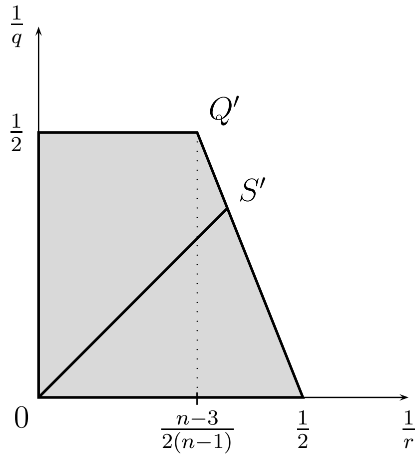
(for the Schrödinger equation) and

$$\|v\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{\dot{H}^\rho} + \|g\|_{\dot{H}^{\rho-1}}$$

(for the wave equation) are valid, coincided at all but the 'endpoint' (see Q and Q' in Figure 1.1 and the commentary below; see also Section 4.7). Moreover,



(a) Schrödinger equation



(b) Wave equation

Figure 1.1: The range of exponents that give valid Strichartz estimates.

if the exponent pair $(\widetilde{q}, \widetilde{r})$ also satisfied these sufficient conditions, then the inhomogeneous Strichartz estimate

$$\|w\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};L^{\widetilde{r}'}(\mathbb{R}^n))} \tag{1.11}$$

(for both wave and Schrödinger equations) was also valid.

In a major contribution, M. Keel and T. Tao [42] showed that both the homogeneous and inhomogeneous Strichartz estimates at the endpoint are valid, except in a special case (depending on the dimension n) where the Strichartz estimates were already known to be false (see [44] and [51]). This completely settled the problem of determining all homogeneous Strichartz estimates for all dimensions n.

Figure 1.1 represents the range of exponents that give valid Strichartz estimates in higher spatial dimensions. If $n > 2$ and u is a (weak) solution to the inhomogeneous Schrödinger equation (1.3), then the closed line segment QR of Figure 1.1 (a) represents exponent pairs (q, r) and $(\widetilde{q}, \widetilde{r})$ for which the Strichartz estimate

$$\|u\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\widetilde{q}'}(\mathbb{R};L^{\widetilde{r}'}(\mathbb{R}^n))} \tag{1.12}$$

is valid. The point Q corresponds to the endpoint determined by Keel and

Tao [42], while the point S corresponds to the original estimate (1.4) of R. Strichartz [67].

Similarly, suppose that $n > 3$ and u is a (weak) solution to the inhomogeneous wave equation (1.9). If the pairs $(1/q, 1/r)$ and $(1/\widetilde{q}, 1/\widetilde{r})$ lie in the closed shaded region of Figure 1.1 (b) and satisfy the 'gap condition'

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \rho = \frac{1}{\widetilde{q}'} + \frac{n}{\widetilde{r}'} - 2,$$

then the Strichartz estimate

$$\|u\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{\dot{H}^\rho(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\rho-1}(\mathbb{R}^n)} + \|F\|_{L^{\widetilde{q}'}(\mathbb{R};L^{\widetilde{r}'}(\mathbb{R}^n))} \tag{1.13}$$

holds. The point Q' corresponds to the endpoint determined by Keel and Tao [42], while the line segment OS' corresponds to the original estimate (1.10) of R. Strichartz [67]. It must also be noted that, when $r = \infty$, the L^r norm in (1.13) must be replaced with a Besov norm, and similarly when $\widetilde{r} = \infty$.

While earlier authors had determined Strichartz estimates by proving operator estimates, Keel and Tao [42] instead proved equivalent bilinear form estimates. This allowed them to employ techniques that are difficult to reproduce in the operator setting. They also stripped assumptions to the bare essentials, assuming only that the family of evolution operators $\{U(t) : t \in \mathbb{R}\}$ associated to the Cauchy problem satisfied an abstract energy estimate

$$\|U(t)h\|_{\mathcal{B}_0} \lesssim \|h\|_{\mathcal{H}} \qquad \forall h \in \mathcal{H} \tag{1.14}$$

and an abstract dispersive estimate

$$\|U(t)U(s)^*h\|_{\mathcal{B}_1^*} \lesssim |t-s|^{-\sigma} \|h\|_{\mathcal{B}_1} \qquad \forall h \in \mathcal{B}_1, \tag{1.15}$$

where $\sigma > 0$, $\mathcal{H}$ is a Hilbert space and $\mathcal{B}_0$ and $\mathcal{B}_1$ are compatible (in the sense of real interpolation) Banach spaces. This allowed a unified treatment of both the wave and Schrödinger equations, as well as, for example, the kinetic transport equation.

In Chapter 4, we present a proof of the abstract generalised Strichartz estimates [42, Theorem 10.1] of Keel and Tao. We do so for two reasons. First,

[42] does not prove this result in its most general form, choosing instead to prove the special case when $(\mathcal{B}_0, \mathcal{B}_1) = (L^2(X), L^1(X))$, where X is a measure space. While this special case is sufficient for obtaining all Strichartz estimates (in classical norms) for the homogeneous wave and Schrödinger equations, it does not yield generalised Strichartz estimates. Second, the techniques presented in our proof of [42, Theorem 10.1] will be used again in Chapter 5, where original results are proved. A statement of [42, Theorem 10.1] may be found in Section 4.2. At the end of Chapter 4, we illustrate how this abstract result recovers all Strichartz estimates for the Schrödinger equation (illustrated in Figure 1.1 (a)) and the generalised Strichartz estimates of [28] for the wave equation.

While all homogeneous Strichartz estimates for the wave and Schrödinger equations were determined by the late 1990s, there remained exponent pairs (q, r) for which the inhomogeneous Strichartz estimate holds but the homogeneous estimate fails. Prior to [42], this phenomenom was observed by D. Oberlin [54] and J. Harmse [31] for the wave equation, and by T. Cazenave and F. Weissler [10] and T. Kato [39] for Schrödinger's equation. Using the techniques of [42], D. Foschi [24] and M. Vilela [75] independently obtained what is currently the largest known range of exponent pairs (q, r) and $(\widetilde{q}, \widetilde{r})$ for which the inhomogeneous Strichartz estimate (1.11) for the Schrödinger equation is valid. While this was a significant advance, the gap between necessary and sufficient conditions on these exponent pairs is still substantial (see the discussion in Section 5.7) and the problem of determining all inhomogeneous Strichartz estimates remains open. It was remarked by Keel and Tao [42] that identifying all inhomogeneous Strichartz estimates is likely to be a very difficult problem. In the case of the wave equation, it is related to unsolved conjectures such as the local smoothing conjecture of C. Sogge [63] and the Bochner–Riesz problem for cone multipliers of J. Bourgain [4].

The result of D. Foschi [24, Theorem 1.4] is of particular interest because it is more general than the result of [75] and can be used to find inhomogeneous Strichartz estimates for both the wave and Schrödinger equations. Even so, the hypothesis of Foschi's theorem is unnecessarily restrictive, requiring that (1.14) and (1.15) both hold when $(\mathcal{B}_0, \mathcal{B}_1) = (L^2(X), L^1(X))$. In Chapter 5 we generalise Foschi's results by removing the restriction on $(\mathcal{B}_0, \mathcal{B}_1)$, thereby obtaining one of the two main results of this thesis (see Theorem 5.1.2). When this new theorem is applied, we recover all inhomogeneous Strichartz estimates contained in [24] and [75] for Schrödinger equation, as well as extending the list of inhomogeneous Strichartz estimates determined by M. Nakamura and T. Ozawa [53] for a class of inhomogeneous Klein–Gordon equations (see Section 5.8). The abstract result also allows us to give a new set of generalised inhomogeneous Strichartz estimates for the wave equation (see Corollaries 5.8.2 and 5.8.3), in the same spirit as the generalised Strichartz estimates presented by Ginibre and Velo [28]. Finally, it is shown in Section 5.9 that our abstract theorem allows one to obtain Strichartz estimates for important nonstandard problems, such as inhomogeneous Schrödinger equations with potential. For potentials of a certain class, the crucial ingredient needed to apply the abstract result is a dispersive estimate provided by K. Yajima [79].

The body of literature examining Strichartz estimates is now very large and the above historical overview only summarises one strand of developments, starting from Strichartz [67], maturing in Keel and Tao [42] and branching off to the inhomogeneous estimates of Foschi [24], Vilela [75] and the present work. Other authors have studied weighted Strichartz estimates, Strichartz estimates on manifolds, Strichartz estimates for inhomogeneous problems with variable coefficients and Strichartz estimates derived from dispersive estimates other than those considered here. Such topics are beyond the scope of the thesis.

While the thesis is not about applications of Strichartz estimates, we are aware that the quest to find Strichartz estimates is driven by the need to solve problems in inhomogeneous partial differential equations and scattering theory. We therefore frame our treatment of Strichartz estimates with two simple examples (see Sections 4.1 and 5.9) highlighting their power as a technical tool.

1.3 The pointwise convergence of symmetric diffusion semigroups

We move now to introduce a special class of semigroups, focusing in particular on two classical results which go back to E. M. Stein's monograph [65]. Connected with these ideas, we give a statement of an original result (Theorem 1.3.5) which extends these classical results to the setting of vector-valued Lebesgue spaces and constitutes the second major result of the thesis. Following this we shall indicate the method of proof of this new theorem, reserving the proof itself for Chapter 2.

Definition 1.3.1. Suppose that $\{T_t : t \geq 0\}$ is a semigroup of operators on $L^2(X)$. We say that

(a) the semigroup $\{T_t : t \geq 0\}$ satisfies the *contraction property* if

$$\|T_t f\|_q \leq \|f\|_q \qquad \forall f \in L^2(X) \cap L^q(X) \tag{1.16}$$

whenever $t \geq 0$ and $q \in [1, \infty]$; and

(b) the semigroup $\{T_t : t \geq 0\}$ is a *symmetric diffusion semigroup* if it satisfies the contraction property and if T_t is selfadjoint on $L^2(X)$ whenever $t \geq 0$.

It is well known that if $1 \leq p < \infty$ and $1 \leq q \leq \infty$ then $L^q(X) \cap L^p(X)$ is dense in $L^p(X)$. Hence, if a semigroup $\{T_t : t \geq 0\}$ acting on $L^2(X)$ has the contraction property then each T_t extends uniquely to a contraction of $L^p(X)$ whenever $p \in [1, \infty)$. By abuse of notation, we shall also denote by $\{T_t : t \geq 0\}$ the unique semigroup extension which acts on $L^p(X)$.

The class of symmetric diffusion semigroups is widely used in applications and includes the Gaussian and Poisson semigroups on $L^2(\mathbb{R}^n)$, where $n \in \mathbb{Z}^+$. Despite the simplicity of Definition 1.3.1, symmetric diffusion semigroups have a rich theory. For example, if $\{T_t : t \geq 0\}$ is a symmetric diffusion semigroup on $L^2(X)$ then the semigroup can also be continued analytically to sectors of the complex plane. To be precise, given a positive angle ψ, let Γ_ψ denote the cone $\{z \in \mathbb{C} : |\arg z| < \psi\}$ and $\overline{\Gamma}_\psi$ its closure. We shall denote the interval $[0, \infty)$ by $\overline{\Gamma}_0$. Using spectral theory and complex interpolation, Stein proved the following result.

Theorem 1.3.2 (Stein [65]). *Suppose that* $1 < p < \infty$*,*

$$\psi/\pi = 1/2 - |1/p - 1/2| > 0,$$

and $\{T_t : t \geq 0\}$ *is a symmetric diffusion semigroup on acting* $L^2(X)$*. Then* $\{T_t : t \geq 0\}$ *extends uniquely to a semigroup* $\{T_z : z \in \overline{\Gamma}_\psi\}$ *of contractions on* $L^p(X)$ *such that the operator-valued function* $z \mapsto T_z$ *is holomorphic in* Γ_ψ *and weak operator topology continuous in* $\overline{\Gamma}_\psi$*.*

We now recall two results of M. Cowling [15], developing the fundamental work of Stein [65]. The first is a useful technical tool. For f in $L^p(X)$, define the maximal function $\mathcal{M}^\psi f$ by

$$\mathcal{M}^\psi f = \sup\{|T_z f| : z \in \overline{\Gamma}_\psi\}.$$

The maximal theorem, stated below, says that the maximal function operator $\mathcal{M}^\psi$ is bounded on $L^p(X)$.

Theorem 1.3.3 (Stein–Cowling [15]). *Suppose that* $1 < p < \infty$ *and that*

$$0 \leq \psi/\pi < 1/2 - |1/p - 1/2|.$$

If $\{T_z : z \in \overline{\Gamma}_\psi\}$ *is the semigroup on* $L^p(X)$ *given by Theorem 1.3.2, then there is a positive constant* $C_{p,\psi}$*, depending only on* p *and* ψ*, such that*

$$\left\|\mathcal{M}^\psi f\right\|_p \leq C_{p,\psi} \|f\|_p \qquad \forall f \in L^p(X).$$

The maximal theorem allows one to deduce a pointwise convergence result for the semigroup $\{T_z : z \in \overline{\Gamma}_\psi\}$.

Corollary 1.3.4 (Stein–Cowling [15]). *Assume the hypotheses of Theorem 1.3.3. If $f \in L^p(X)$ then $(T_z f)(x) \to f(x)$ for almost every x in X as z tends to 0 in $\overline{\Gamma}_\psi$.*

The earliest form of the maximal theorem appeared in Stein [65, p. 73] for the case when $\psi = 0$. From this Stein deduced the pointwise convergence of $T_t f$ to f as $t \to 0^+$. Using a simpler approach, Cowling [15] extended Stein's result to semigroups $\{T_z : z \in \overline{\Gamma}_\psi\}$, holomorphic in the sector Γ_ψ, without additional hypotheses. Given $z \in \overline{\Gamma}_\psi$, Cowling's strategy was to decompose the operator T_z into two parts:

$$T_z f = \frac{1}{t}\int_0^t e^{-sL} f \,\mathrm{d}s + \Big[e^{-zL} f - \frac{1}{t}\int_0^t e^{-sL} f \,\mathrm{d}s\Big], \tag{1.17}$$

where $t = |z|$ and $-L$ is the generator of the semigroup. The L^p norm of the first term on the right-hand side can be controlled by the Hopf–Dunford–Schwartz ergodic theorem. A clever use of the Mellin transform allows the terms in brackets to be controlled by bounds on the imaginary powers of L.

One of the main contributions of this thesis is to observe that, under certain assumptions, the argument in [15] may be adapted to the setting of L^p spaces of Banach-space-valued functions. Several other results contained in Stein's monograph [65] have already been pushed in this direction (see, for example, [77], [50] and [36]). In a broader context, there has been much recent interest in operators which act on such spaces, particularly since tools for solving abstract Cauchy problems, such as vector-valued Laplace transforms and maximal L^p-regularity (see the expositions [1] and [47] and the references therein), require this setting. It is unsurprising that developments related to these methods are pertinent to techniques and results we use in this thesis. For example, several decades ago it was shown in the ground breaking work of J. Bourgain [3] and D. Burkholder [7] that Banach spaces possessing the so-called UMD property (see Section 2.5) were the spaces where classical singular integral and Fourier multiplier theory could be generalised to a vector-valued setting. Following this, extensions of the classical Littlewood–Paley, Marcinkiewicz and Mikhlin multiplier theorems were obtained in the UMD setting by F. Zimmermann [81]. Earlier, Cowling [15] had shown how to construct an H^∞-functional calculus for generators of symmetric diffusion semigroups using the transference methods popularised by R. Coifman and G. Weiss [12]. M. Hieber and J. Prüss [32] combined transference with the vector-valued Mikhlin multiplier theorem to construct an H^∞-functional calculus for generators of positive contraction semigroups which act on L^p spaces of UMD-valued functions. It is noteworthy that we use their method to show that the generator of a UMD-valued extension of a symmetric diffusion semigroups also possesses such a functional calculus.

Other advances in this area that are of interest include studies on bounded imaginary powers of operators (of which the article [20] of G. Dore and A. Venni is now a classic), H^∞-functional calculi for sectorial operators (see especially the fundamental paper [16] of A. McIntosh and his collaborators) and maximal L^p-regularity (see L. Weis [76] and the references therein). The article [47] of P. Kunstmann and L. Weis gives an excellent exposition of the interplay between these motifs in the vector-valued setting as well as an extensive bibliography detailing the key contributions made to the field over the last two decades.

Suppose that $\mathcal{B}$ is a (complex) Banach space and let $L^p(X;\mathcal{B})$ denote the Bochner space of $\mathcal{B}$-valued p-integrable functions on X. Given a symmetric diffusion semigroup $\{T_t : t \geq 0\}$ on $L^2(X)$, its tensor product extension $\{\widetilde{T}_t : t \geq 0\}$ to $L^p(X,\mathcal{B})$ exists by the contraction property (see Section 2.1). If $\{\widetilde{T}_t : t \geq 0\}$ can be continued analytically to some sector $\Gamma_{\psi+\epsilon}$, where $0 < \psi < \pi/2$ and ϵ is a (sufficiently) small positive number, then denote this continuation by $\{\widetilde{T}_z : z \in \Gamma_{\psi+\epsilon}\}$. If such a continuation does not exist, we take ψ to be 0. Given any function F in $L^p(X;\mathcal{B})$, one defines the maximal function $\mathcal{M}^\psi_\mathcal{B} F$ by

$$\mathcal{M}^\psi_\mathcal{B} F = \sup\{|\widetilde{T}_z F|_\mathcal{B} : z \in \overline{\Gamma}_\psi\}. \tag{1.18}$$

The theorem below is one of the main results of the thesis.

Theorem 1.3.5. *Suppose that (X,μ) is a σ-finite measure space and that $\{T_t : t \geq 0\}$ is a symmetric diffusion semigroup on $L^2(X)$. Suppose also that $\mathcal{B}$*

is a Banach space isomorphic to a closed subquotient of a complex interpolation space $(\mathcal{H},\mathcal{U})_{[\theta]}$*, where* $\mathcal{H}$ *is a Hilbert space,* $\mathcal{U}$ *is a UMD space and* $0<\theta<1$*. If* $1<p<\infty$*,* $|2/p-1|<\theta$ *and*

$$0 \leq \psi < \frac{\pi}{2}(1-\theta)$$

then

(a) $\{\widetilde{T}_t : t \geq 0\}$ *has a bounded analytic continuation to the sector* Γ_ψ *in* $L^p(X;\mathcal{B})$*,*

(b) there is a positive constant C *such that*

$$\left\| \mathcal{M}^{\psi}_{\mathcal{B}} F \right\|_{L^p(X)} \leq C \left\| F \right\|_{L^p(X;\mathcal{B})} \qquad \forall F \in L^p(X;\mathcal{B}),$$

and

(c) if $F \in L^p(X;\mathcal{B})$ *then* $\widetilde{T}_z F(x)$ *converges to* $F(x)$ *for almost every* x *in* X *as* z *tends to* 0 *in the sector* $\overline{\Gamma}_\psi$*.*

It is noteworthy that the class of Banach spaces $\mathcal{B}$ satisfying the interpolation hypothesis of Theorem 1.3.5 is a subset of those Banach spaces possessing the UMD property. It includes those classical Lebesgue spaces, Sobolev spaces and Schatten–von Neumann ideals that are reflexive. The reader is directed to Section 2.5 for further remarks on these spaces.

While Theorem 1.3.5 will be proved in Chapter 2, we now indicate the structure of the proof and the main techniques involved. We begin, in Section 2.1, by considering tensor product extensions of operators to vector-valued L^p spaces. It is well known that every positive (that is, positivity-preserving) operator on $L^p(X)$ has such an extension, but less known is the fact that every *subpositive* operator on $L^p(X)$ also has a tensor extension. As its name suggests, subpositivity is a weaker condition than positivity but surprisingly is not exploited in the literature as often as it could be. In Section 2.2, we demonstrate that every measurable semigroup $\{T_t : t \geq 0\}$ on $L^2(X)$ satisfying the contraction property is subpositive on $L^p(X)$ when $1 \leq p < \infty$, and that it is dominated by a measurable positive semigroup on $L^p(X)$ which also satisfies the contraction property. This is an extension of a similar result obtained independently by Y. Kubokawa [46] and C. Kipnis [43] for semigroups acting on $L^1(X)$. Our extension, while not difficult to prove, allows us to deduce that $\{T_t : t \geq 0\}$ has a tensor product extension to $L^p(X; \mathcal{B})$. (We note here that there are other well known methods for achieving the same ends; see Remark 2.2.3.) More importantly however, the our semigroup subpositivity result also allows us to easily deduce a vector-valued version of the Hopf–Dunford–Schwartz ergodic theorem in Section 2.3.

Parts (a) and (b) of Theorem 1.3.5 are proved in Sections 2.4 and 2.5. Following techniques used in [15], we begin by proving a maximal theorem for the tensor product extension $\{\widetilde{T}_t : t \geq 0\}$ to $L^p(X; \mathcal{B})$ of a strongly continuous semigroup $\{T_t : t \geq 0\}$ satisfying the contraction property. Here we assume that $1 < p < \infty$ and $\mathcal{B}$ is any Banach space, provided that the generator $-\widetilde{L}$ of the $\mathcal{B}$-valued extension has bounded imaginary powers on $L^p(X; \mathcal{B})$ with a power angle less than $\pi/2 - \psi$. Section 2.5 discusses circumstances under which this condition holds. In general, it is necessary that $\mathcal{B}$ has the UMD property. Moreover, by exploiting the subpositivity of $\{T_t : t \geq 0\}$ and adapting arguments of M. Hieber and J. Prüss [32], we show that if $\mathcal{B}$ has the UMD property then $\widetilde{L}$ has an H^∞-functional calculus. This, along with spectral theory (where the self-adjointness of each operator T_t is imposed) and interpolation, allows us to remove the bounded imaginary powers hypothesis at the cost of restricting the class of Banach spaces $\mathcal{B}$ for which the maximal theorem is valid.

In Section 2.6 we show that that the pointwise convergence of $\{\widetilde{T}_z : z \in \overline{\Gamma}_\psi\}$ is easily deduced from the pointwise convergence of $\{T_z : z \in \overline{\Gamma}_\psi\}$ and the maximal theorem. This completes the proof of Theorem 1.3.5.

Chapter 2

Symmetric diffusion semigroups in vector-valued L^p spaces

The main goal of this chapter is to give a proof of Theorem 1.3.5. The broad outline of the proof, which also determines the structure of the chapter, was given at the end of Section 1.3 and we will not repeat it here. There are several results presented in the chapter which are of interest independent to the proof of the main theorem. These are the subpositivity theorem for semigroups possessing the contraction property (see Theorem 2.2.1), the vector-valued Hopf–Dunford–Schwartz ergodic theorem (see Corollary 2.3.2) and the fact that the generator of a UMD-valued extension of a symmetric diffusion semigroup has an H^∞-functional calculus (see Theorem 2.5.1).

Throughout this chapter, we will suppose that (X, μ) is a σ-finite measure space.

2.1 Tensor product extensions of subpositive operators

Suppose that $\mathcal{B}$ is a Banach space with norm $|\cdot|_{\mathcal{B}}$ and that (X, μ) is a σ-finite measure space. We assume throughout this section that $p \in [1, \infty)$. Let

$L^p(X) \otimes \mathcal{B}$ denote the set of all finite linear combinations of $\mathcal{B}$-valued functions of the form uf, where $u \in \mathcal{B}$ and $f \in L^p(X)$. It is well known that this set is dense in $L^p(X; \mathcal{B})$ (see, for example, [29]). Many operators acting on scalar-valued function spaces can be extended to act on $\mathcal{B}$-valued function spaces in the following canonical way.

Definition 2.1.1. Suppose that T is a bounded operator on $L^p(X)$. If $I_{\mathcal{B}}$ denotes the identity operator on $\mathcal{B}$ then define the tensor product $T \otimes I_{\mathcal{B}}$ on $L^p(X) \otimes \mathcal{B}$ by

$$T \otimes I_{\mathcal{B}} \left(\sum_{k=1}^{n} u_k f_k \right) = \sum_{k=1}^{n} u_k T f_k$$

whenever $n \in \mathbb{Z}^+$, $u_k \in \mathcal{B}$, $f_k \in L^p(X)$ and $k = 1, \ldots, n$. We say that a bounded operator $\widetilde{T} : L^p(X; \mathcal{B}) \to L^p(X; \mathcal{B})$ is a *$\mathcal{B}$-valued extension of T* if $\widetilde{T} = T \otimes I_{\mathcal{B}}$ on $L^p(X) \otimes \mathcal{B}$. In this case, $\widetilde{T}$ is also called a *tensor product extension of T to $L^p(X; \mathcal{B})$*.

If it exists, a $\mathcal{B}$-valued extension $\widetilde{T}$ of T is necessarily unique, by the density of $L^p(X) \otimes \mathcal{B}$ in $L^p(X; \mathcal{B})$. We now consider a class of operators for which such extensions are possible.

Definition 2.1.2. Suppose that T is a linear operator on $L^p(X)$ and that $\{T_t : t \geq 0\}$ is a semigroup of operators on $L^p(X)$. We say that

(a) the operator T is *positive* if $Tf \geq 0$ whenever $f \geq 0$ for f in $L^p(X)$;

(b) the operator T on $L^p(X)$ is *subpositive* if there is a bounded positive operator S on $L^p(X)$ such that $|Tf| \leq S|f|$ whenever $f \in L^p(X)$, in which case we also say that T is *dominated* by S; and

(c) the semigroup $\{T_t : t \geq 0\}$ is *subpositive* if there is a family $\{S_t : t \geq 0\}$ of bounded positive operators on $L^p(X)$ such that $|T_t f| \leq S_t |f|$ whenever $t \geq 0$ and $f \in L^p(X)$.

It is well known that every bounded positive operator T on $L^p(X)$ has a $\mathcal{B}$-valued extension $\widetilde{T}$ on $L^p(X; \mathcal{B})$ (see, for example, [29, Section 4.5]). We will deduce the same for subpositive operators.

Lemma 2.1.3. *Suppose that T is a subpositive operator on $L^p(X)$ dominated by a bounded positive operator S. Then*

(i) T has a $\mathcal{B}$-valued extension $\widetilde{T}$ on $L^p(X;\mathcal{B})$,

(ii) $|\widetilde{T}F|_{\mathcal{B}} \le S|F|_{\mathcal{B}}$ whenever $F \in L^p(X;\mathcal{B})$, and

(iii) $\|\widetilde{T}\|_{L^p(X;\mathcal{B})\to L^p(X;\mathcal{B})} \le \|S\|_{L^p(X)\to L^p(X)}$.

Although the above lemma is undoubtedly known by the experts, we are not aware of any proof in the literature. We shall therefore give one here. In what follows, we write $\mathcal{B}^*$ for the dual of $\mathcal{B}$ and write $\langle u, v\rangle$ for $v(u)$ when $u \in \mathcal{B}$ and $v \in \mathcal{B}^*$.

Proof. Suppose that T is a subpositive operator dominated by a bounded positive operator S. We begin by noting that for any countable set K,

$$\sup_{k\in K} |Tg_k| \le \sup_{k\in K} S|g_k| \le S\big(\sup_{k\in K} |g_k|\big)$$

whenever $\{g_k\}_{k\in K} \subset L^p(X)$ and $\sup_{k\in K} |g_k| \in L^p(X)$, by the positivity of S. Let $I_{\mathcal{B}}$ denote the identity operator on $\mathcal{B}$. We will show that $T\otimes I_{\mathcal{B}}$ is bounded on $L^p(X)\otimes\mathcal{B}$ and can therefore be extended to a bounded operator $\widetilde{T}$ on $L^p(X;\mathcal{B})$.

Given F in $L^p(X)\otimes\mathcal{B}$, write F as $\sum_{j=1}^n u_j f_j$ where $u_j \in \mathcal{B}$ and $f_j \in L^p(X)$. Since F and $T\otimes I_{\mathcal{B}}F$ both take values in a finite dimensional (and hence separable) subspace of $\mathcal{B}$, there exists a countable subset V of the unit ball of $\mathcal{B}^*$ such that

$$\sup_{v\in V} |\langle \widetilde{T}F(x), v\rangle| = |\widetilde{T}F(x)|_B$$

and

$$\sup_{v\in V} |\langle F(x), v\rangle| = |F|_{\mathcal{B}}(x)$$

for almost every x in X. Therefore

$$\begin{aligned}
|T \otimes I_{\mathcal{B}} F(x)|_{\mathcal{B}} &= \sup_{v \in V} |\langle \widetilde{T} F(x), v \rangle| \\
&= \sup_{v \in V} \left| T\Big(\sum_{j=1}^{n} \langle u_j ,\, v \rangle \, f_j \Big)(x) \right| \\
&\leq S \left(\sup_{v \in V} \left| \sum_{j=1}^{n} \langle u_j ,\, v \rangle \, f_j \right| \right)(x) \\
&= S(|F|_{\mathcal{B}})(x).
\end{aligned}$$

for almost every x in X. Taking the $L^p(X)$ norm of both sides gives

$$\|T \otimes I_{\mathcal{B}} F\|_{L^p(X;\mathcal{B})} \leq \|S\|_{L^p(X) \to L^p(X)} \|F\|_{L^p(X;\mathcal{B})}$$

as required. □

Given a strongly continuous semigroup of subpositive operators, we would like to know when its $\mathcal{B}$-valued extension is also strongly continuous. We remind readers that a family of operators $\{R_z : z \in \Lambda\}$ on $L^p(X;\mathcal{B})$ indexed by z on some subset Λ of the complex plane is said to be *locally uniformly bounded in norm on* Λ if, for each positive r and each z in Λ, there exists a positive number M such that

$$\|R_w F\|_p \leq M \|F\|_p \qquad \forall F \in L^p(X;\mathcal{B})$$

whenever $|w - z| < r$ and $w \in \Lambda$.

Lemma 2.1.4. *Suppose that $\psi \geq 0$ and that $\{T_z : z \in \overline{\Gamma}_\psi\}$ is a semigroup of subpositive operators on $L^p(X)$ such that the mapping $z \mapsto T_z$ is strongly continuous for z in $\overline{\Gamma}_\psi$. If its $\mathcal{B}$-valued extension $\{\widetilde{T}_z : z \in \overline{\Gamma}_\psi\}$ is locally uniformly bounded in norm on $\overline{\Gamma}_\psi$, then the mapping $z \mapsto \widetilde{T}_z$ is strongly continuous for $z \in \overline{\Gamma}_\psi$.*

Proof. Suppose that $z \in \overline{\Gamma}_\psi$, that $F \in L^p(X;\mathcal{B})$ and that $\epsilon > 0$. By the hypothesis there exists a positive number M such that

$$\left\| \widetilde{T}_w G \right\|_p \leq M \|F\|_p \qquad \forall G \in L^p(X;\mathcal{B})$$

whenever $|w - z| < 1$ and $w \in \overline{\Gamma}_\psi$. Choose nonzero G in $L^p(X) \otimes \mathcal{B}$ such that

$$\|F - G\|_p < \frac{\epsilon}{3M}.$$

The function G has the representation $\sum_{k=1}^n u_k g_k$, where $u_k \in \mathcal{B}$ and $g_k \in L^p(X)$ for some finite integer n. Write α for $\sum_{k=1}^n |u_k|_{\mathcal{B}}$. Since the map $z \mapsto T_z$ is strongly continuous, for each k in $\{1, \ldots, n\}$ there is a positive δ_k such that

$$\|T_z g_k - T_w g_k\|_p < \frac{\epsilon}{3\alpha}$$

whenever $|z - w| < \delta_k$ and $w \in \overline{\Gamma}_\psi$. Denote by δ the minimum of the set $\{1, \delta_1, \delta_2, \ldots, \delta_n\}$. If $w \in \overline{\Gamma}_\psi$ and $|z - w| < \delta$ then

$$\begin{aligned}
\left\|\widetilde{T}_z F - \widetilde{T}_w F\right\|_p &\le \left\|\widetilde{T}_z F - \widetilde{T}_z G\right\|_p + \left\|\widetilde{T}_z G - \widetilde{T}_w G\right\|_p + \left\|\widetilde{T}_w G - \widetilde{T}_w F\right\|_p \\
&\le 2M \|F - G\|_p + \sum_{k=1}^n |u_k|_{\mathcal{B}} \|T_z g_k - T_w g_k\|_p \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

Hence $\{\widetilde{T}_z : z \in \overline{\Gamma}_\psi\}$ is strongly continuous on $L^p(X; \mathcal{B})$. □

Remark 2.1.5. If a strongly continuous semigroup $\{T_z : z \in \Gamma_\psi\}$ on $L^p(X)$ is dominated by a positive family $\{S_z : z \in \Gamma_\psi\}$ that is locally uniformly bounded in $L^p(X)$ norm, then the $\mathcal{B}$-valued extension $\{\widetilde{T}_z : z \in \Gamma_\psi\}$ is strongly continuous.

Suppose that a strongly continuous semigroup $\{T_t : t \ge 0\}$ of subpositive operators on $L^p(X)$ has a $\mathcal{B}$-valued extension $\{\widetilde{T}_t : t \ge 0\}$ that is also a strongly continuous semigroup. As usual, the generator B of $\{\widetilde{T}_t : t \ge 0\}$ is given by

$$BF = \lim_{t \to 0^+} \frac{\widetilde{T}_t F - F}{t}$$

for all F in $L^p(X; \mathcal{B})$ for which the limit exists. The collection of such F is the domain of B. Let $-L$ be the generator of $\{T_t : t \ge 0\}$. It is easy to show that $\mathrm{Dom}(L) \otimes \mathcal{B} \subseteq \mathrm{Dom}(B)$ and that $B = -L \otimes I_{\mathcal{B}}$ on $\mathrm{Dom}(L) \otimes \mathcal{B}$. Therefore we denote B by $-\widetilde{L}$.

We close this section by demonstrating that our definition of subpositivity coincides with that implied by R. Coifman, R. Rochberg and G. Weiss [11, p. 54] (strictly speaking, [11] only introduces the notion of a *subpositive contraction*). We later use the equivalence of these definitions to prove Theorem 2.5.1.

If R is an operator on $L^p(X)$ then define $\overline{R}$ by the formula $\overline{R}f = \overline{R\overline{f}}$ whenever $f \in L^p(X)$, and define $\mathrm{Re}(R)$ by $(R + \overline{R})/2$.

Proposition 2.1.6. *If $1 \leq p \leq \infty$ and T is a linear operator on $L^p(X)$ then the following statements are equivalent.*

(i) The operator T is a subpositive operator.

(ii) There exists a bounded positive operator S on $L^p(X)$ such that $S + \mathrm{Re}(e^{i\theta}T)$ is positive whenever $\theta \in \mathbb{R}$.

Proof. Suppose that (i) holds. Then there is a bounded positive operator S such that $Sf \geq |Tf|$ for all nonnegative f in $L^p(X)$. Choose f in $L^p(X)$ such that $f \geq 0$. Then for all real θ,

$$\begin{aligned}\big(S + \mathrm{Re}(e^{i\theta}T)\big)f &= Sf + \mathrm{Re}(e^{i\theta}Tf) \\ &\geq Sf - |Tf| \\ &\geq 0.\end{aligned}$$

Thus (ii) holds.

Conversely, suppose that (ii) holds. We deduce that (i) holds in three steps.

Step 1. Assume that $f \geq 0$. Then

$$Sf + \mathrm{Re}(e^{i\theta}T)f \geq 0 \qquad \forall \theta \in \mathbb{R}.$$

That is,

$$S|f| + \mathrm{Re}(e^{i\theta}Tf) \geq 0 \qquad \forall \theta \in \mathbb{R}.$$

Therefore

$$S|f| + \inf_{\theta \in \mathbb{Q}} \mathrm{Re}(e^{i\theta}Tf) \geq 0.$$

But it is clear that

$$-|Tf| = \inf_{\theta \in \mathbb{Q}} \mathrm{Re}(e^{i\theta} Tf)$$

and hence (i) holds for $f \geq 0$.

Step 2. Suppose that $\alpha > 0$ and that $f \in L^p(X)$, such that f only takes values in the sector Λ^α, where Λ^α is defined by the formula

$$\Lambda^\alpha = \{z \in \mathbb{C} : 0 \leq \arg z < \alpha\} \cup \{0\}.$$

Then $\mathrm{Re}(f) \geq 0$ and

$$0 \leq \mathrm{Im}(f) \leq \tan(\alpha)\mathrm{Re}(f).$$

So using Step 1,

$$\begin{aligned} |Tf| &\leq |T(\mathrm{Re}(f))| + |iT(\mathrm{Im}(f))| \\ &\leq S(\mathrm{Re}(f)) + S(\tan(\alpha)\mathrm{Re}(f)) \\ &\leq (1 + \tan\alpha) S|f|. \end{aligned}$$

Step 3. Let F be any function in $L^p(X)$. Suppose that $\epsilon > 0$ and choose n in $\mathbb{N}$ such that $n > 2\pi/\tan^{-1}(\epsilon)$. Whenever $j = 0, \ldots, n-1$, denote by Λ_j the sector

$$\{e^{i2\pi j/n} z : z \in \Lambda^{2\pi/n}\}$$

and define f_j by

$$f_j(x) = \begin{cases} f(x) & \text{if } f(x) \in \Lambda_j \\ 0 & \text{otherwise.} \end{cases}$$

Then $f = \sum_{j=0}^{n-1} f_j$, $|f| = \sum_{j=0}^{n-1} |f_j|$ and

$$|Tf| \leq \sum_{j=0}^{n-1} |Tf_j| = \sum_{j=0}^{n-1} |T(e^{-i2\pi j/n} f_j)|. \tag{2.1}$$

But for each j in $\{0, \ldots, n-1\}$, the function $e^{-i2\pi j/n} f_j$ takes values only in $\Lambda^{2\pi/n}$. Continuing from (2.1) with an application of the result of Step 2, we have

$$
\begin{aligned}
|Tf| &\leq \sum_{j=0}^{n-1} \big(1 + \tan(2\pi/n)\big)\, S|e^{-i2\pi j/n} f_j| \\
&= \big(1 + \tan(2\pi/n)\big)\, S\Big(\sum_{j=0}^{n-1} |f_j|\Big) \\
&< (1 + \epsilon)\, S|f|.
\end{aligned}
$$

Since ϵ is arbitrary, $|Tf| \leq S|f|$ as required. □

2.2 Subpositivity for contraction semigroups

The goal of this section is to prove that every semigroup on $L^2(X)$ with the contraction property is, when extended to a semigroup on $L^p(X)$ for p in $[1, \infty)$, dominated by a positive contraction semigroup on $L^p(X)$.

We begin with a few preliminaries. Suppose that $1 \leq p < \infty$ and T is a bounded linear operator on $L^p(X)$. If $1 \leq q < \infty$ and $\|Tf\|_q \leq C\,\|f\|_q$ for all f in $L^q(X) \cap L^p(X)$ then, by a density argument, T has a unique bounded linear extension acting on $L^q(X)$ and by abuse of notation we will also denote this extension by T.

We say that a family of operators $\{T_t : t \geq 0\}$ is *(strongly) measurable* on $L^p(X)$ if, for every f in $L^p(X)$, the $L^p(X)$-valued map $t \mapsto T_t f$ is measurable with respect to Lebesgue measure on $[0, \infty)$. The family is said to be *weakly measurable* if the complex-valued map $t \mapsto \langle T_t f,\, g\rangle$ is measurable with respect to Lebesgue measure on $[0, \infty)$ whenever $f \in L^p(X)$ and $g \in L^{p'}(X)$. If $1 \leq p < \infty$ then $L^p(X)$ is a separable Banach space and hence strong measurability and weak measurability coincide, by the Pettis measurability theorem (see [21, Theorem III.6.11]).

The main result of this section is the following theorem.

Theorem 2.2.1. *Suppose that $\{T_t : t \geq 0\}$ is a semigroup on $L^2(X)$ satisfying the contraction property. Then there exists a positive semigroup $\{S_t : t \geq 0\}$*

on $L^2(X)$, satisfying the contraction property, such that

$$|T_t f| \leq S_t |f| \qquad \forall f \in L^p(X)$$

whenever $1 \leq p < \infty$ and $t \geq 0$. If $\{T_t : t \geq 0\}$ is a measurable semigroup on $L^2(X)$ then $\{S_t : t \geq 0\}$ extends to a measurable semigroup on $L^p(X)$ whenever $1 \leq p < \infty$.

If $\{T_t : t \geq 0\}$ is a strongly continuous semigroup, it is natural to ask whether the positive semigroup $\{S_t : t \geq 0\}$ of Theorem 2.2.1 is also continuous. Under certain circumstances one can answer in the affirmative (see Corollary 2.2.9). However, such a result is unnecessary for our applications. What we do use is the following corollary, which is immediately deduced from the theorem and Lemma 2.1.4.

Corollary 2.2.2. *Suppose that $\mathcal{B}$ is a Banach space, that $1 \leq p < \infty$ and that $\{T_t : t \geq 0\}$ is a strongly continuous semigroup on $L^2(X)$ satisfying the contraction property. Then T_t has a $\mathcal{B}$-valued extension to $L^p(X;\mathcal{B})$ and the family $\{\widetilde{T}_t : t \geq 0\}$ is a strongly continuous semigroup of contractions on $L^p(X;\mathcal{B})$.*

Remark 2.2.3. It is well known that any operator T that acts as a contraction on $L^q(X)$ for all $q \in [1,\infty]$ has a contractive tensor extension to $L^p(X;\mathcal{B})$ whenever $1 \leq p < \infty$. This is not hard to show if $p = 1$; for other values of p the result is obtained by duality and interpolation. This provides an alternate proof of Corollary 2.2.2. We point out that Theorem 2.2.1 is still needed to prove our vector-valued version of the Hopf–Dunford–Schwartz ergodic theorem.

We turn now to the proof of Theorem 2.2.1, which shall be achieved via a sequence of lemmata. The final stage of the proof draws heavily on the work of Y. Kubokawa [46] and C. Kipnis [43], who independently proved a similar result for L^1 contraction semigroups.

Lemma 2.2.4. *If $1 \leq p < \infty$ then $L^p_+(X)$ is a closed subset of $L^p(X)$.*

Proof. Suppose that $\{f_n\}_{n\in\mathbb{N}} \subset L^p_+(X)$ and that $f_n \to f$ in $L^p(X)$ as $n \to \infty$.

We will first show that $\mathrm{Im} f = 0$. Suppose that this is not the case. Then there exists a subset E of X with positive measure such that $|\mathrm{Im} f| > 0$ on E. If

$$E_n = \{x \in E : 1/n \le |\mathrm{Im} f(x)| < n\}$$

whenever $n \ge 1$, then $E_n \subseteq E_{n+1}$ whenever $n \ge 1$ and

$$\bigcup_{n=1}^{\infty} E_n = E.$$

Since

$$0 < \mu(E) = \lim_{n\to\infty} \mu(E_n),$$

there exists a positive integer j such that $\mu(E_j) > 0$. But then

$$\|f_n - f\|_p \ge \|\mathrm{Im}(f_n - f)\| = \|\mathrm{Im} f\| \ge \mu(E_j)/j$$

for every n in $\mathbb{N}$, contradicting the hypothesis that $f_n \to f$ in $L^p(X)$. Hence $\mathrm{Im} f = 0$.

It is also easy to show, using an argument similar to that above, that the negative part of f is 0. Hence $f \ge 0$. $\square$

Lemma 2.2.5. *Suppose that $1 < p < \infty$. Assume also that T and S are bounded operators on $L^1(X)$ such that $\|Tf\|_p \le \|f\|_p$ and $\|Sf\|_p \le \|f\|_p$ whenever $f \in L^1(X) \cap L^p(X)$. If S is positive and dominates T on $L^1(X)$ then*

$$|Tf| \le S|f| \qquad \forall f \in L^p(X).$$

Proof. Assume the hypotheses of the lemma and suppose that $f \in L^p(X)$. Since $L^1(X)\cap L^p(X)$ is dense in $L^p(X)$, there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in $L^1(X) \cap L^p(X)$ such that $f_n \to f$ in $L^p(X)$. By continuity, $|Tf_n| \to |Tf|$ in $L^p(X)$ and similarly $S|f_n| \to S|f|$ in $L^p(X)$. Moreover, $|Tf_n| \le S|f_n|$ for all n. If $g_n = S|f_n| - |Tf_n|$ and $g = S|f| - |Tf|$ then each g_n is nonnegative and $g_n \to g$ in $L^p(X)$. By the previous lemma, this implies that $g \ge 0$, completing the proof. $\square$

Lemma 2.2.6. *Suppose that p and q both lie in the interval $[1,\infty)$ and that $\{T_t : t \geq 0\}$ is a family of operators on $L^2(X)$ satisfying the contraction property. If $\{T_t : t \geq 0\}$ is measurable on $L^p(X)$ then $\{T_t : t \geq 0\}$ is measurable on $L^q(X)$.*

Proof. Assume the hypotheses and suppose that $f \in L^q(X)$ and $g \in L^{q'}(X)$. It suffices to show that the map $\phi : [0,\infty) \to \mathbb{C}$, defined by

$$\phi(t) = \langle T_t f,\, g\rangle\,,$$

is measurable. We will construct a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of measurable functions converging pointwise to ϕ. By [21, Corollary III.6.14], this will complete the proof.

Since (X,μ) is a σ-finite measure space, there is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of measurable sets such that

$$\bigcup_{n\in\mathbb{N}} X_n = X$$

and $X_n \subset X_{n+1}$ whenever $n \in \mathbb{N}$. When $q \neq 1$, there is a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $L^{q'}(X) \cap L^{p'}(X)$ such that

$$\|g - g_n\|_{q'} \|f\|_q \leq \frac{1}{n}$$

for all n in $\mathbb{N}$. When $q = 1$, set g_n equal to g whenever $n \in \mathbb{N}$. Now find a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^p(X) \cap L^q(X)$ such that

$$\|f - f_n\|_q \|g_n\|_{q'} \leq \frac{1}{n}$$

for all n in $\mathbb{N}$. For n in $\mathbb{N}$, define $\phi_n : [0,\infty) \to \mathbb{C}$ by $\phi_n(t) = \langle T_t f_n,\, 1_{X_n} g_n\rangle$, where 1_{X_n} denotes the characteristic function of the set X_n. Now $f \in L^p(X)$, the semigroup $\{T_t : t \geq 0\}$ is measurable on $L^p(X)$ and $1_{X_n} g_n \in L^{p'}(X)$, so ϕ_n is measurable. Given positive ϵ and nonnegative t, choose $N > 3/\epsilon$ such that

$$\left\|1_{X\setminus X_n} T_t f\right\|_q \|g\|_{q'} < \frac{\epsilon}{3}$$

whenever $n > N$. Then

$$\begin{aligned}|\phi(t) - \phi_n(t)| &\leq |\langle T_t f,\, g - 1_{X_n} g, \rangle| + |\langle T_t f,\, 1_{X_n}(g - g_n)\rangle| \\ &\qquad + |\langle T_t f - T_t f_n,\, 1_{X_n} g_n\rangle| \\ &\leq \left\|1_{X\setminus X_n} T_t f\right\|_q \|g\|_{q'} + \|f\|_q \|g - g_n\|_{q'} + \|f - f_n\|_q \|g_n\|_{q'} \\ &< \epsilon\end{aligned}$$

whenever $n > N$. This completes the proof. $\square$

Lemma 2.2.7. *Suppose that T is a linear contraction on $L^1(X)$ and that*

$$\|Tf\|_q \leq \|f\|_q$$

whenever $f \in L^q(X) \cap L^1(X)$ and $1 \leq q \leq \infty$. Then there is a unique bounded linear positive operator $\mathbf{T}$ on $L^1(X)$ such that

(i) the operator norms of T and $\mathbf{T}$ on $L^1(X)$ are equal,

(ii) $\|\mathbf{T}f\|_q \leq \|f\|_q$ whenever $f \in L^q(X) \cap L^1(X)$ and $1 \leq q \leq \infty$,

(iii) $|Tf| \leq \mathbf{T}|f|$ whenever $f \in L^p(X)$ and $1 \leq p \leq \infty$, and

(iv) $\mathbf{T}f = \sup\{|Tg| : g \in L^1(X), |g| \leq f\}$ whenever $f \in L^1_+(X)$.

Proof. For the existence of a unique operator $\mathbf{T}$ satisfying properties (i), (iii) (in the case when $p = 1$) and (iv), see, for example, [45, Theorem 4.1.1]. Property (ii) holds by [21, Lemma VIII.6.4]. We can now deduce property (iii), in the case when $1 < p < \infty$, from Lemma 2.2.5. $\square$

The operator $\mathbf{T}$ introduced in the lemma is called the *linear modulus* of T. If $\{T_t : t \geq 0\}$ is a bounded semigroup on $L^1(X)$ then $\mathbf{T}_{s+t} \leq \mathbf{T}_s\mathbf{T}_t$ for all nonnegative s and t. However, equality may not hold so the family $\{\mathbf{T}_t : t \geq 0\}$ of bounded positive operators will not, in general, be a semigroup. However Kubokawa [46] and Kipnis [43] (see [45, Theorems 4.1.1 and 7.2.7] for a more recent exposition) showed that the linear modulus $\mathbf{T}_t$ could be used to construct a positive semigroup $\{S_t : t \geq 0\}$, known as the *modulus semigroup*, which dominates $\{T_t : t \geq 0\}$. The following proof uses this construction.

Proof of Theorem 2.2.1. Assume the hypothesis of Theorem 2.2.1 and suppose that $t > 0$. Let $\mathcal{D}_t$ denote the family of all finite subdivisions (s_i) of $[0, t]$ satisfying

$$0 = s_0 < s_1 < s_2 < \ldots < s_n = t.$$

If $\mathbf{s} = (s_i)$ and $\mathbf{s}' = (s'_j)$ are two elements of $\mathcal{D}_t$ then we write $\mathbf{s} < \mathbf{s}'$ whenever $\mathbf{s}'$ is a refinement of $\mathbf{s}$. With this partial order, $\mathcal{D}_t$ is an increasingly filtered set. For f in $L^1_+(X)$ put

$$\Phi(\mathbf{s}, f) = \mathbf{T}_{s_1}\mathbf{T}_{s_2-s_1}\ldots\mathbf{T}_{s_n-s_{n-1}}f,$$

where $\mathbf{T}_\alpha$ is the linear modulus of T_α whenever $\alpha \geq 0$. It follows from $\mathbf{T}_{\alpha+\beta} \leq \mathbf{T}_\alpha\mathbf{T}_\beta$ that $\Phi(\mathbf{s}, f) \leq \Phi(\mathbf{s}', f)$ when $\mathbf{s} < \mathbf{s}'$. Since the operator $\mathbf{T}_\alpha$ is contraction whenever $\alpha \geq 0$, we have $\|\Phi(\mathbf{s}, f)\|_1 \leq \|f\|_1$. We now define S_t on $L^1_+(X)$ by

$$S_t f = \sup\{\Phi(\mathbf{s}, f) : \mathbf{s} \in \mathcal{D}_t\}.$$

Note that

$$\sup\{\Phi(\mathbf{s}, f) : \mathbf{s} \in \mathcal{D}_t\} = \lim_{\mathbf{s}\in\mathcal{D}_t} \Phi(\mathbf{s}, f),$$

so S_t is well-defined by the monotone convergence theorem for increasing filtered families.

It is easy to check that $S_t(f + g) = S_t f + S_t g$ and $S_t(\lambda f) = \lambda S_t f$ whenever f and g belong to $L^1_+(X)$ and $\lambda \geq 0$. Moreover, $\|S_t f\|_1 \leq \|f\|_1$ if $f \in L^1_+(X)$. Therefore S_t can now be defined for all f in $L^1(X)$ as a linear contraction of $L^1(X)$. We define S_0 as the identity operator on $L^1(X)$.

We now show that $\{S_t : t \geq 0\}$ is a semigroup. Suppose that t and t' are both positive. If

$$0 = s_0 < s_1 < s_2 < \ldots < s_n = t$$

and

$$0 = s'_0 < s'_1 < s'_2 < \ldots < s'_n = t'$$

form subdivisions of $[0, t]$ and $[0, t']$ then

$$0 = s_0 < s_1 < s_2 < \ldots < s_n = s_n + s'_0 < s_n + s'_1 < s_n + s'_2 < \ldots < s_n + s'_n = t + t'$$

forms a subdivision of $[0, t+t']$. Conversely every subdivision of $[0, t+t']$ which is fine enough to contain t is of this form. This yields $S_{t+t'} = S_t S_{t'}$.

By Lemma 2.2.7 (ii), it is easy to check that $\{S_t : t \geq 0\}$ extends to a contraction semigroup on $L^2(X)$ which satisfies the contraction property. We now show that $|T_t f| \leq S_t |f|$ whenever $f \in L^p(X)$ and $t \geq 0$. For the case when $p = 1$, consider any finite subdivision (s_i) of $[0, t]$ satisfying

$$0 = s_0 < s_1 < s_2 < \ldots < s_n = t.$$

Then

$$\begin{aligned} |T_t f| &= |T_{s_1} T_{s_2 - s_1} \ldots T_{s_n - s_{n-1}} f| \\ &\leq \mathbf{T}_{s_1} \mathbf{T}_{s_2 - s_1} \ldots \mathbf{T}_{s_n - s_{n-1}} |f|. \end{aligned}$$

Hence $|T_t f| \leq S_t |f|$. For the case when $1 < p < \infty$, apply Lemma 2.2.5.

It remains to show that if $\{T_t : t \geq 0\}$ is measurable on $L^2(X)$ then $\{S_t : t \geq 0\}$ is measurable on $L^p(X)$ whenever $1 \leq p < \infty$. In view of Lemma 2.2.6, we may assume that $\{T_t : t \geq 0\}$ is measurable on $L^1(X)$ and it suffices to show that $\{S_t : t \geq 0\}$ is measurable on $L^1(X)$. Fix f in $L^1(X)$ and define $\phi : [0, \infty) \to L^1(X)$ by $\phi(t) = S_t f$. We will construct a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ of measurable functions converging pointwise to ϕ, completing the proof.

Since f can be decomposed as a linear combination of four nonnegative functions (the positive and negative parts of $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$) and each S_t is linear, we may assume, without loss of generality, that $f \geq 0$.

When $t > 0$ and $n \in \mathbb{N}$, let m be the smallest integer such that $m \geq 2^n t$ and $\mathbf{s}(n, t)$ denote the subdivision $(s_k(n, t))_{k=0}^{m}$ of $[0, t]$ given by

$$s_k(n, t) = \begin{cases} k 2^{-n} & \text{if } k = 0, 1, \ldots, m-1 \\ t & \text{if } k = m. \end{cases}$$

Now define $\phi_n : [0, \infty) \to L^1(X)$ by

$$\phi_n(t) = \Phi(\mathbf{s}(n, t), f)$$

when $t > 0$ and $\phi_n(0) = f$. By the definition of S_t, $|\phi(t) - \phi_n(t)| \to 0$ as $n \to \infty$ for each $t \geq 0$.

Our task is to demonstrate that ϕ_n is measurable for each n in $\mathbb{N}$. Note that $\phi_n(t)$, when t is restricted to the interval $[k2^{-n}, (k+1)2^{-n})$, is of the form

$$B_{k,n}\mathbf{T}_{t-k2^{-n}}f$$

where $B_{k,n}$ is a contraction on $L^1(X)$. It follows that if E is an open set of $L^1(X)$ then

$$\phi_n^{-1}(E) = \bigcup_{k=1}^{\infty} \{t \in [k2^{-n}, (k+1)2^{-n}) : B_{k,n}\mathbf{T}_{t-k2^{-n}}f \in E\}.$$

Hence if the map $\varphi : [0, 2^{-n}) \to L^1(X)$, defined by

$$\varphi(t) = \mathbf{T}_t f,$$

is measurable then $\phi_n^{-1}(E)$ can be written as a countable union of measurable sets and consequently ϕ_n is measurable. But by Lemma 2.2.7 (iv) there is a sequence $\{f_j\}_{j\in\mathbb{N}}$ in $L^1_+(X)$ such that $|\mathbf{T}_t f - T_t f_j| \to 0$ as $j \to \infty$. In other words, φ is the pointwise limit of a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ of measurable functions, defined by $\varphi_j(t) = T_t f_j$, and hence φ is measurable. □

The following lemma, which is needed later, fits into the theme of this section.

Lemma 2.2.8. *Suppose that $\{T_t : t \geq 0\}$ is a semigroup on $L^2(X)$ with the contraction property. If $\{T_t : t \geq 0\}$ is strongly continuous on $L^p(X)$ for some p in $[1, \infty)$ then $\{T_t : t \geq 0\}$ is strongly continuous on $L^q(X)$ for all q in $(1, \infty)$.*

Proof. Suppose that $1 \leq p < \infty$, $1 < q < \infty$ and $\{T_t : t \geq 0\}$ is strongly continuous on $L^p(X)$. It suffices to show that $\{T_t : t \geq 0\}$ is weakly continuous on $L^q(X)$ (see, for example, [18, Chapter 1, Section 2]).

Suppose that $f \in L^q(X)$ and $g \in L^{q'}(X)$. We aim to show that

$$\lim_{s\to t} |\langle T_s f - T_t f,\, g\rangle| = 0$$

whenever $t \geq 0$. The proof is trivial if $f = 0$, so suppose otherwise. Fix positive ϵ. Choose positive nonzero g_1 in $L^{q'}(X) \cap L^{p'}(X)$ such that

$$\|g - g_1\|_{q'} < \frac{\epsilon}{6\,\|f\|_q}$$

and choose f_1 in $L^q(X) \cap L^p(X)$ such that

$$\|f - f_1\|_q < \frac{\epsilon}{6\,\|g_1\|_{q'}}.$$

Since $\{T_t : t \geq 0\}$ is weakly continuous on $L^p(X)$, there exists a positive δ such that

$$|\,\langle T_s f_1 - T_t f_1,\, g_1\rangle\,| < \epsilon/3$$

whenever $|s - t| < \delta$. Hence

$$\begin{aligned}
|\,\langle T_s f - T_t f,\, g\rangle\,| &\leq |\,\langle T_s f,\, g - g_1\rangle\,| + |\,\langle T_t f,\, g - g_1\rangle\,| + |\,\langle T_s f - T_t f,\, g_1\rangle\,| \\
&\leq 2\,\|f\|_q\,\|g - g_1\|_{q'} + |\,\langle T_s f - T_s f_1,\, g_1\rangle\,| \\
&\qquad + |\,\langle T_s f_1 - T_t f_1,\, g_1\rangle\,| + |\,\langle T_t f_1 - T_t f,\, g_1\rangle\,| \\
&< \frac{\epsilon}{3} + 2\,\|f_1 - f\|_q\,\|g_1\|_{q'} + \frac{\epsilon}{3} \\
&< \epsilon
\end{aligned}$$

whenever $|s - t| < \delta$. $\square$

Corollary 2.2.9. *If $\{T_t : t \geq 0\}$ is a semigroup on $L^2(X)$ satisfying the contraction property whose extension to $L^1(X)$ is strongly continuous, then the positive semigroup $\{S_t : t \geq 0\}$ on $L^2(X)$ of Theorem 2.2.1 extends to a strongly continuous semigroup on $L^p(X)$ for all p in $[1, \infty)$.*

Proof. If $\{T_t : t \geq 0\}$ is strongly continuous on $L^1(X)$ then $\{S_t : t \geq 0\}$ is strongly continuous on $L^1(X)$ (see, for example, the proof of this fact in [45, pp. 246–247]). Now apply Lemma 2.2.8. $\square$

2.3 A vector-valued ergodic theorem

We now obtain a vector-valued version of the Hopf–Dunford–Schwartz ergodic theorem for use in Section 2.4. If $\mathcal{T}$ is a bounded strongly measurable semigroup $\{T_s : s \geq 0\}$ on $L^p(X)$ then define the operator $A(\mathcal{T}, t)$, for positive t, by the formula

$$A(\mathcal{T}, t)f = \frac{1}{t} \int_0^t T_s f \, \mathrm{d}s \qquad \forall f \in L^p(X).$$

For f in $L^p(X)$, we may then define a maximal ergodic function $\mathcal{A}^{\mathcal{T}} f$ by

$$\mathcal{A}^{\mathcal{T}} f = \sup_{t>0} |A(\mathcal{T}, t)f|. \tag{2.2}$$

A simplified version of the classical Hopf–Dunford–Schwartz ergodic theorem may be stated as follows.

Theorem 2.3.1. *[21, Theorem VIII.7.7] Suppose that $\mathcal{T}$ is a measurable semigroup on $L^2(X)$ satisfying the contraction property and assume that $p \in (1, \infty)$. Then the maximal ergodic function operator $\mathcal{A}^{\mathcal{T}}$ satisfies the inequality*

$$\left\| \mathcal{A}^{\mathcal{T}} f \right\|_p \leq 2 \left(\frac{p}{p-1} \right)^{1/p} \|f\|_p \qquad \forall f \in L^p(X).$$

We will now develop a vector-valued version of this theorem. Fix p in the interval $(1, \infty)$. Suppose that $\mathcal{T}$ is a strongly continuous semigroup $\{T_t : t \geq 0\}$ on $L^2(X)$ satisfying the contraction property. By Lemma 2.2.8, the semigroup $\mathcal{T}$ is a strongly continuous semigroup of contractions when viewed as acting on $L^p(X)$. We first show that the bounded linear operator $A(\mathcal{T}, t)$ on $L^p(X)$ has an extension to $L^p(X; \mathcal{B})$ for all positive t. By Theorem 2.2.1 there is a measurable semigroup $\{S_t : t \geq 0\}$ of positive contractions on $L^p(X)$, which we denote by $\mathcal{S}$, dominating $\mathcal{T}$ on $L^p(X)$. Hence $A(\mathcal{S}, t)$ is also a positive contraction on $L^p(X)$ for each positive t. Moreover,

$$|A(\mathcal{T}, t)f| \leq \frac{1}{t} \int_0^t |T_s f| \, \mathrm{d}s \leq \frac{1}{t} \int_0^t S_s |f| \, \mathrm{d}s = A(\mathcal{S}, t)|f|$$

whenever $f \in L^p(X)$. It follows that $A(\mathcal{T}, t)$ has a tensor product extension to $L^p(X; \mathcal{B})$ for all positive t by Lemma 2.1.3. We can now define a maximal ergodic function operator $\mathcal{A}^{\mathcal{T}}_{\mathcal{B}}$ by the formula

$$\mathcal{A}^{\mathcal{T}}_{\mathcal{B}} F = \sup_{t>0} |\widetilde{A}(\mathcal{T}, t)F|_{\mathcal{B}} \qquad \forall F \in L^p(X; \mathcal{B}). \tag{2.3}$$

Moreover, if $F \in L^p(X;\mathcal{B})$ then $\mathcal{A}^{\mathcal{T}}_{\mathcal{B}}F$ is measurable. To see this, observe that since the mapping $t \mapsto A(\mathcal{T},t)f$ is continuous from $(0,\infty)$ to $L^p(X)$ and the operator norm of $\widetilde{A}(\mathcal{T},t)$ is locally uniformly bounded in t, the vector-valued mapping $t \mapsto \widetilde{A}(\mathcal{T},t)F$ is continuous from $(0,\infty)$ to $L^p(X;\mathcal{B})$ by Lemma 2.1.4. This implies that $t \mapsto |\widetilde{A}(\mathcal{T},t)F|_{\mathcal{B}}$ is continuous from $(0,\infty)$ to $L^p(X)$. Therefore the measurable function $\sup_{t\in\mathbb{Q}^+}|\widetilde{A}(\mathcal{T},t)F|_{\mathcal{B}}$, where $\mathbb{Q}^+$ denotes the set of positive rationals, coincides with $\sup_{t>0}|\widetilde{A}(\mathcal{T},t)F|_{\mathcal{B}}$.

Corollary 2.3.2. *Suppose that $\mathcal{B}$ is a Banach space and that $\mathcal{T}$ is a symmetric diffusion semigroup on $L^2(X)$. If $1 < p < \infty$ then the maximal ergodic function operator $\mathcal{A}^{\mathcal{T}}_{\mathcal{B}}$, defined by (2.3), satisfies the inequality*

$$\left\|\mathcal{A}^{\mathcal{T}}_{\mathcal{B}}F\right\|_{L^p(X)} \leq 2\left(\frac{p}{p-1}\right)^{1/p}\|F\|_{L^p(X;\mathcal{B})} \qquad \forall f \in L^p(X;\mathcal{B}).$$

Proof. Fix p in $(1,\infty)$ and let $\mathcal{S}$ denote the semigroup dominating $\mathcal{T}$ which was introduced in the discussion preceding the statement of the corollary. Now for F in $L^p(X;\mathcal{B})$,

$$\begin{aligned}\mathcal{A}^{\mathcal{T}}_{\mathcal{B}}F &= \sup_{t>0}|\widetilde{A}(\mathcal{T},t)F|_{\mathcal{B}}\\ &\leq \sup_{t>0}A(\mathcal{S},t)|F|_{\mathcal{B}}\\ &= \mathcal{A}^{\mathcal{S}}|F|_{\mathcal{B}}.\end{aligned}$$

The result follows upon taking the $L^p(X)$ norm of both sides and applying Theorem 2.3.1. ☐

2.4 A vector-valued maximal theorem

The main result of this section is a vector-valued version of Theorem 1.3.3. It gives an L^p estimate for the maximum function $\mathcal{M}^{\psi}_{\mathcal{B}}F$ (defined by (1.18)) under the assumption that the generator $-\widetilde{L}$ of $\{\widetilde{T}_t : t \geq 0\}$ has bounded imaginary powers.

Theorem 2.4.1. *Suppose that (X, μ) is a σ-finite measure space, $\mathcal{B}$ is a Banach space, $1 < p < \infty$ and $\{T_t : t \geq 0\}$ is a strongly continuous semigroup on $L^2(X)$ with the contraction property. If there exists ω less than $\pi/2 - \psi$ and a positive constant K such that $\widetilde{L}$ has bounded imaginary powers satisfying the norm estimate*

$$\|\widetilde{L}^{iu} F\|_{L^p(X;\mathcal{B})} \leq K e^{\omega|u|} \|F\|_{L^p(X;\mathcal{B})} \qquad \forall F \in L^p(X;\mathcal{B}) \quad \forall u \in \mathbb{R}, \tag{2.4}$$

then $\{\widetilde{T}_t : t \geq 0\}$ has a bounded analytic continuation in $L^p(X;\mathcal{B})$ to the sector Γ_ψ and there is a constant C such that the maximal function operator $\mathcal{M}_{\mathcal{B}}^{\psi}$ satisfies the inequality

$$\left\|\mathcal{M}_{\mathcal{B}}^{\psi} F\right\|_{L^p(X)} \leq C \|F\|_{L^p(X;\mathcal{B})} \qquad \forall F \in L^p(X;\mathcal{B}). \tag{2.5}$$

Proof. Assume the hypotheses of the theorem. Since $\widetilde{L}$ has bounded imaginary powers satisfying (2.4), $-\widetilde{L}$ generates a uniformly bounded semigroup on $L^p(X;\mathcal{B})$ with analytic continuation to any sector Γ_{ψ_0}, where

$$\psi_0 < \frac{\pi}{2} - \omega,$$

by a result of J. Prüss and H. Sohr [57, Theorem 2]. Hence the operator $\mathcal{M}_{\mathcal{B}}^{\psi}$ is well-defined. It remains to show (2.5).

Take F in $L^p(X;\mathcal{B})$ and z in $\overline{\Gamma}_\psi \backslash \{0\}$. Write z as $e^{i\theta} t$, where $|\theta| \leq \psi$ and $t > 0$. The key idea of the proof is to decompose $\widetilde{T}_z F$ into two parts:

$$\widetilde{T}_z F = \frac{1}{t} \int_0^t e^{-s\widetilde{L}} F \,\mathrm{d}s + \Big[e^{-z\widetilde{L}} F - \frac{1}{t} \int_0^t e^{-s\widetilde{L}} F \,\mathrm{d}s \Big]. \tag{2.6}$$

Define the function m_θ on $(0, \infty)$ by

$$m_\theta(\lambda) = \exp(-e^{i\theta}\lambda) - \int_0^1 e^{-s\lambda} \,\mathrm{d}s \qquad \forall \lambda > 0. \tag{2.7}$$

Then (2.6) can be rewritten formally as

$$\widetilde{T}_z F = \frac{1}{t} \int_0^t e^{-s\widetilde{L}} F \,\mathrm{d}s + m_\theta(t\widetilde{L}) F,$$

whence

$$\sup_{z \in \overline{\Gamma}_\psi \backslash \{0\}} |\widetilde{T}_z F|_{\mathcal{B}} \leq \sup_{t>0} \left| \frac{1}{t} \int_0^t e^{-s\widetilde{L}} F \,\mathrm{d}s \right|_{\mathcal{B}} + \sup_{t>0} \sup_{|\theta| \leq \psi} |m_\theta(t\widetilde{L}) F|_{\mathcal{B}}.$$

If we take the $L^p(X)$ norm of both sides then we have, formally at least,

$$\left\|\mathcal{M}_{\mathcal{B}}^{\psi} F\right\|_p \leq \left\|\mathcal{A}_{\mathcal{B}}^{\mathcal{T}} F\right\|_p + \left\|\sup_{t>0} \sup_{|\theta|\leq\psi} |m_\theta(t\widetilde{L})F|_{\mathcal{B}}\right\|_p, \tag{2.8}$$

where $\mathcal{T}$ denotes the semigroup $\{T_t : t \geq 0\}$ and $\mathcal{A}_{\mathcal{B}}^{\mathcal{T}}$ is the operator defined by (2.3). By Corollary 2.3.2, the first term on the right-hand side is majorised by $2[p/(1-p)]^{1/p} \|F\|_{L^p(X;\mathcal{B})}$. We need to control the second term.

Write n_θ for $m_\theta \circ \exp$ and observe that

$$m_\theta(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{n}_\theta(u) \lambda^{iu} \, \mathrm{d}u, \tag{2.9}$$

where $\hat{n}_\theta$ denotes the Fourier transform of n_θ. Calculation using complex analysis shows that

$$\hat{n}_\theta(u) = \big(e^{-\theta u} - (1+iu)^{-1}\big)\Gamma(iu) \qquad \forall u \in \mathbb{R},$$

and the theory of the Γ-function (see, for example, [70, p. 151]) gives the estimate

$$|\hat{n}_\theta(u)| \leq C_0 \exp\big((|\theta| - \pi/2)|u|\big) \qquad \forall u \in \mathbb{R},$$

where C_0 is a constant independent of u and θ. Thus, the existence of bounded imaginary powers of $\widetilde{L}$ gives

$$\begin{aligned} \sup_{t>0} \sup_{|\theta|\leq\psi} |m_\theta(t\widetilde{L})F|_{\mathcal{B}} &\leq \sup_{t>0} \sup_{|\theta|\leq\psi} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{n}_\theta(u)| \, |(t\widetilde{L})^{iu} F|_{\mathcal{B}} \, \mathrm{d}u \\ &\leq \sup_{t>0} \sup_{|\theta|\leq\psi} \frac{1}{2\pi} \int_{-\infty}^{\infty} C_0 e^{(|\theta|-\pi/2)|u|} |t^{iu}| \, |\widetilde{L}^{iu} F|_{\mathcal{B}} \, \mathrm{d}u \\ &\leq \frac{C_0}{2\pi} \int_{-\infty}^{\infty} e^{(\psi-\pi/2)|u|} |\widetilde{L}^{iu} F|_{\mathcal{B}} \, \mathrm{d}u. \end{aligned}$$

Taking the $L^p(X)$ norm of both sides of the above inequality and applying (2.4) gives

$$\begin{aligned} \left\|\sup_{t>0} \sup_{|\theta|\leq\psi} |m_\theta(t\widetilde{L})F|_{\mathcal{B}}\right\|_p &\leq \frac{C_0}{2\pi} \int_{-\infty}^{\infty} e^{(\psi-\pi/2)|u|} \left\|\widetilde{L}^{iu} F\right\|_p \mathrm{d}u \\ &\leq \frac{C_0 K}{2\pi} \int_{-\infty}^{\infty} e^{(\psi-\pi/2)|u|} e^{\omega|u|} \|F\|_p \, \mathrm{d}u \\ &< C_1 \|F\|_{L^p(X;\mathcal{B})}, \end{aligned}$$

since $\psi - \pi/2 + \omega < 0$, and where C_1 is a positive constant independent of F. Now (2.8) and Corollary 2.3.2 yields (2.5) for some positive constant C.

The opening formal calculations can be justified by working backwards, provided that the function

$$\sup_{t>0} \sup_{|\theta|\leq\psi} |m_\theta(t\widetilde{L})F|_{\mathcal{B}} \tag{2.10}$$

is measurable. Since the map $z \mapsto \widetilde{T}_z F$ is continuous from $\overline{\Gamma}_\psi$ to $L^p(X;\mathcal{B})$, the map $(t,\theta) \mapsto |m_\theta(t\widetilde{L})F|_{\mathcal{B}}$ is continuous from $(0,\infty)\times[-\psi,\psi]$ to $L^p(X)$. Hence

$$\sup_{t>0} \sup_{|\theta|\leq\psi} |m_\theta(t\widetilde{L})F|_{\mathcal{B}} = \sup_{(t,\theta)\in R} |m_\theta(t\widetilde{L})F|_{\mathcal{B}},$$

where R is the denumerable set $\big((0,\infty)\times[-\psi,\psi]\big)\cap\mathcal{Q}^2$. Since each $m_\theta(t\widetilde{L})F$ is measurable in $L^p(X;\mathcal{B})$ it follows that (2.10) is measurable in $L^p(X)$. $\square$

2.5 Bounded imaginary powers of the generator

In this section we examine circumstances under which the bounded imaginary power estimate (2.4), one of the hypotheses of the preceding theorem and corollary, is satisfied. A fruitful (and in our context, necessary) setting is when the Banach space $\mathcal{B}$ has the *UMD property*. A Banach space $\mathcal{B}$ is said to be a *UMD space* if one of the following equivalent statements hold:

(a) The Hilbert transform is bounded on $L^p(X;\mathcal{B})$ for one (and hence all) p in $(1,\infty)$.

(b) If $1 < p < \infty$ then $\mathcal{B}$-valued martingale difference sequences on $L^p(X;\mathcal{B})$ converge unconditionally.

(c) If $1 < p < \infty$ then $(-\Delta)^{iu}\otimes I_{\mathcal{B}}$ extends to a bounded operator on $L^p(\mathbb{R},\mathcal{B})$ for every u in $\mathbb{R}$ (a result due to S. Guerre-Delabrière [30]).

Several other characterisations of UMD spaces exist (see, for example, [6] and the survey in [60]) but those cited here are, for different reasons, the most relevant to our discussion. If the Hilbert transform, which corresponds to the multiplier function $u \mapsto i\,\mathrm{sgn}(u)$, is bounded on $L^p(X;\mathcal{B})$ then one can establish vector-valued versions of some Fourier multiplier theorems (such as Mikhlin's multiplier theorem [81]). This fact is used below to establish Theorem 2.5.1. The second characterisation gave rise to the name UMD. The third characterisation shows that, in general, $\mathcal{B}$ must be a UMD space if $\widetilde{L}$ is to have bounded imaginary powers, since $-\Delta$ generates the Gaussian semigroup. Examples of UMD spaces include, when $1 < p < \infty$, the classical $L^p(X)$ spaces and the Schatten–von Neumann ideals $\mathcal{C}^p$. Moreover, if $\mathcal{B}$ is a UMD space then its dual $\mathcal{B}^*$, closed subspaces of $\mathcal{B}$, quotient spaces of $\mathcal{B}$ and $L^p(X;\mathcal{B})$ when $1 < p < \infty$ also inherit the UMD property.

It was shown by Hieber and Prüss [32] that when $1 < q < \infty$ the generator of a UMD-valued extension of a bounded strongly continuous positive semigroup on $L^q(X)$ has a bounded H^∞-functional calculus. The next result says that the same is true if the positivity condition is relaxed to subpositivity (assuming that the UMD-valued extension of the semigroup is bounded), though it is convenient in the present context to state it for semigroups possessing the contraction property. First we introduce some notation. If $\sigma \in (0, \pi]$ then let $H^\infty(\Gamma_\sigma)$ denote the Banach space of all bounded analytic functions defined on Γ_σ with norm

$$\|f\|_{H^\infty(\Gamma_\sigma)} = \sup_{z\in\Gamma_\sigma} |f(z)|.$$

Theorem 2.5.1. *Suppose that $1 < q < \infty$ and $\mathcal{B}$ is a UMD space. If $\{T_t : t \geq 0\}$ is a strongly continuous semigroup on $L^2(X)$ satisfying the contraction property and $-\widetilde{L}$ is the generator of its tensor extension $\{\widetilde{T}_t : t \geq 0\}$ to $L^q(X;\mathcal{B})$, then $\widetilde{L}$ has a bounded $H^\infty(\Gamma_\sigma)$-calculus for all σ in $(\pi/2, \pi]$. Consequently, for every $\sigma \in (\pi/2, \pi]$ there exists a positive constant $C_{q,\sigma}$ such that*

$$\|\widetilde{L}^{iu} F\|_{L^q(X;\mathcal{B})} \leq C_{q,\sigma} e^{\sigma|u|}\, \|F\|_{L^q(X;\mathcal{B})} \qquad \forall F \in L^q(X;\mathcal{B}) \quad \forall u \in \mathbb{R}. \qquad (2.11)$$

Proof. Since the semigroup $\{T_t : t \geq 0\}$ can be extended to a subpositive strongly continuous semigroup of contractions on $L^q(X)$, it has a dilation to a bounded c_0-group on $L^q(X')$ for some measure space (X', μ'). In other words, there exists a measure space (X', μ'), a strongly continuous group $\{U_t : t \in \mathbb{R}\}$ of subpositive contractions on $L^q(X')$, a positive isometric embedding $D : L^q(X) \to L^q(X')$ and a subpositive contractive projection $P : L^q(X') \to L^q(X')$ such that

$$DT_t = PU_tD \qquad \forall t \geq 0$$

(see the result of G. Fendler [22, pp. 737–738] which extends the work of Coifman, Rochberg, and Weiss [11]). Lifting this identity to its $\mathcal{B}$-valued extension, we see that the semigroup $\{\widetilde{T}_t : t \geq 0\}$ on $L^q(X; \mathcal{B})$ has a dilation to a bounded c_0-group $\{\widetilde{U}_t : t \in \mathbb{R}\}$ on $L^q(X', \mathcal{B})$.

Let $-\widetilde{L}$ denote the generator of $\{\widetilde{T}_t : t \geq 0\}$. Then the dilation implies that $\widetilde{L}$ has a bounded $H^\infty(\Gamma_\sigma)$-calculus for all σ in $(\pi/2, \pi]$ (see [32] or the exposition in [47, pp. 212–214], where the H^∞-calculus is first constructed for the generator of the group $\{\widetilde{U}_t : t \in \mathbb{R}\}$ using the vector-valued Mikhlin multiplier theorem in conjunction with the transference principle, and then projected back to the generator $-\widetilde{L}$ of $\{\widetilde{T}_t : t \geq 0\}$ via the dilation).

The bounded $H^\infty(\Gamma_\sigma)$-calculus gives a positive constant $C_{q,\sigma}$ such that

$$\|f(\widetilde{L})\|_{L^q(X;\mathcal{B})} \leq C_{q,\sigma} \|f\|_{H^\infty(\Gamma_\sigma)} \qquad \forall f \in H^\infty(\Gamma_\sigma).$$

If $f(z) = z^{iu}$ for u in $\mathbb{R}$ then (2.11) follows. $\square$

The theorem above suggests that the problem of finding bounded imaginary powers of $\widetilde{L}$ is critical to $L^2(X; \mathcal{B})$. That is, if

$$\|\widetilde{L}^{iu} F\|_{L^2(X;\mathcal{B})} \leq Ce^{\omega|u|} \|F\|_{L^2(X;\mathcal{B})} \qquad \forall F \in L^2(X; \mathcal{B})\, \forall u \in \mathbb{R}$$

for some ω less than $\pi/2 - \psi$ then one could interpolate between the L^2 estimate and (2.11) to obtain (2.4). Unfortunately, suitable $L^2(X; \mathcal{B})$ bounded imaginary power estimates, where $\mathcal{B}$ is a nontrivial UMD space, appear to be absent in the literature, even when $\widetilde{L}$ is the Laplacian. However, if $\mathcal{B}$ is a Hilbert space, such estimates are available via spectral theory.

Lemma 2.5.2. *Suppose that $\mathcal{H}$ is a Hilbert space. If $\{T_t : t \geq 0\}$ is a symmetric diffusion semigroup on $L^2(X)$ then the generator $-\widetilde{L}$ of the $\mathcal{H}$-valued extension $\{\widetilde{T}_t : t \geq 0\}$ to $L^2(X, \mathcal{H})$ satisfies*

$$\|\widetilde{L}^{iu} F\|_{L^2(X,\mathcal{H})} \leq \|F\|_{L^2(X,\mathcal{H})} \qquad \forall F \in L^2(X, \mathcal{H}) \quad \forall u \in \mathbb{R}. \tag{2.12}$$

Proof. It is not hard to check that the tensor product extension to $L^2(X, \mathcal{H})$ of the semigroup $\{T_t : t \geq 0\}$ is a semigroup of selfadjoint contractions on $L^2(X, \mathcal{H})$. Its generator $-\widetilde{L}$ is therefore selfadjoint on $L^2(X, \mathcal{H})$ and hence $\widetilde{L}$ has nonnegative spectrum. Spectral theory now gives estimate (2.12). $\square$

To obtain (2.4) we shall interpolate between (2.11) and (2.12). Hence we consider the class of UMD spaces whose members $\mathcal{B}$ are isomorphic to closed subquotients of a complex interpolation space $(\mathcal{H}, \mathcal{U})_{[\theta]}$, where $\mathcal{H}$ is a Hilbert space, $\mathcal{U}$ is a UMD space and $0 < \theta < 1$. Members of this class include the UMD function lattices on a σ-finite measure space (such as the reflexive $L^p(X)$ spaces) by a result of Rubio de Francia (see [60, Corollary, p. 216]), the reflexive Sobolev spaces (which are subspaces of products of L^p spaces) and the reflexive Schatten–von Neumann ideals. This class can be further extended to include many operator ideals by combining Rubio de Francia's theorem with results due to P. Dodds, T. Dodds and B. de Pagter [19] which show that the interpolation properties of noncommutative spaces coincide with those of their commutative counterparts under fairly general conditions. It was asked in [60] whether the described class of UMD spaces includes all UMD spaces. It appears that this is still an open question.

Corollary 2.5.3. *Suppose that $\mathcal{B}$ is a UMD space isomorphic to a closed subquotient of a complex interpolation space $(\mathcal{H}, \mathcal{U})_{[\theta]}$, where $\mathcal{H}$ is a Hilbert space, $\mathcal{U}$ is a UMD space and $0 < \theta < 1$. Suppose also that $\{T_t : t \geq 0\}$ is a symmetric diffusion semigroup on $L^2(X)$ and denote by $-\widetilde{L}$ the generator of its tensor* *extension to $L^p(X;\mathcal{B})$, where $1 < p < \infty$. If*

$$|2/p - 1| < \theta \tag{2.13}$$

and

$$0 \leq \psi < \frac{\pi}{2}(1-\theta)$$

then there exists ω less than $\pi/2-\psi$ such that $\widetilde{L}$ has bounded imaginary powers on $L^p(X;\mathcal{B})$ satisfying estimate (2.4).

Proof. Assume the hypotheses of the corollary. Note that

$$\frac{\pi}{2} < \frac{1}{\theta}\left(\frac{\pi}{2} - \psi\right)$$

so that if σ is the arithmetic mean of $\pi/2$ and $(\pi/2-\psi)/\theta$ then $\sigma > \pi/2$ and $\sigma\theta < \pi/2 - \psi$. Now choose q such that

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}.$$

Inequality (2.13) guarantees that $1 < q < \infty$. Interpolating between (2.12) and (2.11) (for the space $L^q(X,\mathcal{U})$) gives

$$\|\widetilde{L}^{iu}\|_{L^p(X;\mathcal{B})} \leq C^{\theta}_{q,\sigma} e^{\sigma\theta|u|} \|F\|_{L^p(X;\mathcal{B})} \qquad \forall F \in L^p(X;\mathcal{B}) \quad \forall u \in \mathbb{R}.$$

If $\omega = \sigma\theta$ then (2.4) follows, completing the proof. $\square$

2.6 Proof of Theorem 1.3.5

In this final section of Chapter 2, we complete the proof of Theorem 1.3.5. Suppose the hypotheses of the theorem. Parts (a) and (b) follow immediately from Theorem 2.4.1 and Corollary 2.5.3. Part (c) will be deduced from the vector-valued maximal theorem and the pointwise convergence of $\{T_t : t \geq 0\}$ (see Corollary 1.3.4).

For ease of notation, write $z \to 0$ as shorthand for $z \to 0$ with z in $\overline{\Gamma}_\psi$.

Suppose that $F \in L^p(X;\mathcal{B})$ and $\epsilon > 0$. There exists a function G in $L^p(X) \otimes \mathcal{B}$ such that $\|G - F\|_{L^p(X;\mathcal{B})} < \epsilon$. Write G as $\sum_{k=1}^n u_k f_k$, where n is a positive integer, $\{u_k\}_{k=1}^n$ is contained in $\mathcal{B}$ and $\{f_k\}_{k=1}^n$ is contained in $L^p(X)$. Hence, for almost every x in X,

$$\begin{aligned}\limsup_{z\to 0} |\widetilde{T}_z F(x) - F(x)|_{\mathcal{B}} &\le \limsup_{z\to 0} |\widetilde{T}_z F(x) - \widetilde{T}_z G(x)|_{\mathcal{B}} + |G(x) - F(x)|_{\mathcal{B}} \\ &\quad + \limsup_{z\to 0} |\widetilde{T}_z G(x) - G(x)|_{\mathcal{B}} \\ &\le \sup_{z\in\overline{\Gamma}_\psi} |\widetilde{T}_z (F-G)(x)|_{\mathcal{B}} + |G(x) - F(x)|_{\mathcal{B}} \\ &\quad + \sum_{k=1}^n \big|u_k\big|_{\mathcal{B}} \limsup_{z\to 0} \big|T_z f_k(x) - f_k(x)\big| \\ &\le 2\mathcal{M}_{\mathcal{B}}^{\psi}(G-F)(x),\end{aligned}$$

since Corollary 1.3.4 implies that

$$\lim_{z\to 0} |T_z f_k(x) - f_k(x)| = 0$$

for each k and for almost every x in X. By taking the $L^p(X)$ norm and applying Theorem 2.4.1 we obtain

$$\left\|\limsup_{z\to 0} |\widetilde{T}_z F - F|_{\mathcal{B}}\right\|_p \le 2\left\|\mathcal{M}_{\mathcal{B}}^{\psi}(G-F)\right\|_p < 2C\epsilon,$$

where the positive constant C is independent of F and G. Since ϵ is an arbitrary positive number,

$$\limsup_{z\to 0} |\widetilde{T}_z F(x) - F(x)|_{\mathcal{B}} = 0$$

for almost every x in X, proving the theorem.

Chapter 3

Miscellany

The results contained in this chapter serve as a reference point for tools used later chapters. It is thereby hoped that our treatment of Strichartz estimates is self-contained. If the reader is unfamiliar with basic L^p inequalities and Banach space interpolation, then they may want to peruse Sections 3.1, 3.2 and 3.3 before starting Chapter 4. Otherwise, readers may wish to skip this chapter, returning to it only when knowledge of its contents is required.

3.1 Inequalities in L^p spaces

The following L^p inequalities will be used in the proof of Strichartz estimates in Chapters 4 and 5. We only state the inequalities here, referring readers to [59] or other standard texts for proofs. For us, by far the most used of these inequalities is Hölder's inequality.

Theorem 3.1.1 (Hölder's inequality). *Suppose that X is a measure space and $p, q, r \in [1, \infty]$. If $1/r = 1/p + 1/q$, $f \in L^p(X)$ and $g \in L^q(X)$ then $fg \in L^r(X)$ and*

$$\|fg\|_r \le \|f\|_p \|g\|_q \,.$$

If (X, μ) is a measure space, then the convolution $f * g$ of two measurable complex-valued functions f and g is formally given by

$$f * g(x) = \int_X f(x-y)g(y)\,\mathrm{d}\mu(y) \qquad \forall x \in X.$$

The following theorem gives integrability conditions on f and g so that their convolution is well-defined.

Theorem 3.1.2 (Young's inequality). *Suppose that X is a measure space and $p,q,r \in [1,\infty]$. If $1+1/r = 1/p+1/q$, $f \in L^p(X)$ and $g \in L^q(X)$ then $f*g$ is defined and*

$$\|f*g\|_r \leq \|f\|_p \|g\|_q .$$

By specialising the above theorem to the ℓ^p sequence spaces, one may obtain the following result.

Lemma 3.1.3 (Young's inequality for convolution of sequences). *Suppose that a, b and c are sequences $\{a_n\}_{n\in\mathbb{Z}}$, $\{b_n\}_{n\in\mathbb{Z}}$ and $\{c_n\}_{n\in\mathbb{Z}}$ of nonnegative numbers. If $p,q,r \in [1,\infty]$ and*

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \geq 2$$

then

$$\sum_{m,n\in\mathbb{Z}} a_m b_n c_{m-n} \leq \|a\|_{\ell^p} \|b\|_{\ell^q} \|c\|_{\ell^r} .$$

The following theorem may be proved using an extended version of Young's inequality (see [58, pp. 31–32]).

Theorem 3.1.4 (The Hardy–Littlewood–Sobolev inequality). *Suppose that $0<\lambda<n$ and $q,r \in (1,\infty)$, where $1/q+1/r+\lambda/n = 2$. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(s)||g(t)|}{|t-s|^\lambda}\,\mathrm{d}^n s\,\mathrm{d}^n t \lesssim \|f\|_q \|g\|_r \qquad \forall f \in L^q(\mathbb{R}^n) \quad \forall g \in L^r(\mathbb{R}^n).$$

3.2 Interpolation spaces

Interpolation of Banach spaces is a subject of immense importance to mathematical analysis in general, and its importance to this thesis is no exception.

Of the many mathematicians who have made contributions to this theory, A. Calderón, J. Lions and J. Peetre deserve to be mentioned. In this section we give a brief outline of the philosophy behind Banach space interpolation and introduce the complex and real methods of interpolation. Our treatment is rudimentary so readers who wish to pursue the subject in greater depth are directed to [72] or [2].

Suppose that $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{C}}$ are Hausdorff topological vector spaces and that $S : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{C}}$ is a linear operator. In general S will not be bounded, but it may be possible to find Banach spaces $\mathcal{B}_0$ and $\mathcal{B}_1$ contained in $\widetilde{\mathcal{B}}$ and Banach spaces $\mathcal{C}_0$ and $\mathcal{C}_1$ contained in $\widetilde{\mathcal{C}}$ such that the restriction maps $S : \mathcal{B}_0 \to \mathcal{C}_0$ and $S : \mathcal{B}_1 \to \mathcal{C}_1$ are bounded. In this case, we ask if there are other Banach spaces $\mathcal{B} \subset \widetilde{\mathcal{B}}$ and $\mathcal{C} \subset \widetilde{\mathcal{C}}$ such that the restriction map $S : \mathcal{B} \to \mathcal{C}$ is also bounded. To develop these ideas further, we introduce the notions of Banach couples, intermediate spaces and interpolation spaces.

A *Banach couple* $(\mathcal{B}_0, \mathcal{B}_1)$ is a pair of Banach spaces $\mathcal{B}_0$ and $\mathcal{B}_1$ such that both $\mathcal{B}_0$ and $\mathcal{B}_1$ can be algebraically and topologically embedded in a Hausdorff topological vector space $\widetilde{\mathcal{B}}$.

If $\mathcal{B}_0 \cap \mathcal{B}_1 \subseteq \mathcal{B} \subseteq \mathcal{B}_0 + \mathcal{B}_1$ with continuous inclusions, then $\mathcal{B}$ is said to be an *intermediate space* for the Banach couple $(\mathcal{B}_0, \mathcal{B}_1)$.

Suppose that $\mathcal{B}$ and $\mathcal{C}$ are intermediate spaces for $(\mathcal{B}_0, \mathcal{B}_1)$ and $(\mathcal{C}_0, \mathcal{C}_1)$ respectively and that the restriction maps $S : \mathcal{B}_0 \to \mathcal{C}_0$ and $S : \mathcal{B}_1 \to \mathcal{C}_1$ are both bounded. If this implies that the restriction map $S : \mathcal{B} \to \mathcal{C}$ is bounded then we say that $\mathcal{B}$ and $\mathcal{C}$ are *interpolation spaces* with respect to $(\mathcal{B}_0, \mathcal{B}_1)$ and $(\mathcal{C}_0, \mathcal{C}_1)$.

We remark here that if the Banach spaces $\mathcal{B}_0$ and $\mathcal{B}_1$ are known but $\widetilde{\mathcal{B}}$ has not been specified, we can always take $\widetilde{\mathcal{B}}$ to be the Banach space $\mathcal{B}_0+\mathcal{B}_1$, where

$$\mathcal{B}_0 + \mathcal{B}_1 = \{a_0 + a_1 : a_0 \in \mathcal{B}_0, a_1 \in \mathcal{B}_1\}$$

with norm

$$\|a\|_{\mathcal{B}_0+\mathcal{B}_1} = \inf\{\|a_0\|_{\mathcal{B}_0} + \|a_1\|_{\mathcal{B}_1} : a = a_0 + a_1, a_0 \in \mathcal{B}_0, a_1 \in \mathcal{B}_1\}$$

(see [72, 1.2.1] for more details).

There are two main methods of Banach space interpolation: real and complex. The complex method usually finds intermediate spaces by considering vector-valued functions which are continuous and bounded on the strip

$$\Lambda = \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$$

and analytic on the open strip $\Lambda_0 = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$. Constructions are based on various complex analytic theorems, of which Hadamard's celebrated three-lines lemma is a prototype.

Given a Banach couple $(\mathcal{B}_0, \mathcal{B}_1)$, one way of constructing an interpolation space $\mathcal{B}$ using the complex interpolation method is as follows. Let $\mathcal{F}$ denote the space of all functions f with values in $\mathcal{B}_0 + \mathcal{B}_1$ which are bounded and continuous on Λ, analytic on Λ_0 and satisfy the following property: for each j in $\{0, 1\}$ the functions $t \mapsto f(j + it)$ are continuous functions from $\mathbb{R}$ into $\mathcal{B}_j$ and tend to zero as $|t| \to \infty$. We equip $\mathcal{F}$ with the norm

$$\|f\|_{\mathcal{F}} = \max\big(\sup_{t\in\mathbb{R}} \|f(it)\|_{\mathcal{B}_0}, \sup_{t\in\mathbb{R}} \|f(1 + it)\|_{\mathcal{B}_1}\big).$$

It turns out that $\mathcal{F}$ is a Banach space. We now define, for θ in $(0, 1)$, the interpolation space $\mathcal{B}$ to be the space of all $a \in \mathcal{B}_0 + \mathcal{B}_1$ such that $a = f(\theta)$ for some $f \in \mathcal{F}$. We denote this space $\mathcal{B}$ by $(\mathcal{B}_0, \mathcal{B}_1)_{[\theta]}$ and give it the norm

$$\|a\|_{[\theta]} = \inf\big\{ \|f\|_{\mathcal{F}} : f(\theta) = a,\ f \in \mathcal{F}\big\}.$$

Basic inclusion and density properties associated to the interpolation space $(\mathcal{B}_0, \mathcal{B}_1)_{[\theta]}$ are given by the following lemma. All inclusions below are continuous.

Theorem 3.2.1. *[2, Theorems 4.2.1 and 4.2.2][72, Theorem 1.9.3] Suppose that* $(\mathcal{B}_0, \mathcal{B}_1)$ *is a Banach interpolation couple. Then*

(i) $(\mathcal{B}_0, \mathcal{B}_1)_{[\theta]} = (\mathcal{B}_1, \mathcal{B}_0)_{[1-\theta]}$ *(with equal norms) whenever* $0 \leq \theta \leq 1$*,*

(ii) $\mathcal{B}_0 \subset \mathcal{B}_1$ *implies that*

$$\mathcal{B}_0 \subset (\mathcal{B}_0, \mathcal{B}_1)_{[\theta_0]} \subset (\mathcal{B}_0, \mathcal{B}_1)_{[\theta_1]} \subset \mathcal{B}_1$$

whenever $0 < \theta_0 < \theta_1 < 1$,

(iii) $(\mathcal{B}_0, \mathcal{B}_0)_{[\theta]} = \mathcal{B}_0$ *if* $0 < \theta < 1$, *and*

(iv) $\mathcal{B}_0 \cap \mathcal{B}_1$ *is dense in* $(\mathcal{B}_0, \mathcal{B}_1)_{[\theta]}$.

Moreover, suppose that $0 < \theta < 1$, $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)_{[\theta]}$ *and* $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)_{[\theta]}$. *If* S *is a linear operator such that*

$$\|Sb_0\|_{\mathcal{C}_0} \leq M_0 \|b_0\|_{\mathcal{B}_0} \qquad \forall b_0 \in \mathcal{B}_0$$

and

$$\|Sb_1\|_{\mathcal{C}_1} \leq M_1 \|b_1\|_{\mathcal{B}_1} \qquad \forall b_1 \in \mathcal{B}_1,$$

then S *is a bounded linear operator from* $\mathcal{B}$ *to* $\mathcal{C}$ *satisfying*

$$\|Sb\|_{\mathcal{C}} \leq M_0^{1-\theta} M_1^{\theta} \|b\|_{\mathcal{B}} \qquad \forall b \in \mathcal{B}.$$

The last part of the above theorem has a generalisation to multilinear forms. In this thesis we shall often use the bilinear version below.

Theorem 3.2.2. *[2, Theorem 4.4.1] Suppose that the pairs* $(\mathcal{A}_0, \mathcal{A}_1)$, $(\mathcal{B}_0, \mathcal{B}_1)$ *and* $(\mathcal{C}_0, \mathcal{C}_1)$ *are Banach interpolation couples. Assume that* $S : A_0 \cap \mathcal{A}_1 \times \mathcal{B}_0 \cap \mathcal{B}_1 \to \mathcal{C}_0 \cap \mathcal{C}_1$ *is bilinear and that for every* (a, b) *in* $\mathcal{A}_0 \cap \mathcal{A}_1 \times \mathcal{B}_0 \cap \mathcal{B}_1$ *the inequalities*

$$\|S(a, b)\|_{\mathcal{C}_0} \leq M_0 \|a\|_{\mathcal{A}_0} \|b\|_{\mathcal{B}_0}$$

and

$$\|S(a, b)\|_{\mathcal{C}_1} \leq M_1 \|a\|_{\mathcal{A}_1} \|b\|_{\mathcal{B}_1}$$

hold. If $0 < \theta < 1$ *and* $0 < q \leq \infty$ *then* S *extends uniquely to a bilinear mapping from* $(\mathcal{A}_0, \mathcal{A}_1)_{[\theta]} \times (\mathcal{B}_0, \mathcal{B}_1)_{[\theta]}$ *to* $(\mathcal{C}_0, \mathcal{C}_1)_{[\theta]}$ *with norm at most* $M_0^{1-\theta} M_1^{\theta}$.

Applications of complex interpolation to L^p spaces are given in the next section. Now, however, we introduce real interpolation. One way of constructing a real interpolation space $\mathcal{B}$ from an interpolation couple $(\mathcal{B}_0, \mathcal{B}_1)$ is known as *the K-method* [72, 1.3]. When $0 < t < \infty$ and $b \in \mathcal{B}_0 + \mathcal{B}_1$, define K by the formula

$$K(t, b, \mathcal{B}_0, \mathcal{B}_1) = \inf_{b=b_0+b_1} (\|b_0\|_{\mathcal{B}_0} + t\,\|b_1\|_{\mathcal{B}_1}). \tag{3.1}$$

If $\theta \in (0,1)$ and $q \in [1, \infty)$, then we construct the interpolation space $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}$ by

$$(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q} = \left\{ b \in \mathcal{B}_0 + \mathcal{B}_1 : \|b\|_{(\mathcal{B}_0,\mathcal{B}_1)_{\theta,q}} < \infty \right\}$$

where

$$\|b\|_{(\mathcal{B}_0,\mathcal{B}_1)_{\theta,q}} = \left(\int_0^\infty \left(t^{-\theta} K(t, b, \mathcal{B}_0, \mathcal{B}_1) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q}. \tag{3.2}$$

One can also define the interpolation space $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,\infty}$, but we shall not do so here.

Some basic properties of real interpolation spaces are given in the next theorem.

Theorem 3.2.3. *[2, Theorems 3.1.2 and 3.4.1] Suppose that $(\mathcal{B}_0, \mathcal{B}_1)$ is a Banach interpolation couple, $0 < \theta < 1$ and $1 \le q \le \infty$. Then the following properties hold:*

(i) $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q} = (\mathcal{B}_1, \mathcal{B}_0)_{1-\theta,q}$ *with equal norms,*

(ii) if $1 \le q \le r \le \infty$ *then*

$$(\mathcal{B}_0, \mathcal{B}_1)_{\theta,1} \subseteq (\mathcal{B}_0, \mathcal{B}_1)_{\theta,q} \subseteq (\mathcal{B}_0, \mathcal{B}_1)_{\theta,r} \subseteq (\mathcal{B}_0, \mathcal{B}_1)_{\theta,\infty},$$

(iii) $(\mathcal{B}_0, \mathcal{B}_0)_{\theta,q} = \mathcal{B}_0$ *with equivalent norms, and*

(iv) if $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *are complete then so is* $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}$.

Moreover, suppose that $0 < \theta < 1$, $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}$ and $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)_{\theta,q}$. If S is a linear operator such that

$$\|Sb_0\|_{\mathcal{C}_0} \le M_0 \|b_0\|_{\mathcal{B}_0} \qquad \forall b_0 \in \mathcal{B}_0$$

and

$$\|Sb_1\|_{\mathcal{C}_1} \leq M_1 \|b_1\|_{\mathcal{B}_1} \qquad \forall b_1 \in \mathcal{B}_1$$

then S is a bounded linear operator from $\mathcal{B}$ to $\mathcal{C}$ satisfying

$$\|Sb\|_{\mathcal{C}} \leq M_0^{1-\theta} M_1^{\theta} \|b\|_{\mathcal{B}} \qquad \forall b \in \mathcal{B}.$$

Some bilinear results for real interpolation spaces are now given.

Theorem 3.2.4. *[2, pp. 76–77] Suppose that $(\mathcal{A}_0, \mathcal{A}_1)$, $(\mathcal{B}_0, \mathcal{B}_1)$ and $(\mathcal{C}_0, \mathcal{C}_1)$ are interpolation couples.*

(i) Suppose that for every (a, b) in $\mathcal{A}_0 \cap \mathcal{B}_0 \times \mathcal{A}_1 \cap \mathcal{B}_1$ the inequalities

$$\|S(a, b)\|_{\mathcal{C}_0} \leq M_0 \|a\|_{\mathcal{A}_0} \|b\|_{\mathcal{B}_0}$$

and

$$\|S(a, b)\|_{\mathcal{C}_1} \leq M_1 \|a\|_{\mathcal{A}_1} \|b\|_{\mathcal{B}_1}$$

hold. If $0 < \theta < 1$ and $1/r + 1 = 1/p + 1/q$ with $1 \leq r \leq \infty$, then S extends uniquely to a bilinear mapping from $(\mathcal{A}_0, \mathcal{A}_1)_{\theta,p} \times (\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}$ to $(\mathcal{C}_0, \mathcal{C}_1)_{\theta,r}$ with norm at most $M_0^{1-\theta} M_1^{\theta}$.

(ii) Suppose that the bilinear operator S acts as a bounded transformation as indicated below:

$$S : \mathcal{A}_0 \times \mathcal{B}_0 \to \mathcal{C}_0$$
$$S : \mathcal{A}_0 \times \mathcal{B}_1 \to \mathcal{C}_1$$
$$S : \mathcal{A}_1 \times \mathcal{B}_0 \to \mathcal{C}_1.$$

If $\theta_0, \theta_1 \in (0, 1)$ and $p, q, r \in [1, \infty]$ such that $1 \leq 1/p + 1/q$ and $\theta_0 + \theta_1 < 1$, then S also acts as a bounded transformation in the following way:

$$S : (\mathcal{A}_0, \mathcal{A}_1)_{\theta_0,pr} \times (\mathcal{B}_0, \mathcal{B}_1)_{\theta_1,qr} \to (\mathcal{C}_0, \mathcal{C}_1)_{\theta_0+\theta_1,r}.$$

3.3 Interpolation of L^p spaces

In this section, we compile a list of results that are used in later chapters. We begin by noting the elegance of complex interpolation when applied to the vector-valued L^p spaces.

Theorem 3.3.1. *[2, Theorem 5.1.2] Suppose that (X, μ) is a measure space, $(\mathcal{B}_0, \mathcal{B}_1)$ is a Banach interpolation couple, $p_0, p_1 \in [1, \infty]$ and $0 < \theta < 1$. If $1/p = (1-\theta)/p_0 + \theta/p_1$ and $\mathcal{B}_{[\theta]} = (\mathcal{B}_0, \mathcal{B}_1)_{[\theta]}$ then*

$$\big(L^{p_0}(X; \mathcal{B}_0), L^{p_1}(X; \mathcal{B}_1)\big)_{[\theta]} = L^p(X, \mathcal{B}_{[\theta]}).$$

If $p_i = \infty$ for some i in $\{1, 2\}$, then L^{p_i} must be replaced with the space L_0^∞ of bounded functions with compact support.

Proof. This is a simple application of [2, Theorems 5.1.1 and 5.1.2]. $\square$

The situation with real interpolation of L^p spaces is more complicated. This is partly due to the extra interpolation parameter and also to the fact that, in general, real interpolation of L^p spaces gives Lorentz spaces rather than L^p spaces.

Definition 3.3.2. Suppose that (X, μ) is a measure space, $\mathcal{B}$ is a Banach space and $1 < p < \infty$. If $1 \leq q < \infty$ then the *Lorentz space* $L^{p,q}(X)$ is given by

$$L^{p,q}(X; \mathcal{B}) = \{F \in L^1(X; \mathcal{B}) + L^\infty(X; \mathcal{B}) : \|F\|_{L^{p,q}(X;\mathcal{B})} < \infty\},$$

where

$$\|F\|_{L^{p,q}(X;\mathcal{B})} = \left(\int_0^\infty \left(t^{1/p} F^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}$$

and F^* is the measure-preserving rearrangement function of F (see [72, 1.18.6] for further details).

Theorem 3.3.3. *[2, Theorem 5.2.1] Suppose that X is a measure space, $\mathcal{B}$ is a Banach space, $1 \leq p_0 < p_1 \leq \infty$, $p_0 < q \leq \infty$ and $0 < \theta < 1$. If $1/p = (1-\theta)/p_0 + \theta/p_1$ then*

$$\big(L^{p_0}(X; \mathcal{B}), L^{p_1}(X; \mathcal{B})\big)_{\theta,q} = L^{p,q}(X; \mathcal{B}) \tag{3.3}$$

with equivalent norms.

The right-hand side of the interpolation formula (3.3) can be replaced with an L^p space in certain circumstances by applying the following embedding results.

Lemma 3.3.4. *[2, p. 8] Suppose that X is a measure space and $\mathcal{B}$ is a Banach space.*

(i) If $1 \leq r_1 < r_2 \leq \infty$ and $1 < p < \infty$ then $L^{p,r_1}(X;\mathcal{B}) \subseteq L^{p,r_2}(X;\mathcal{B})$.

(ii) If $1 \leq p \leq \infty$ then $L^{p,p}(X;\mathcal{B}) = L^p(X;\mathcal{B})$ with equal norms.

A consequence of the lemma is that

$$L^p(X;\mathcal{B}) \subseteq \big(L^{p_0}(X;\mathcal{B}), L^{p_1}(X;\mathcal{B})\big)_{\theta,q} = L^{p,q}(X;\mathcal{B})$$

whenever $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $p \leq q$. One would hope that if we consider mixed Banach spaces $\mathcal{B}_0$ and $\mathcal{B}_1$ then the above formula would generalise to

$$\big(L^{p_0}(X;\mathcal{B}_0), L^{p_1}(X;\mathcal{B}_1)\big)_{\theta,q} = L^{p,q}\big(X;(\mathcal{B}_0,\mathcal{B}_1)_{\theta,q}\big).$$

Unfortunately, M. Cwikel [17] showed that this is not the case if p is different from q, even when $p_0 = p_1$. However, there is a positive result.

Theorem 3.3.5. *[2, p. 130][72, p. 128] Suppose that X is a measure space, $p_0, p_1 \in [1,\infty)$, $\theta \in (0,1)$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. If $(\mathcal{B}_0,\mathcal{B}_1)$ is a Banach interpolation couple then*

$$\big(L^{p_0}(X;\mathcal{B}_0), L^{p_1}(X;\mathcal{B}_1)\big)_{\theta,p} = L^{p}\big(X;(\mathcal{B}_0,\mathcal{B}_1)_{\theta,p}\big). \tag{3.4}$$

A little more flexibility is gained by mixing up real and complex interpolation.

Lemma 3.3.6. *Suppose that $p_0, p_1 \in [1,\infty]$, $1 < \theta_0, < \theta_1 < 1$, $\eta \in [0,1]$, $q_0, q_1 \in [1,\infty]$, X is a measure space and $(\mathcal{B}_0,\mathcal{B}_1)$ is a Banach interpolation couple. Denote $(\mathcal{B}_0,\mathcal{B}_1)_{\theta_i,q_i}$ by $\mathcal{B}_{\theta_i,q_i}$ whenever $i = 1,2$. Then*

$$\big(L^{p_0}(X;\mathcal{B}_{\theta_0,q_0}), L^{p_1}(X;\mathcal{B}_{\theta_1,q_1})\big)_{[\eta]} = L^{p}(X;\mathcal{B}_{\theta,q})$$

where

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}, \qquad \frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1}, \qquad \theta = (1-\eta)\theta_0 + \eta\theta_1. \tag{3.5}$$

If $p_i = \infty$ for some i in $\{1, 2\}$, then L^{p_i} must be replaced with L_0^∞.

Proof. The lemma is a simple consequence of Theorem 3.3.1 and the formula

$$(\mathcal{B}_{\theta_0,q_0}, \mathcal{B}_{\theta_1,q_1})_{[\eta]} = \mathcal{B}_{\theta,q},$$

valid whenever (3.5) holds, which connects the real and complex methods of interpolation (see [2, Theorem 4.7.2]). □

We shall also use interpolation results for weighted Lebesgue sequence spaces. Whenever $s \in \mathbb{R}$ and $1 < q < \infty$, let ℓ_s^q denote the space of all scalar-valued sequences $\{a_j\}_{j\in\mathbb{Z}}$ such that

$$\|\{a_j\}_{j\in\mathbb{Z}}\|_{\ell_s^q} = \Big(\sum_{j\in\mathbb{Z}} 2^{js}|a_j|^q\Big)^{1/q} < \infty. \tag{3.6}$$

If $q = \infty$ then the norm is defined by

$$\|\{a_j\}_{j\in\mathbb{Z}}\|_{\ell_s^\infty} = \sup_{j\in\mathbb{Z}} 2^{js}|a_j|.$$

We have the following interpolation theorem.

Theorem 3.3.7. *[2, Theorem 5.6.1] Assume that $0 < q_0 \le \infty$, $0 < q_1 \le \infty$, $0 < \theta < 1$ and $s_0 \ne s_1$. If $0 < q \le \infty$ then*

$$(\ell_{s_0}^{q_0}, \ell_{s_1}^{q_1})_{\theta,q} = \ell_s^q$$

where $s = (1-\theta)s_0 + \theta s_1$.

In Chapters 4 and 5 we will use the special case

$$(\ell_{s_0}^\infty, \ell_{s_1}^\infty)_{\theta,1} = \ell_s^1. \tag{3.7}$$

3.4 Besov spaces

When finding Strichartz estimates for the wave equation, Littlewood–Paley dyadic decompositions of functions seem unavoidable. Besov spaces consist of functions whose Littlewood–Paley decomposition is bounded in the Besov norm. From another perspective, Besov spaces arise naturally as the real interpolants of Sobolev spaces. There are also embedding relations between Sobolev, Besov and classical Lebesgue spaces. We mainly consider the so-called *homogeneous Besov spaces* rather than Besov spaces. Both share similar properties but the symmetry of the former's norm will be helpful in our context. For a treatment of these spaces in greater depth, we refer the reader to [2] and [72].

We begin with a brief introduction to Littlewood–Paley dyadic decompositions. Suppose that $\psi \in C_0^\infty(\mathbb{R}^n)$ and that its Fourier transform $\hat{\psi}$ satisfies the properties $0 \leq \hat{\psi} \leq 1$, $\hat{\psi}(\xi) = 1$ whenever $|\xi| \leq 1$ and $\hat{\psi}(\xi) = 0$ whenever $|\xi| \geq 2$. If $j \in \mathbb{Z}$ then define φ_j by

$$\hat{\varphi}_0(\xi) = \hat{\psi}(\xi) - \hat{\psi}(2\xi)$$

and $\hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi)$. This means that

$$\operatorname{supp}(\hat{\varphi}_j) \subseteq \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$$

and

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1$$

for any ξ in $\mathbb{R}^n \backslash \{0\}$, with at most two nonvanishing terms in the sum. When $j \in \mathbb{Z}$, define $\widetilde{\varphi}_j$ by the formula

$$\widetilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1} \tag{3.8}$$

This implies that $\hat{\varphi}_j = \hat{\widetilde{\varphi}}_j \hat{\varphi}_j$, thereby allowing for the use of the standard trick

$$\varphi_j * u = \widetilde{\varphi}_j * \varphi_j * u \tag{3.9}$$

for any tempered distribution u.

If $\rho \in \mathbb{R}$, $1 \leq r \leq \infty$, $1 \leq s \leq \infty$ and u is a tempered distribution then define $\|u\|_{\dot{B}^{\rho}_{r,s}}$ by

$$\|u\|_{\dot{B}^{\rho}_{r,s}} = \left(\sum_{j \in \mathbb{Z}} 2^{\rho j} \|\varphi * u\|^{s}_{L^r(\mathbb{R}^n)} \right)^{1/s} . \tag{3.10}$$

We note that if u is a polynomial then $\operatorname{supp}(\hat{u}) = \{0\}$ and hence $\|u\|_{\dot{B}^{\rho}_{r,s}} = 0$. Conversely if $\|u\|_{\dot{B}^{\rho}_{r,s}} = 0$ then u is a polynomial. We therefore define the homogeneous Besov space $\dot{B}^{\rho}_{r,s}$ to be the completion in $\|\cdot\|_{\dot{B}^{\rho}_{r,s}}$ of the set of equivalence classes of tempered distributions u, modulo polynomials, such that $\|u\|_{\dot{B}^{\rho}_{r,s}} < \infty$.

The Besov norm of u corresponds to taking the $L^r(\mathbb{R}^n)$ norm of each $\varphi_j * u$ and then the weighted ℓ^s norm in the j variable. Therefore the following interpolation result should come as no surprise. In what follows, $\mathcal{S}'(\mathbb{R}^n)$ denotes the set of tempered distributions on $\mathbb{R}^n$ and $\mathcal{P}(\mathbb{R}^n)$ denotes the set of polynomials on $\mathbb{R}^n$.

Lemma 3.4.1. *[72, Section 2.4] Suppose that $\rho_0, \rho_1 \in \mathbb{R}$, $\rho_0 \neq \rho_1$, $r_0, r_1 \in [1, \infty)$, $r_0 \neq r_1$, $s_0, s_1 \in (1, \infty)$ and $\theta \in (0, 1)$. Then*

$$\left(\dot{B}^{\rho_0}_{r_0,s_0}, \dot{B}^{\rho_0}_{r_0,s_0} \right)_{\theta,s} = \dot{B}^{\rho}_{r,s,(s)}$$

where

$$\rho = (1-\theta)\rho_0 + \theta\rho_1, \qquad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \qquad \frac{1}{s} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}$$

and

$$\dot{B}^{\rho}_{r,s,(s)} = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \backslash \mathcal{P}(\mathbb{R}^n) : \left\| \left\{ 2^{\rho j} \|\varphi_j * u\|_{L^{r,s}(\mathbb{R}^n)} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^s} < \infty \right\}.$$

If $s = 2$ then we can use the above lemma together with the continuous embedding

$$L^p = L^{p,p} \subseteq L^{p,2} \qquad (p \leq 2)$$

to obtain the continuous inclusion

$$\dot{B}^{\rho}_{r,2} \subset \dot{B}^{\rho}_{r,2,(2)} = \left(\dot{B}^{\rho_0}_{r_0,2}, \dot{B}^{\rho_0}_{r_0,2} \right)_{\theta,s} \tag{3.11}$$

whenever $r \leq 2$.

A few other continuous embedding results will be needed. The first one is an easy consequence of Young's inequality and the definition of homogeneous Besov spaces. The proof is short so we present it here.

Lemma 3.4.2. *[2, Section 6.5] Suppose that* $1 \leq r_2 \leq r_1 \leq \infty$*,* $1 \leq s \leq \infty$*,* $\rho_1, \rho_2 \in \mathbb{R}$ *and* $\rho_1 - n/r_1 = \rho_2 - n/r_2$*. Then* $\dot{B}^{\rho_2}_{r_2,s} \subseteq \dot{B}^{\rho_1}_{r_1,s}$ *and*

$$\|u\|_{\dot{B}^{\rho_1}_{r_1,s}} \leq C \|u\|_{\dot{B}^{\rho_2}_{r_2,s}} \tag{3.12}$$

for some positive constant C*.*

Proof. By (3.9) and Young's inequality,

$$\|\varphi_j * u\|_{r_1} \leq \|\widetilde{\varphi}_j\|_p \|\varphi_j * u\|_{r_2}$$

where $n/p' = n/r_1 - n/r_2 = \rho_2 - \rho_1$. But by scaling, $\|\widetilde{\varphi}_j\|_p = 2^{jn/p} \|\widetilde{\varphi}_0\|_p$ and hence

$$\|\varphi_j * u\|_{r_1} \leq 2^{j(\rho_2 - \rho_1)} \|\widetilde{\varphi}_0\|_p \|\varphi_j * u\|_{r_2}.$$

Substituting this into (3.10) gives (3.12) with $C = \|\widetilde{\varphi}_0\|_p$. □

The homogeneous Sobolev spaces $\dot{H}^\rho_r$ may be defined in terms of Riesz potentials. Briefly, whenever $1 < r < \infty$ and $\rho \in \mathbb{R}$, the space $\dot{H}^\rho_r$ coincides with the space $(-\Delta)^{-\rho/2} L^2(\mathbb{R}^n)$ with norm

$$\|u\|_{\dot{H}^\rho} \approx \left\|(-\Delta)^{-\rho/2} u\right\|_{L^2(\mathbb{R}^n)}$$

(see [2] or [73] for further details). The homogeneous Besov spaces and homogeneous Sobolev spaces are related by interpolation and in particular by the continuous embeddings

$$\dot{B}^\rho_{r,2} \subseteq \dot{H}^\rho_r \quad \text{when } 2 \leq r < \infty; \qquad \dot{B}^\rho_{r,2} \supseteq \dot{H}^\rho_r \quad \text{when } 1 < r \leq 2, \tag{3.13}$$

whenever $\rho \in \mathbb{R}$. When $r = 2$ it is customary to write $\dot{H}^\rho$ instead of $\dot{H}^\rho_2$. In this case (3.13) reduces to $\dot{H}^\rho = \dot{B}^\rho_{2,2}$.

We shall use Besov spaces to prove one corollary in Section 5.8. For the definition of the Besov space $B^{\rho}_{r,p}$, see [2] or [72]. We need only mention here that Lemmas 3.4.1 and 3.4.2 have exact analogies for the 'non-dotted' Besov spaces $B^{\rho}_{r,p}$.

3.5 Translation invariant operators

We use a translation invariance argument in Chapters 4 and 5 to demonstrate some necessary conditions for the validity of various Strichartz estimates. The basis of this argument is found below.

Definition 3.5.1. Suppose that X is a measure space, that $\mathcal{B}$ and $\mathcal{C}$ are Banach spaces and that p and q are Lebesgue exponents in $[1,\infty]$. We say that an operator $A : L^p(X;\mathcal{B}) \to L^q(X;\mathcal{C})$ is translation invariant if

$$\tau_y A f = A \tau_y f \qquad \forall y \in X \quad \forall f \in L^p(X;\mathcal{B}),$$

where the translation operator τ_y is given by $\tau_y f(x) = f(x-y)$.

The following lemma is a simple vector-valued extension of a result due to L. Hörmander [34, Theorem 1.1]. It asserts that no nontrivial translation invariant operator can map from a higher Lebesgue exponent space to a lower Lebesgue exponent space.

Lemma 3.5.2. *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are Banach spaces and that the linear operator $A : L^p(\mathbb{R};\mathcal{B}) \to L^q(\mathbb{R};\mathcal{C})$ is bounded and translation invariant. If $q < p < \infty$ then $A = 0$ and if $q < p = \infty$ then $A = 0$ on $L^\infty_0(\mathbb{R};\mathcal{B})$.*

Proof. Suppose first that $q < p < \infty$. We begin by showing that

$$\lim_{y\to\infty} \|f + \tau_y f\|_{L^p(\mathbb{R};\mathcal{B})} = 2^{1/p} \|f\|_{L^p(\mathbb{R};\mathcal{B})} \qquad \forall f \in L^p(\mathbb{R};\mathcal{B}). \tag{3.14}$$

To see this, suppose that $\epsilon > 0$ and write f as $g + h$ where g has compact support and $\|h\|_{L^p(\mathbb{R};\mathcal{B})} < \epsilon$. For $|y|$ sufficiently large, the supports of g and $\tau_y g$ do not meet and hence

$$\|g + \tau_y g\|_{L^p(\mathbb{R};\mathcal{B})} = 2^{1/p} \|g\|_{L^p(\mathbb{R};\mathcal{B})}.$$

Now since $\big| \|f\|_{L^p(\mathbb{R};\mathcal{B})} - \|g\|_{L^p(\mathbb{R};\mathcal{B})} \big| < \epsilon$,

$$\big| \|f + \tau_y f\|_{L^p(\mathbb{R};\mathcal{B})} - \|g + \tau_y g\|_{L^p(\mathbb{R};\mathcal{B})} \big| < 2\epsilon$$

and ϵ is arbitrary, we obtain (3.14).

Assume now that

$$\|Af\|_{L^q(\mathbb{R};\mathcal{C})} \leq C\,\|f\|_{L^p(\mathbb{R};\mathcal{B})} \qquad \forall f \in L^p(\mathbb{R};\mathcal{B})$$

where the nonnegative constant C is chosen to be as small as possible. By the linearity and translation invariance of A,

$$\begin{aligned} \|Af + \tau_y Af\|_{L^q(\mathbb{R};\mathcal{B})} &= \|A(f + \tau_y f)\|_{L^q(\mathbb{R};\mathcal{C})} \\ &\leq C\,\|f + \tau_y f\|_{L^p(\mathbb{R};\mathcal{B})} \qquad \forall y \in \mathbb{R} \quad \forall f \in L^p(\mathbb{R};\mathcal{B}). \end{aligned}$$

If we let y approach infinity and apply (3.14) to both sides then

$$\|Af\|_{L^q(\mathbb{R};\mathcal{C})} \leq C\,2^{1/p-1/q}\,\|f\|_{L^p(\mathbb{R};\mathcal{B})} \qquad \forall f \in L^p(\mathbb{R};\mathcal{B}).$$

Since $p > q$ we must have $C = 0$, otherwise $C\,2^{1/p-1/q} < C$ and a contradiction results. Hence $A = 0$.

If $q < p = \infty$ then the same arguments apply provided that we replace L^∞ with L^∞_0. $\square$

Chapter 4

Strichartz estimates

In this chapter we prove and apply the abstract Strichartz theorem of M. Keel and T. Tao [42, Theorem 10.1]. The theorem is stated in Section 4.2 and proved over the four sections following. In Section 4.7, we show how the theorem completely solves the problem of determining all homogeneous Strichartz estimates for the Schrödinger equation. The following section then puts the theorem to work on the wave equation. Though there are no new results in this chapter, its content serves as fundamental background to new material contained in Chapter 5. Before attending to the theorem and its proof, we shall illustrate why Strichartz estimates are sought after by those working with inhomogeneous Cauchy problems.

4.1 A motivating example

In this section, we illustrate the power of Strichartz estimates for answering questions concerning nonlinear dispersive partial differential equations. The theorem given below is a variant on results that go back as far as the 1980s (see, for example, [26], [40], [9] and the references therein) but the proof is essentially the same. A particularly readable account may be obtained from lecture notes [68] of T. Tao. For ease of notation, write L^p_x for the space $L^p(\mathbb{R}^2)$ and $L^p_{t,x}$ for the space $L^p(\mathbb{R}; L^p(\mathbb{R}^2))$.

Theorem 4.1.1. *[9] Consider the initial value problem*

$$\begin{cases} iu'(t) + \Delta u(t) = \lambda |u(t)|^2 u(t) & \forall t \geq 0 \\ u(0) = f & , \end{cases} \tag{4.1}$$

where Δ is the Laplacian on L^2_x, λ is a complex constant and $\|f\|_{L^2_x} = 1$. If $|\lambda|$ is sufficiently small then there is a global solution u to (4.1) such that $\|u(t)\|_{L^2_x} \lesssim 1$ for every time t. Furthermore the solution u satisfies the spacetime estimate

$$\|u\|_{L^4_{t,x}} \lesssim 1, \tag{4.2}$$

is unique subject to the above conditions, and depends continuously in $L^4_{t,x}$ on the initial data f in L^2_x. Finally we have scattering in the sense that there exists some initial data f_+ in L^2_x such that

$$\lim_{t\to\infty} \left\| u(t) - e^{it\Delta} f_+ \right\|_{L^2_x} = 0. \tag{4.3}$$

Equation (4.1), known as the *meson equation*, is a perturbation of the free (or homogeneous) Schrödinger equation $iu' + \Delta u = 0$. With two spatial dimensions, it is an L^2 critical perturbation in the sense that, unlike other powers of u, the cubic forcing term cannot vanish under a rescaling of dimensions if one requires that $u(t)$ is dimensionless with constant L^2 norm (as happens in physical applications where $u(t)$ is interpreted probabilistically).

Our principal reason for considering the above theorem and its proof is to illustrate the use of Strichartz estimates. A sketch of the argument will be in given certain places to avoid obscuring the main ideas. For convenience, we state the contraction mapping principle which is used in the proof of Theorem 4.1.1.

Theorem 4.1.2 (The contraction mapping principle). *Suppose that X is a complete metric space with metric d and that $N : X \to X$ is a contraction mapping satisfying*

$$d(N(u), N(v)) \leq c\, d(u, v) \qquad \forall u, v \in X,$$

where the contractivity coefficient c lies in $[0,1)$. Suppose that $u_0 \in X$ and inductively define the sequence $\{u_k\}_{k=0}^{\infty}$ by $u_{k+1} = N(u_k)$ whenever $k > 0$. Then N has a unique fixed point u, the sequence $\{u_k\}_{k=0}^{\infty}$ converges to u and

$$d(u,u_k) \leq c^k d(u,u_0) \qquad \forall k \geq 0.$$

Sketch proof of Theorem 4.1.1. It is well known that if we want to solve an equation of the form

$$iu'(t) + \Delta u(t) = F(t), \qquad u(0) = f,$$

then the only solution is

$$u(t) = e^{it\Delta} f - i \int_0^t e^{i(t-s)\Delta} F(s)\, \mathrm{d}s$$

by Duhamel's principle and spectral theory. Therefore we rewrite the inhomogeneous equation (4.1) in its integral form

$$u(t) = e^{it\Delta} f - i\lambda \int_0^t e^{i(t-s)\Delta} \big(|u(s)|^2 u(s) \big)\, \mathrm{d}s. \tag{4.4}$$

To find a solution to (4.4) we shall construct a sequence $\{u_k\}_{k=0}^{\infty}$ of approximate solutions which converges to the solution. First approximate u by the solution u_0, given by

$$u_0(t) = e^{it\Delta} f,$$

of the free Schrödinger initial value problem. Now make a better approximation u_1, given by

$$u_1(t) = e^{it\Delta} f - i\lambda \int_0^t e^{i(t-s)\Delta} \big(|u_0(s)|^2 u_0(s) \big)\, \mathrm{d}s.$$

More generally, define the inhomogeneous map $u \mapsto N_f(u)$ by

$$N_f(u)(t) = e^{it\Delta} f - i\lambda \int_0^t e^{i(t-s)\Delta} \big(|u(s)|^2 u(s) \big)\, \mathrm{d}s$$

and define $\{u_k\}_{k=0}^{\infty}$ by the iteration $u_{k+1} = N_f(u_k)$ whenever $k > 0$.

To prove that (4.1) has a unique global solution, it suffices to show that the sequence $\{u_k\}_{k=0}^{\infty}$ has a limit u satisfying $u = N_f(u)$ and that u is the only fixed point of N_f. This will be achieved by applying the contraction mapping principle to N_f for a suitable contraction space X. Our choice of X is determined by the existence of the estimates

$$\left\|e^{it\Delta}f\right\|_{L^4_{t,x}} \lesssim \|f\|_{L^2_x} \qquad \forall f \in L^2_x, \tag{4.5}$$

$$\left\|\int_0^\infty e^{-is\Delta}F(s)\,\mathrm{d}s\right\|_{L^2_x} \lesssim \|F\|_{L^{4/3}_{t,x}} \qquad \forall F \in L^{4/3}_{t,x} \tag{4.6}$$

and

$$\left\|\int_0^t e^{i(t-s)\Delta}F(s)\,\mathrm{d}s\right\|_{L^4_{t,x}} \lesssim \|F\|_{L^{4/3}_{t,x}} \qquad \forall F \in L^{4/3}_{t,x}. \tag{4.7}$$

These three estimates are *Strichartz estimates* associated to the initial value problem (4.1). Theorem 4.2.2, which is given in the next section, will imply these estimates (see Remark 4.7.3); for the moment, we simply assume that they are true and use them to complete the proof of Theorem 4.1.1.

Define the metric space X by

$$X = \{u : \|u\|_{L^4_{t,x}} \leq C\}$$

where C is an absolute constant (sufficiently large for the argument below to hold) and with metric induced from the $L^4_{t,x}$ norm. To apply the contraction mapping principle, it suffices to show that $u_0 \in X$,

$$\|N_f(u) - N_f(v)\|_{L^4_{t,x}} \leq \frac{1}{2}\|u - v\|_{L^4_{t,x}} \qquad \forall u, v \in X \tag{4.8}$$

and $N(u) \in X$ whenever $u \in X$.

First, (4.5) implies that $u_0 \in X$ since $\|f\|_2 = 1$. To show (4.8), we first note that

$$N_f(u) - N_f(v) = -i\lambda\int_0^t e^{i(t-s)\Delta}\left(|u(s)|^2u(s) - |v(s)|^2v(s)\right)\mathrm{d}s$$

and hence (4.7) gives

$$\|N_f(u) - N_f(v)\|_{L^4_{t,x}} \lesssim |\lambda|\left\||u|^2u - |v|^2v\right\|_{L^{4/3}_{t,x}}.$$

By a pointwise estimate

$$|u|^2u - |v|^2v = O(|u|^2|u-v|) + O(|v|^2|u-v|)$$

which gives

$$\begin{aligned}\|N_f(u) - N_f(v)\|_{L^4_{t,x}} &\lesssim |\lambda| \left(\left\| |u|^2 |u-v| \right\|_{L^{4/3}_{t,x}} + \left\| |v|^2 |u-v| \right\|_{L^{4/3}_{t,x}} \right) \\ &\lesssim |\lambda| \left(\|u\|^2_{L^4_{t,x}} \|u-v\|_{L^4_{t,x}} + \|v\|^2_{L^4_{t,x}} \|u-v\|_{L^4_{t,x}} \right)\end{aligned}$$

by Hölder's inequality. Since u and v both belong to X, their $L^4_{t,x}$ norms are bounded and it follows that (4.8) holds if λ is sufficiently close to zero. Finally, if $u \in X$ then it may be easily deduced that $N_f(u) \in X$ by (4.8) (see, for example, the argument at (5.69)).

So far we have shown the existence and uniqueness of the solution u to (4.1). Since $u \in X$ we also have (4.2). We turn now to show the continuous dependence of the solution on the initial data. Suppose that v is a solution to (4.1) with initial data g satisfying $\|g\|_{L^2_x} = 1$. Then

$$\begin{aligned}\|u-v\|_{L^4_{t,x}} &= \|N_f(u) - N_g(v)\|_{L^4_{t,x}} \\ &\leq \left\| e^{it\Delta} f - e^{it\Delta} g \right\|_{L^4_{t,x}} + \|N_f(u) - N_f(v)\|_{L^4_{t,x}} \\ &\leq C_1 \|f-g\|_{L^2_x} + \frac{1}{2} \|u-v\|_{L^4_{t,x}}\end{aligned}$$

by (4.5) and (4.8). Rearranging the estimate gives

$$\|u-v\|_{L^4_{t,x}} \leq 2C_1 \|f-g\|_{L^2_x}$$

as required.

To show that $\|u(t)\|_{L^2_x} \lesssim 1$ for all t, observe from (4.4) that

$$\|u(t)\|_{L^2_x} \lesssim \left\| e^{it\Delta} f \right\|_{L^2_x} + \left\| e^{it\Delta} \int_0^t e^{-is\Delta} \big(|u(s)|^2 u(s) \big) \, \mathrm{d}s \right\|_{L^2_x}$$

provided that both terms on the right-hand side are bounded. It is a standard fact of Fourier analysis (and in particular a consequence of Plancherel's theorem) that

$$\left\| e^{it\Delta} f \right\|_{L^2_x} = \|f\|_{L^2_x}. \tag{4.9}$$

Looking now at the second term, (4.6) gives

$$\left\| \int_0^t e^{-is\Delta} \big(|u(s)|^2 u(s) \big) \, \mathrm{d}s \right\|_{L^2_x} \lesssim \left\| |u|^2 u \right\|_{L^{4/3}_{t,x}} = \|u\|^3_{L^4_{t,x}} \leq C.$$

By combining this with (4.9) we obtain the uniform L_x^2 boundedness of $u(t)$.

It remains to show scattering. Define f_+ by

$$f_+ = f - i\lambda \int_0^\infty e^{-is\Delta}\big(|u(s)|^2 u(s)\big)\,\mathrm{d}s.$$

From (4.6) and the argument just given, we see that $f_+ \in L_x^2$. By (4.4),

$$u(t) - e^{it\Delta} f_+ = i\lambda e^{it\Delta} \int_t^\infty e^{-is\Delta}\big(|u(s)|^2 u(s)\big)\,\mathrm{d}s$$

and hence

$$\big\|u(t) - e^{it\Delta} f_+\big\|_{L_x^2} \lesssim \big\|1_{(t,\infty)}|u|^2 u\big\|_{L_{t,x}^{4/3}}$$

by (4.6) and (4.9), where $1_{(t,\infty)}$ denotes the characteristic function of the interval (t,∞). The limit (4.3) now follows from the monotone convergence theorem. □

The above proof illustrates how impressively Strichartz estimates perform as a technical tool for solving inhomogeneous Cauchy problems. However, we have yet to give a proof of the Strichartz estimates (4.5), (4.6) and (4.7) themselves. These estimates will be a consequence of some abstract Strichartz estimates proved in the ensuing sections. In anticipation of later developments, we highlight that, if we define a family $\{U(t) : t \in \mathbb{R}\}$ of operators on L_x^2 by $U(t) = e^{it\Delta}$, then we have conservation of probability

$$\|U(t)f\|_{L_x^2} = \|f\|_{L_x^2} \qquad \forall f \in L_x^2 \quad \forall t \in \mathbb{R}$$

and a *dispersive estimate*

$$\|U(s)U(t)^* g\|_\infty \lesssim |t-s|^{-1}\, \|g\|_1 \qquad \forall g \in L_x^1 \cap L_x^2 \quad \forall \text{ real } s \neq t.$$

The dispersive estimate may be easily derived from the integral representation of $U(t)$ (see Section 4.7) while conservation of probability, an example of what we later term an *energy estimate*, is simply a restatement of (4.9).

Remark 4.1.3. Theorem 4.1.1 states that the solution u to (4.1) is unique subject to the conditions

$$\|u\|_{L_{t,x}^4} \lesssim 1 \qquad \text{and} \qquad \|u(t)\|_{L_x^2} \lesssim 1 \quad \forall t \in \mathbb{R}.$$

In fact, one can strengthen uniqueness by showing that u is a unique solution subject only to the condition $\|u(t)\|_{L^2_x} \lesssim 1$ for all t. This type of 'unconditional uniqueness' result goes back to [40] and [41]. See [13, Section 16] for a recent discussion .

4.2 Abstract Strichartz estimates

In this section we develop a framework which will enable us to write down Strichartz estimates in a very general form. This leads to the statement of Theorem 4.2.2 which gives a range of spacetime exponents for which abstract Strichartz estimates hold under some very simple hypotheses. The section concludes with a broad outline of how the theorem will be proved while the proof itself begins in Section 4.3.

Suppose that $\mathcal{H}$ is a Hilbert space, $(\mathcal{B}_0, \mathcal{B}_1)$ is a Banach interpolation couple and $\sigma > 0$. Suppose also that for each time t in $\mathbb{R}$ we have an operator $U(t) : \mathcal{H} \to \mathcal{B}_0^*$. Its adjoint $U(t)^*$ is an operator from $\mathcal{B}_0$ to $\mathcal{H}$. We will assume that the family $\{U(t) : t \in \mathbb{R}\}$ satisfies the *energy estimate*

$$\|U(t)f\|_{\mathcal{B}_0^*} \lesssim \|f\|_{\mathcal{H}} \qquad \forall f \in \mathcal{H} \quad \forall t \in \mathbb{R}, \tag{4.10}$$

and either the *untruncated decay estimate*

$$\|U(s)U(t)^*g\|_{\mathcal{B}_1^*} \lesssim |t-s|^{-\sigma} \|g\|_{\mathcal{B}_1} \qquad \forall g \in \mathcal{B}_1 \cap \mathcal{B}_0 \, \forall \text{ real } s \neq t \tag{4.11}$$

or the *truncated decay estimate*

$$\|U(s)U(t)^*g\|_{\mathcal{B}_1^*} \lesssim (1+|t-s|)^{-\sigma} \|g\|_{\mathcal{B}_1} \qquad \forall g \in \mathcal{B}_1 \cap \mathcal{B}_0 \quad \forall s,t \in \mathbb{R} \tag{4.12}$$

The two decay estimates are sometimes referred to as *dispersive estimates*.

The energy estimate allows us to consider the operator $T : \mathcal{H} \to L^\infty(\mathbb{R}; \mathcal{B}_0^*)$ defined by the formula

$$Tf(t) = U(t)f \qquad \forall f \in \mathcal{H} \quad \forall t \in \mathbb{R}.$$

Its formal adjoint is the operator $T^* : L^1(\mathbb{R};\mathcal{B}_0) \to \mathcal{H}$ given by the $\mathcal{H}$-valued integral

$$T^*F = \int_{\mathbb{R}} U(s)^* F(s)\,\mathrm{d}s.$$

The composition $TT^* : L^1(\mathbb{R};\mathcal{B}_0) \to L^\infty(\mathbb{R};\mathcal{B}_0^*)$ is the operator given by

$$TT^*F(t) = \int_{\mathbb{R}} U(t)U(s)^* F(s)\,\mathrm{d}s, \tag{4.13}$$

which can be decomposed as the sum of retarded and advanced parts

$$(TT^*)_R F(t) = \int_{s<t} U(t)U(s)^* F(s)\,\mathrm{d}s$$

and

$$(TT^*)_A F(t) = \int_{s>t} U(t)U(s)^* F(s)\,\mathrm{d}s.$$

In many applications (see Sections 4.1, 4.7, 4.8 and 5.9 for examples) $\{U(t) : t \in \mathbb{R}\}$ is the evolution group associated to a homogeneous differential equation. The operator T solves the initial value problem of the homogeneous equation while $(TT^*)_R$ is used, by Duhamel's principle, to solve the corresponding inhomogeneous problem with zero initial data. Spacetime estimates for the functions Tf and $(TT^*)_R F$ therefore correspond to homogeneous and inhomogeneous Strichartz estimates.

When $\theta \in [0,1]$, let $\mathcal{B}_\theta$ denote the real interpolation space $(\mathcal{B}_0,\mathcal{B}_1)_{\theta,2}$. In this chapter we will show that, for certain exponent pairs (q,θ) and $(\widetilde{q},\widetilde{\theta})$, the family $\{U(t) : t \in \mathbb{R}\}$ has Strichartz estimates of the following form:

(i) the homogeneous Strichartz estimate

$$\|Tf\|_{L^q(\mathbb{R};\mathcal{B}_\theta^*)} \lesssim \|f\|_{\mathcal{H}} \qquad \forall f \in \mathcal{H}, \tag{4.14}$$

(ii) its dual estimate

$$\|T^*F\|_{\mathcal{H}} \lesssim \|F\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \qquad \forall F \in L^{q'}(\mathbb{R};\mathcal{B}_\theta) \cap L^1(\mathbb{R};\mathcal{B}_0) \tag{4.15}$$

and

(iii) the inhomogeneous (or retarded) Strichartz estimate

$$\|(TT^*)_R F\|_{L^q(\mathbb{R};\mathcal{B}^*_\theta)} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})} \qquad \forall F \in L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}}) \cap L^1(\mathbb{R};\mathcal{B}_0). \tag{4.16}$$

These estimates will be obtained by first interpolating between a bilinear version of the energy estimate (4.10) and one of the dispersive estimates (4.11) or (4.12). This, and further manipulation, impose certain conditions on the exponents q, $\widetilde{q}$, θ and $\widetilde{\theta}$, giving rise to the following definition.

Definition 4.2.1. Suppose that $\sigma > 0$. We say that a pair of exponents (q, θ) is *σ-admissible* if $(q, \theta, \sigma) \neq (2, 1, 1)$, $2 \leq q \leq \infty$, $0 \leq \theta \leq 1$ and

$$\frac{1}{q} \leq \frac{\sigma\theta}{2}. \tag{4.17}$$

We say that a pair of exponents (q, θ) is *sharp σ-admissible* if equality holds in (4.17) and *nonsharp σ-admissible* otherwise.

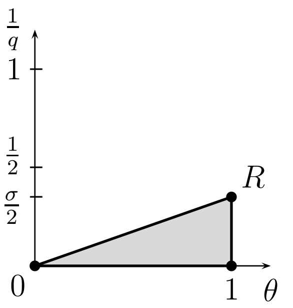
It is natural to interpret σ-admissible pairs (q, θ) as those corresponding to the points $(1/q, \theta)$ in $[0, 1] \times [0, 1]$. Figure 4.1 illustrates regions which contain these points for different values of σ. The closed line segments OQ and OR correspond to the sharp admissible pairs in each case. The point $Q = (1/2, 1/\sigma)$ corresponds to the sharp endpoint P, which is given by the formula

$$P = (2, 1/\sigma) \tag{4.18}$$

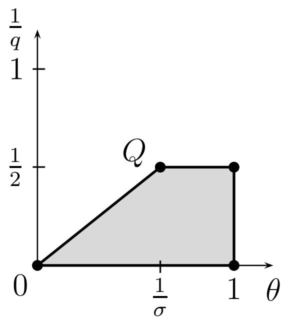
when $\sigma > 1$. This endpoint will be of notable interest later in the chapter.

The main theorem is this chapter is the following result.

Theorem 4.2.2 (Keel–Tao [42]). *Suppose that $\sigma > 0$. If $U(t)$ satisfies the energy estimate (4.10) and the untruncated decay estimate (4.11) then the abstract Strichartz estimates (4.14), (4.15) and (4.16) hold for all sharp σ-admissible pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$. Furthermore, if the decay hypothesis is strengthened to (4.12), then the Strichartz estimates (4.14), (4.15) and (4.16) hold for all σ-admissible pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$.*



(a) $\sigma < 1$



(b) $\sigma > 1$

Figure 4.1: σ-admissible pairs (q, θ) for different values of σ.

The theorem above, when announced in the late 1990s, was a breakthrough in several respects. First, the abstract setting allowed both the wave and Schrödinger equations to be treated in a unified manner. Second, when applied to the wave and Schrödinger equations in higher spatial dimensions, the theorem gave previously unknown Strichartz estimates. These new estimates occur precisely when $\sigma > 1$ and (q, θ) or $(\widetilde{q}, \widetilde{\theta})$ are the endpoint P. Consequently this completely solved the problem of determining all possible homogeneous Strichartz estimates for the wave and Schrödinger equations in higher dimensions. Third, while previous publications in the field largely followed techniques used by R. Strichartz [67], Keel and Tao [42] provided new techniques for obtaining Strichartz estimates. These have been adopted in recent papers (see, for example, [24] and [75]) and will also be exploited in this thesis.

In Section 4.7, we apply Theorem 4.2.2 to obtain Strichartz estimates for the Schrödinger equation. As a by-product, this application gives the estimates (4.5), (4.6) and (4.7) used in Section 4.1 to solve the meson equation (see Remark 4.7.3). An application of the theorem to the wave equation is explored in Section 4.8, by a route alluded to in [42, Section 10].

The rest of the chapter is dedicated to the proof of Theorem 4.2.2. This is novel in the sense that the article [42] of Keel and Tao only gives a proof of this theorem when $\mathcal{B}_0$ and $\mathcal{B}_1$ are respectively specialised to $L^2(X)$ and $L^1(X)$, where X is a measure space. Moreover, details familiar to the experts are sometimes omitted in the presentation of [42]. It is therefore hoped that our proof of Theorem 4.2.2 will be welcomed at least by the uninitiated.

The structure of the proof is as follows. Section 4.3 explores the symmetry inherent in the Strichartz estimates and their bilinear equivalents. The results stated there are used repeatedly in proofs throughout this chapter and the next. In Section 4.4, we prove homogeneous Strichartz estimates for all σ-admissible exponent pairs excluding the endpoint P. The basic strategy will be to interpolate between bilinear versions of the energy and dispersive estimates and then to estimate further via a classical L^p inequality. The task of obtaining homogeneous Strichartz estimates for the endpoint P is more delicate because the classical L^p inequality breaks down in this case. Section 4.5 addresses this difficulty by decomposing a bilinear version of the Strichartz estimate into a dyadic sum. While this sum diverges if each term is estimated individually, some summability is obtained by slightly perturbing the spatial exponents. We then interpolate between the perturbed estimates to prove the endpoint estimate. Finally, in Section 4.6 we prove inhomogeneous Strichartz estimates for σ-admissible pairs. This is achieved by proving inhomogeneous estimates for some extreme exponents and then interpolating between these and the sharp σ-admissible exponents. The proofs for these extreme cases utilise the homogeneous Strichartz estimates proved in earlier sections.

Taken together, Theorems 4.4.1, 4.4.2, 4.5.1, 4.6.1 and 4.6.3 give Theorem 4.2.2. For the retarded Strichartz estimate (4.16), the conclusion of Theorem 4.2.2 also holds for a wider class of exponent pairs. We will devote our attention to this in Chapter 5.

4.3 Equivalence, symmetry and invariance

The results of this section constitute a tool kit for simplifying the proof of Theorem 4.2.2. We show that there is a bilinear form B such that each of the Strichartz estimates (4.14), (4.15) and (4.16) is implied by a corresponding estimate on B. Not only will this give a more unified approach to proving the estimates, but, as we shall see later sections, the bilinear form estimates yield a manipulative flexibility that is harder to extract from their operator estimate counterparts. Critically, the proof of the homogeneous Strichartz estimates at the endpoint P relies on a result of real bilinear interpolation. In this section we also prove that certain key estimates are invariant under the exponent symmetry $(q, \theta) \leftrightarrow (\widetilde{q}, \widetilde{\theta})$. It ends by demonstrating the invariance of the theorem's hypothesis (in the sharp σ-admissible case) under a particular set of scaling transformations.

We set the tone with a formal calculation of the dual of $(TT^*)_R$. For clarity, denote by $\langle f,\, g\rangle_{\mathcal{B}}$ the action of a linear functional g on an element f of $\mathcal{B}$. (Note that $\langle\,\cdot\,,\,\cdot\,\rangle$ with no subscript denotes the inner product on $\mathcal{H}$.) Suppose that F and G are in $L^1(\mathbb{R}; \mathcal{B}_0)$. Then

$$\begin{aligned}
\langle (TT^*)_R F,\, G\rangle_{L^\infty(\mathbb{R};\mathcal{B}_0^*)} &= \int_{\mathbb{R}} \left\langle U(t) \int_{-\infty}^{t} U(s)^* F(s)\,\mathrm{d}s,\, G(t) \right\rangle_{\mathcal{B}_0^*} \mathrm{d}t \\
&= \iint_{s<t} \langle U(s)^* F(s),\, U(t)^* G(t)\rangle_{\mathcal{H}}\, \mathrm{d}s\,\mathrm{d}t \qquad (4.19)\\
&= \int_{\mathbb{R}} \left\langle F(s),\, U(s) \int_{s}^{\infty} U(t)^* G(t)\,\mathrm{d}t \right\rangle_{\mathcal{B}_0} \mathrm{d}s \\
&= \langle F,\, (TT^*)_A G\rangle_{L^1(\mathbb{R};\mathcal{B}_0)}
\end{aligned}$$

So formally, $((TT^*)_R)^* = (TT^*)_A$. Inspired by (4.19), define the bilinear form B on $L^1(\mathbb{R}; \mathcal{B}_0) \times L^1(\mathbb{R}; \mathcal{B}_0)$ by

$$B(F, G) = \iint_{s<t} \langle U(s)^* F(s),\, U(t)^* G(t)\rangle\; \mathrm{d}s\,\mathrm{d}t. \qquad (4.20)$$

Lemma 4.3.1. *Suppose that $q, \widetilde{q} \in [1, \infty]$ and $\theta, \widetilde{\theta} \in [0, 1]$. Then the retarded Strichartz estimate (4.16) is equivalent to the bilinear estimate*

$$\begin{aligned}
&|B(F, G)| \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \\
&\qquad \forall F \in L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}}) \cap L^1(\mathbb{R}; \mathcal{B}_0) \quad \forall G \in L^{q'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}}) \cap L^1(\mathbb{R}; \mathcal{B}_0), \quad (4.21)
\end{aligned}$$

where the bilinear form B is given by (4.20).

Proof. By (4.19),

$$B(F,G) = \langle (TT^*)_R F,\, G\rangle\,.$$

Hence if (4.21) holds then, by taking the supremum over all functions G in $L^{q'}(\mathbb{R};\mathcal{B}_\theta)\cap L^1(\mathbb{R};\mathcal{B}_0)$ such that $\|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \leq 1$, we obtain

$$\|(TT^*)_R F\|_{L^q(\mathbb{R};\mathcal{B}^*_\theta)} = \sup_G |B(F,G)| \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})}$$

whenever $F \in L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})\cap L^1(\mathbb{R};\mathcal{B}_0)$.

On the other hand, suppose that (4.16) holds. Then

$$\begin{aligned} |B(F,G)| &\leq \|(TT^*)_R F\|_{L^q(\mathbb{R};\mathcal{B}^*_\theta)}\, \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \\ &\lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})}\, \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)}\,, \end{aligned}$$

by duality. □

The next lemma explains why the conditions on the exponents of the Strichartz estimate (4.16) appearing in Theorem 4.2.2 must be invariant under the symmetry $(q,\theta)\leftrightarrow(\widetilde{q},\widetilde{\theta})$. We exploit the facts that $((TT^*)_R)^* = (TT^*)_A$ and that $(TT^*)_A$ becomes $(TT^*)_R$ when we invert the direction of time.

Lemma 4.3.2. *Suppose that (q,θ) and $(\widetilde{q},\widetilde{\theta})$ are two exponent pairs. If the energy estimate (4.10) and one of the dispersive estimates (4.11) or (4.12) imply the retarded Strichartz estimate (4.16), then*

$$\|(TT^*)_R F\|_{L^{\widetilde{q}}(\mathbb{R};\mathcal{B}^*_{\widetilde{\theta}})} \lesssim \|F\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \qquad \forall F\in L^{q'}(\mathbb{R};\mathcal{B}_\theta)\cap L^1(\mathbb{R};\mathcal{B}_0). \tag{4.22}$$

Proof. Suppose that $\{U(t) : t\in\mathbb{R}\}$ satisfies the energy estimate and one of the dispersive estimates. If $V(t) = U(-t)$ whenever $t\in\mathbb{R}$ then the family $\{V(t) : t\in\mathbb{R}\}$ also satisfies the energy estimate and dispersive estimate. Hence if the operator S is given by $Sf(t) = V(t)f$ then

$$\|(SS^*)_R F\|_{L^q(\mathbb{R};\mathcal{B}^*_{\widetilde{\theta}})} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})} \qquad \forall F\in L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})\cap L^1(\mathbb{R};\mathcal{B}_0),$$

or equivalently

$$\|(SS^*)_A F\|_{L^{\widetilde{q}}(\mathbb{R};\mathcal{B}^*_{\widetilde{\theta}})} \lesssim \|F\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \qquad \forall F\in L^{q'}(\mathbb{R};\mathcal{B}_\theta)\cap L^1(\mathbb{R};\mathcal{B}_0) \tag{4.23}$$

by duality. But

$$\begin{aligned}((SS^*)_A F)(t) &= \int_t^\infty V(t)V(s)^* F(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{-t} V(t)V(-s)^* F(-s)\,\mathrm{d}s \\ &= \int_{-\infty}^{-t} U(-t)U(s)^* F_0(s)\,\mathrm{d}s \\ &= ((TT^*)_R F_0)(-t),\end{aligned}$$

where $F_0(s) = F(-s)$. Therefore (4.23) implies (4.22). $\square$

Later we decompose $B(F,G)$ dyadically as $\sum_{j\in\mathbb{Z}} B_j(F,G)$, where

$$B_j(F,G) = \iint_{t-2^{j+1}<s<t-2^j} \langle U(s)^*F(s),\, U(t)^*G(t)\rangle \,\mathrm{d}s\,\mathrm{d}t. \tag{4.24}$$

Using the same techniques as above, one can prove the following lemma.

Lemma 4.3.3. *Suppose that (q,θ) and $(\widetilde{q},\widetilde{\theta})$ are two exponent pairs and that $C(j)$ is a positive constant for every j in $\mathbb{Z}$. If the energy estimate (4.10) and one of the dispersive estimates (4.11) or (4.12) imply the bilinear estimate*

$$\begin{aligned}&|B_j(F,G)| \le C(j)\,\|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})}\,\|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \\ &\qquad\qquad \forall F\in L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})\cap L^1(\mathbb{R};\mathcal{B}_0) \quad \forall G\in L^{q'}(\mathbb{R};\mathcal{B}_\theta)\cap L^1(\mathbb{R};\mathcal{B}_0)\end{aligned}$$

then

$$\begin{aligned}&|B_j(F,G)| \le C(j)\,\|F\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)}\,\|G\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})} \\ &\qquad\qquad \forall F\in L^{q'}(\mathbb{R};\mathcal{B}_\theta)\cap L^1(\mathbb{R};\mathcal{B}_0) \quad \forall G\in L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})\cap L^1(\mathbb{R};\mathcal{B}_0).\end{aligned}$$

Proof. We adopt the notation of the proof of Lemma (4.3.2) and follow the same strategy. The key observation is that

$$\left|\iint_{t-2^{j+1}<s<t-2^j} \langle V(s)^*F(s),\, V(t)^*G(t)\rangle \,\mathrm{d}s\,\mathrm{d}t\right| = |B_j(G_0,F_0)|$$

where $F_0(s) = F(-s)$ and $G_0(t) = G(-t)$. The details are left to the reader. $\square$

So far we have shown that an appropriate estimate on the bilinear form B implies the inhomogeneous Strichartz estimate (4.16). The same can be said for the homogeneous Strichartz estimates.

Lemma 4.3.4. *Suppose that $q \in [1, \infty]$ and $\theta \in [0, 1]$. If*

$$|B(F, G)| \lesssim \|F\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \qquad \forall F, G \in L^{q'}(\mathbb{R}; \mathcal{B}_\theta) \cap L^1(\mathbb{R}; \mathcal{B}_0) \tag{4.25}$$

then the homogeneous Strichartz estimates (4.14) and (4.15) hold.

Proof. We first show that the bilinear estimate

$$\left| \iint_{\mathbb{R}^2} \langle U(s)^* F(s),\, U(t)^* G(t) \rangle \,\mathrm{d}s\, \mathrm{d}t \right| \lesssim \|F\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \qquad \forall F, G \in L^{q'}(\mathbb{R}; \mathcal{B}_\theta) \cap L^1(\mathbb{R}; \mathcal{B}_0) \tag{4.26}$$

is equivalent to (4.15) (and by duality to (4.14) also). If (4.26) holds then estimate (4.15) is easily derived by taking F equal to G. Conversely, if (4.15) holds then the Cauchy–Schwarz inequality gives (4.26).

Now observe that

$$\left| \iint_{\mathbb{R}^2} \langle U(s)^* F(s),\, U(t)^* G(t) \rangle \,\mathrm{d}s\, \mathrm{d}t \right| \le |B(F, G)| + |B(G, F)|.$$

Hence (4.25) implies (4.26) and the proof is complete. □

The next lemma will help with the transparency of future proofs.

Lemma 4.3.5. *Suppose that I is an interval of the real line. If (4.15) holds then*

$$\left\| \int_I U(s)^* F(s)\, \mathrm{d}s \right\|_{\mathcal{H}} \lesssim \|F\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \qquad \forall F \in L^{q'}(\mathbb{R}; \mathcal{B}_\theta). \tag{4.27}$$

Proof. Let 1_I denote the characteristic function of I on $\mathbb{R}$. Replacing F by $1_I F$ in (4.15) yields the result. □

The following lemma will later be used in conjunction with Lemma 3.5.2 to demonstrate the necessity of some exponent conditions.

Lemma 4.3.6. *If $\{U(t) : t \in \mathbb{R}\}$ has the group property*

$$U(t)U(s)^* = U(t-s) \qquad \forall s,t \in \mathbb{R} \tag{4.28}$$

then $(TT^*)_R$ is translation invariant.

Proof. If (4.28) holds then

$$\begin{aligned}\tau_y (TT^*)_R F(t) &= \int_{-\infty}^{t-y} U(t-y-s)F(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{t} U(t-v)F(v-y)\,\mathrm{d}v \\ &= (TT^*)_R \tau_y F(t)\end{aligned}$$

for any translation τ_y, where $y \in \mathbb{R}$. □

In later sections, scaling arguments will be used to simplify the proof of Theorem 4.2.2 for the sharp σ-admissible case. The hypotheses for this case (that is, estimates (4.10) and (4.11)) are invariant under many rescalings; we regard the one introduced below to be the simplest.

Proposition 4.3.7. *If $\lambda > 0$ then the estimates (4.10) and (4.11) are invariant under the scaling*

$$\begin{cases} U(t) & \leftarrow U(t/\lambda) \\ \langle f,\, g\rangle & \leftarrow \langle f,\, g\rangle \\ \|f\|_{\mathcal{B}_0} & \leftarrow \|f\|_{\mathcal{B}_0} \\ \|f\|_{\mathcal{B}_1} & \leftarrow \lambda^{\sigma/2}\,\|f\|_{\mathcal{B}_1}\,. \end{cases} \tag{4.29}$$

Before attending to the proof, we give a two helpful lemmata related to the rescaling of Banach space norms.

Lemma 4.3.8. *Suppose that $\mathcal{B}$ is a Banach space and $\lambda > 0$. Then the scaling*

$$\|f\|_{\mathcal{B}} \leftarrow \lambda\,\|f\|_{\mathcal{B}} \qquad \forall f \in \mathcal{B} \tag{4.30}$$

induces the scaling

$$\|\phi\|_{\mathcal{B}^*} \leftarrow \lambda^{-1}\,\|\phi\|_{\mathcal{B}^*} \qquad \forall \phi \in \mathcal{B}^*.$$

Proof. Let $\mathcal{B}'$ denote the Banach space induced from $\mathcal{B}$ under scaling (4.30). Then

$$\begin{aligned}
\|\phi\|_{(\mathcal{B}')^*} &= \sup\left\{ |\langle f,\, \phi\rangle| : \|f\|_{\mathcal{B}'} = 1 \right\} \\
&= \sup\left\{ |\langle f,\, \phi\rangle| : \|\lambda f\|_{\mathcal{B}} = 1 \right\} \\
&= \sup\left\{ |\left\langle \lambda^{-1} g,\, \phi\right\rangle| : \|g\|_{\mathcal{B}} = 1 \right\} \\
&= \lambda^{-1} \|\phi\|_{\mathcal{B}^*}
\end{aligned}$$

as required. ☐

Lemma 4.3.9. *If $\theta \in [0,1]$ then scaling (4.29) implies that*

$$\|f\|_{\mathcal{B}_\theta} \leftarrow \lambda^{\sigma\theta/2} \|f\|_{\mathcal{B}_\theta}.$$

Proof. Whenever $\vartheta \in [0,1]$, let $\mathcal{B}'_\vartheta$ denote the Banach space induced from $\mathcal{B}_\vartheta$ under scaling (4.29). Suppose that $f \in \mathcal{B}_\theta$. Recalling (3.1), we have

$$\begin{aligned}
K(t, f, \mathcal{B}'_0, \mathcal{B}'_1) &= \inf\left\{ \|f_0\|_{\mathcal{B}_0} + t\lambda^{\sigma/2} \|f_1\|_{\mathcal{B}_1} : f_0 + f_1 = f \right\} \\
&= K(\lambda^{\sigma/2} t, f, \mathcal{B}_0, \mathcal{B}_1)
\end{aligned}$$

whenever $t > 0$. Hence, by (3.2),

$$\begin{aligned}
\|f\|_{\mathcal{B}'_\theta} &= \left(\int_0^\infty \left(t^{-\theta} K(t, f, \mathcal{B}'_0, \mathcal{B}'_1) \right)^2 \frac{\mathrm{d}t}{t} \right)^{1/2} \\
&= \left(\int_0^\infty \left(t^{-\theta} K(\lambda^{\sigma/2} t, f, \mathcal{B}_0, \mathcal{B}_1) \right)^2 \frac{\mathrm{d}t}{t} \right)^{1/2} \\
&= \left(\int_0^\infty \left(\left(\lambda^{-\sigma/2} s\right)^{-\theta} K(s, f, \mathcal{B}_0, \mathcal{B}_1) \right)^2 \frac{\mathrm{d}s}{s} \right)^{1/2} \\
&= \lambda^{\sigma\theta/2} \|f\|_{\mathcal{B}_\theta}
\end{aligned}$$

and the lemma follows. ☐

Proof of Proposition 4.3.7. It is obvious that (4.10) is invariant with respect to the rescaling (4.29). For the second estimate (4.11), there is a constant C such that

$$\|U(t)U(s)^* g\|_{\mathcal{B}_1^*} \le C|t-s|^{-\sigma} \|g\|_{\mathcal{B}_1} \qquad \forall g \in \mathcal{B}_1 \cap \mathcal{B}_0 \quad \forall \text{ real } s \neq t.$$

When scaling (4.29) is applied,

$$\lambda^{-\sigma/2} \left\| U(t/\lambda)U(s/\lambda)^* g \right\|_{\mathcal{B}_1^*} \leq C|t-s|^{-\sigma} \lambda^{\sigma/2} \left\| g \right\|_{\mathcal{B}_1} \quad \forall g \in \mathcal{B}_1 \cap \mathcal{B}_0 \quad \forall \text{ real } s \neq t.$$

The substitutions $s \mapsto s/\lambda$ and $t \mapsto t/\lambda$ yield

$$\left\| U(t)U(s)^* g \right\|_{\mathcal{B}_1^*} \leq C|\lambda t - \lambda s|^{-\sigma} \lambda^{\sigma} \left\| g \right\|_{\mathcal{B}_1} \qquad \forall g \in \mathcal{B}_1 \cap \mathcal{B}_0 \quad \forall \text{ real } s \neq t,$$

from which we recover (4.11). $\square$

4.4 Proof of the homogeneous estimates

We now begin the proof of Theorem 4.2.2. In this section, we derive homogeneous Strichartz estimates under the assumption that the family $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate (4.10) and one of the decay estimates (4.11) or (4.12). The basic strategy is as follows. We need to find conditions under which the bilinear estimate (4.25) holds. We begin by interpolating between the energy estimate and one of the decay estimates. This yields a new estimate, which we further manipulate via a classical L^p inequality, to establish (4.26). This last step imposes conditions on the spacetime exponent pair (q, θ); these are precisely the σ-admissibility criteria defined in Section 4.2. The classical inequalities used are the Young, Hölder and Hardy–Littlewood–Sobolev inequalities; these are stated in Section 3.1.

Recall that the exponent endpoint P is $(2, 1/\sigma)$ whenever $\sigma > 1$.

Theorem 4.4.1. *Suppose that $\sigma > 0$ and that $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate (4.10) and the untruncated decay estimate (4.11). If (q, θ) is sharp σ-admissible and $(q, \theta) \neq P$ then the homogeneous Strichartz estimates (4.14) and (4.15) and the bilinear estimate (4.25) hold.*

Proof. In view of Lemma 4.3.4, it suffices to establish (4.25). Suppose that F

and G belong to $L^{q'}(\mathbb{R};\mathcal{B}_1)\cap L^1(\mathbb{R};\mathcal{B}_0)$. On the one hand, (4.11) implies that

$$\begin{aligned} |\langle U(s)^*F(s),\, U(t)^*G(t)\rangle| &= |\langle F(s),\, U(s)U(t)^*G(t)\rangle| \\ &\le \|F(s)\|_{\mathcal{B}_1}\|U(s)U(t)^*G(t)\|_{\mathcal{B}_1^*} \\ &\lesssim |t-s|^{-\sigma}\|F(s)\|_{\mathcal{B}_1}\|G(t)\|_{\mathcal{B}_1}, \end{aligned} \tag{4.31}$$

while on the other hand, the dual

$$\|U(t)^*g\|_{\mathcal{H}} \lesssim \|g\|_{\mathcal{B}_0} \qquad \forall t\in\mathbb{R} \quad \forall g\in\mathcal{B}_0 \tag{4.32}$$

of the energy estimate (4.10) implies that

$$|\langle U(s)^*F(s),\, U(t)^*G(t)\rangle| \lesssim \|F(s)\|_{\mathcal{B}_0}\|G(t)\|_{\mathcal{B}_0}. \tag{4.33}$$

Integrating (4.33) with respect to s and t gives (4.25) when $(q,\theta)=(\infty,0)$. Suppose now that $(q,\theta)\neq(\infty,0)$. Real interpolation (see Theorem 3.2.4) between (4.31) and (4.33) gives

$$|\langle U(s)^*F(s),\, U(t)^*G(t)\rangle| \lesssim |t-s|^{-\sigma\theta}\|F(s)\|_{\mathcal{B}_\theta}\|G(t)\|_{\mathcal{B}_\theta}, \tag{4.34}$$

where $\theta\in(0,1]$. By an application of the triangle inequality,

$$\left|\iint_{s<t}\langle U(s)^*F(s),\, U(t)^*G(t)\rangle\,\mathrm{d}s\,\mathrm{d}t\right| \lesssim \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\|F(s)\|_{\mathcal{B}_\theta}\|G(t)\|_{\mathcal{B}_\theta}}{|t-s|^{\sigma\theta}}\,\mathrm{d}s\,\mathrm{d}t. \tag{4.35}$$

To deduce (4.26), we need only estimate the right hand side of (4.35) by the Hardy–Littlewood–Sobolev inequality. We require that $0<\sigma\theta<1$ and that $2/q'+\sigma\theta=2$; this is guaranteed by one of the hypotheses. Hence

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\|F(s)\|_{\mathcal{B}_\theta}\|G(t)\|_{\mathcal{B}_\theta}}{|t-s|^{\sigma\theta}}\,\mathrm{d}s\,\mathrm{d}t \lesssim \|F\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)}\|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)}$$

which, together with (4.35), establishes (4.25) as required. $\square$

Note that the method of the proof above covers sharp admissible pairs (q,θ) when $2<q\le\infty$. However, the Hardy–Littlewood–Young inequality cannot handle the case when $q=2$ (that is, when (q,θ) is the endpoint P).

Establishing homogeneous Strichartz estimates for this case is a more delicate problem and we defer this task to Section 4.5.

By strengthening the decay hypothesis, homogeneous Strichartz estimates are obtained for a larger set of exponent pairs (q, θ).

Theorem 4.4.2. *Suppose that $\sigma > 0$ and that $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate (4.10) and the truncated decay estimate (4.12). If (q, θ) is nonsharp σ-admissible, then the homogeneous Strichartz estimates (4.14) and (4.15) and the bilinear estimate (4.25) hold.*

Proof. Suppose that F and G are functions in $L^{q'}(\mathbb{R}; \mathcal{B}_1) \cap L^1(\mathbb{R}; \mathcal{B}_0)$. To begin we follow the method of the first half of the proof of Theorem 4.4.1. However, since we are using the truncated estimate, we obtain

$$|\langle U(s)^*F(s), U(t)^*G(t)\rangle| \lesssim (1 + |t - s|)^{-\sigma\theta} \|F(s)\|_{\mathcal{B}_\theta} \|G(t)\|_{\mathcal{B}_\theta} \tag{4.36}$$

rather than (4.34) when $\theta \in [0, 1]$. Suppose now that $f(t) = \|F(t)\|_{\mathcal{B}_\theta}$, $g(t) = \|G(t)\|_{\mathcal{B}_\theta}$ and $h(t) = (1 + |t|)^{-\sigma\theta}$. Successive applications of the triangle inequality, estimate (4.36), Hölder's inequality and Young's inequality give

$$\begin{aligned}
\left|\iint_{s<t} \langle U(s)^*F(s), U(t)^*G(t)\rangle \,\mathrm{d}s\,\mathrm{d}t\right| &\lesssim \|h * f \,.\, g\|_1 \\
&\le \|h * f\|_q \|g\|_{q'} \\
&\le \|h\|_{q/2} \|f\|_{q'} \|g\|_{q'} \\
&\lesssim \|F\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)},
\end{aligned}$$

provided that $h \in L^{q/2}(\mathbb{R})$. This occurs precisely when $q\sigma\theta/2 > 1$, which is guaranteed by one of the hypotheses. Hence (4.25) is satisfied. □

4.5 Proof of the endpoint estimate

The methods of the previous section were not able to prove the homogeneous Strichartz estimate for the endpoint P. We now deal with this special case.

Throughout this section, suppose that $\sigma > 1$,

$$P = (q, \theta) = (2, 1/\sigma) \tag{4.37}$$

and $\{U(t) : t \in \mathbb{R}\}$ satisfies both the energy estimate (4.10) and the untruncated decay estimate (4.11). Our goal is to prove the following.

Theorem 4.5.1. *Under the assumptions above, the homogeneous Strichartz estimates (4.14) and (4.15) and the bilinear estimate (4.25) hold.*

In view of Lemma 4.3.4, it suffices to show (4.25), where $B(F, G)$ is defined by (4.20). To do this, we decompose $B(F, G)$ dyadically as $\sum_{j\in\mathbb{Z}} B_j(F, G)$, where each B_j is defined by (4.24). Note that if scaling (4.29) is applied to the estimate

$$|B(F, G)| \leq C \|F\|_{L^2(\mathbb{R};\mathcal{B}_a)} \|G\|_{L^2(\mathbb{R};\mathcal{B}_b)},$$

then

$$|B(F_0, G_0)| \leq C\lambda^{\frac{\sigma}{2}(a+b)-1} \|F_0\|_{L^2(\mathbb{R};\mathcal{B}_a)} \|G_0\|_{L^2(\mathbb{R};\mathcal{B}_b)}$$

where $F_0(s) = F(\lambda s)$ and $G_0(t) = G(\lambda t)$ (see Step 1 of the proof below where we rehearse the necessary calculations). Inspired by this observation, define the quantity $\beta(a, b)$ by the formula

$$\beta(a, b) = \frac{\sigma}{2}(a + b) - 1 \tag{4.38}$$

whenever $a, b \in [0, 1]$. For positive ϵ, define the set Ψ_ϵ by

$$\Psi_\epsilon = \{(a, b) \in [0, 1] \times [0, 1] : 0 \leq |a - 1/\sigma| < \epsilon, 0 \leq |b - 1/\sigma| < \epsilon\}.$$

The following two-parameter family of estimates for the dyadic parts B_j will be proved.

Lemma 4.5.2. *There is a positive ϵ such that, for every j in $\mathbb{Z}$,*

$$|B_j(F, G)| \lesssim 2^{-j\beta(a,b)} \|F\|_{L^2(\mathbb{R};\mathcal{B}_a)} \|G\|_{L^2(\mathbb{R};\mathcal{B}_b)}$$
$$\forall F \in L^2(\mathbb{R}; \mathcal{B}_a) \cap L^1(\mathbb{R}; \mathcal{B}_0) \quad \forall G \in L^2(\mathbb{R}; \mathcal{B}_b) \cap L^1(\mathbb{R}; \mathcal{B}_0) \tag{4.39}$$

whenever $(a, b) \in \Psi_\epsilon$.

Proof. The proof will take place in three steps, the first two of which will dramatically simplify the problem. In Step 1, we show that if (4.39) holds when $j = 0$ then it holds for any j in $\mathbb{Z}$. In Step 2, we take j equal to 0 and show that if (4.39) holds for all F and G which have compact support in an interval of length 2, then (4.39) holds for all F and G. To complete the proof it suffices to show that there is a positive ϵ such that, when $j = 0$ and every function F and G has support on an interval of length 2, (4.39) holds whenever $(a, b) \in \Psi_\epsilon$. This is achieved in Step 3.

Step 1. Suppose that the hypotheses of this section implies that (4.39) holds when $j = 0$. By Proposition 4.3.7, the hypotheses are invariant under the rescaling (4.29). Hence the hypotheses also imply that the rescaled version of (4.39) (when $j = 0$), given by

$$\begin{aligned}
&\left| \iint_{t-2<s<t-1} \langle U(s/\lambda)^* F(s),\, U(t/\lambda)^* G(t)\rangle \,\mathrm{d}s\,\mathrm{d}t \right| \\
&\qquad\qquad \le C\,\lambda^{\sigma a/2}\,\|F\|_{L^2(\mathbb{R};\mathcal{B}_a)}\,\lambda^{\sigma b/2}\,\|G\|_{L^2(\mathbb{R};\mathcal{B}_b)} \\
&\qquad\qquad \forall F \in L^2(\mathbb{R};\mathcal{B}_a)\cap L^1(\mathbb{R};\mathcal{B}_0) \quad \forall G \in L^2(\mathbb{R};\mathcal{B}_b)\cap L^1(\mathbb{R};\mathcal{B}_0),
\end{aligned}$$

holds. If $F_0(s) = F(\lambda s)$ and $G_0(t) = G(\lambda t)$, then a change of variables on the left-hand side gives

$$\begin{aligned}
&\lambda^2 \left| \iint_{t-2\lambda^{-1}<s<t-\lambda^{-1}} \langle U(s)^* F_0(s),\, U(t)^* G_0(t)\rangle \,\mathrm{d}s\,\mathrm{d}t \right| \\
&\qquad\qquad \le C\,\lambda^{\sigma(a+b)/2+1}\,\|F_0\|_{L^2(\mathbb{R};\mathcal{B}_a)}\,\|G_0\|_{L^2(\mathbb{R};\mathcal{B}_b)} \\
&\qquad\qquad \forall F_0 \in L^2(\mathbb{R};\mathcal{B}_a)\cap L^1(\mathbb{R};\mathcal{B}_0) \quad \forall G_0 \in L^2(\mathbb{R};\mathcal{B}_b)\cap L^1(\mathbb{R};\mathcal{B}_0).
\end{aligned}$$

Now cancel λ^2 from both sides and set λ equal to 2^{-j}. This gives

$$\begin{aligned}
&|B_j(F_0, G_0)| \le C\,2^{-j\beta(a,b)}\,\|F_0\|_{L^2(\mathbb{R};\mathcal{B}_a)}\,\|G_0\|_{L^2(\mathbb{R};\mathcal{B}_j)} \\
&\qquad\qquad \forall F_0 \in L^2(\mathbb{R};\mathcal{B}_a)\cap L^1(\mathbb{R};\mathcal{B}_0) \quad \forall G_0 \in L^2(\mathbb{R};\mathcal{B}_b)\cap L^1(\mathbb{R};\mathcal{B}_0),
\end{aligned}$$

completing Step 1.

Step 2. Suppose that $j = 0$ and that (4.39) holds with the modification that every F and G has support on an interval of length 2. Now take any F

in $L^2(\mathbb{R};\mathcal{B}_a) \cap L^1(\mathbb{R};\mathcal{B}_0)$ and any G in $L^2(\mathbb{R};\mathcal{B}_b) \cap L^1(\mathbb{R};\mathcal{B}_0)$ (not necessarily with compact support). We will show that

$$|B_0(F,G)| \lesssim \|F\|_{L^2(\mathbb{R};\mathcal{B}_a)} \|G\|_{L^2(\mathbb{R};\mathcal{B}_b)} .$$

Decompose F and G each as a sum

$$F(t) = \sum_{m\in\mathbb{Z}} \phi_m(t)F(t), \qquad G(t) = \sum_{m\in\mathbb{Z}} \phi_m(t)G(t),$$

where $\phi_0 \in C_0^\infty(\mathbb{R})$, $0 \le \phi_0 \le 1$, $\phi_0 = 0$ outside the interval $[-1,1]$, $\phi_m(t) = \phi_0(t-m)$ whenever $m \in \mathbb{Z}$ and $t \in \mathbb{R}$, and $\sum_{m\in\mathbb{Z}} \phi_m = 1$. (Such a sequence $\{\phi_m\}_{m\in\mathbb{Z}}$ of functions exists by the usual construction of a *partition of unity*.) Then

$$\begin{aligned}
B_0(F,G) &= \iint_{t-2<s<t-1} \langle U(s)^*F(s),\, U(t)^*G(t)\rangle \,\mathrm{d}s\,\mathrm{d}t \\
&= \sum_{m,n\in\mathbb{Z}} \iint_{t-2<s<t-1} \langle U(s)^*\phi_m(s)F(s),\, U(t)^*\phi_n(t)G(t)\rangle \,\mathrm{d}s\,\mathrm{d}t \\
&= \sum_{|m-n|\le 3} \iint_{t-2<s<t-1} \langle U(s)^*\phi_m(s)F(s),\, U(t)^*\phi_n(t)G(t)\rangle \,\mathrm{d}s\,\mathrm{d}t \\
&= \sum_{j=-3}^{3}\sum_{m\in\mathbb{Z}} \iint_{t-2<s<t-1} \langle U(s)^*\phi_m(s)F(s),\, U(t)^*\phi_{m+j}(t)G(t)\rangle \,\mathrm{d}s\,\mathrm{d}t \\
&= \sum_{j=-3}^{3}\sum_{m\in\mathbb{Z}} B_0(\phi_m F, \phi_{m+j} G).
\end{aligned}$$

Since $\phi_m F$ and $\phi_{m+j} G$ both have support on an interval of length 2, there exists a positive constant C such that

$$\begin{aligned}
|B_0(F,G)| &\le C \sum_{j=-3}^{3}\sum_{m\in\mathbb{Z}} \|\phi_m F\|_{L^2(\mathbb{R};\mathcal{B}_a)} \|\phi_{m+j} G\|_{L^2(\mathbb{R};\mathcal{B}_b)} \\
&\le C \sum_{j=-3}^{3} \left(\sum_{m\in\mathbb{Z}} \|\phi_m F\|^2_{L^2(\mathbb{R};\mathcal{B}_a)}\right)^{1/2} \left(\sum_{m\in\mathbb{Z}} \|\phi_{m+j} G\|^2_{L^2(\mathbb{R};\mathcal{B}_b)}\right)^{1/2} \\
&\le C' \left(\sum_{m\in\mathbb{Z}} \|\phi_m F\|^2_{L^2(\mathbb{R};\mathcal{B}_a)}\right)^{1/2} \left(\sum_{m\in\mathbb{Z}} \|\phi_m G\|^2_{L^2(\mathbb{R};\mathcal{B}_b)}\right)^{1/2},
\end{aligned}$$

where the second estimate is justified by Hölder's inequality. By further estimation,

$$\sum_{m\in\mathbb{Z}} \|\phi_m F\|^2_{L^2(\mathbb{R};\mathcal{B}_a)} = \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \phi_m(t)^2 \|F(t)\|^2_{\mathcal{B}_a} \,\mathrm{d}t \le \|F\|^2_{L^2(\mathbb{R};\mathcal{B}_a)},$$

since

$$0 < \sum_{m\in\mathbb{Z}} \phi_m^2 \le 1.$$

Therefore

$$|B_0(F,G)| \le C' \|F\|_{L^2(\mathbb{R};\mathcal{B}_a)} \|G\|_{L^2(\mathbb{R};\mathcal{B}_b)},$$

as required.

Step 3. If $\mathcal{B}$ is a Banach space, let $L^p_c(\mathbb{R};\mathcal{B})$ denote the subset of functions of $L^p(\mathbb{R};\mathcal{B})$ whose supports are contained in intervals of length 2. In light of previous steps, our task is to find a positive ϵ such that

$$|B_0(F,G)| \lesssim \|F\|_{L^2(\mathbb{R};\mathcal{B}_a)} \|G\|_{L^2(\mathbb{R};\mathcal{B}_b)}$$
$$\forall F \in L^2_c(\mathbb{R};\mathcal{B}_a) \cap L^1(\mathbb{R};\mathcal{B}_0) \quad \forall G \in L^2_c(\mathbb{R};\mathcal{B}_b) \cap L^1(\mathbb{R};\mathcal{B}_0) \tag{4.40}$$

whenever $(a,b) \in \Psi_\epsilon$. Our approach is to prove (4.40) for the pair (a,b) in the following cases:

(i) $a = b = 1$;

(ii) $0 < a < 1/\sigma, \qquad b = 0$;

(iii) $0 < b < 1/\sigma, \qquad a = 0$; and

(iv) $a = b = 0$.

Since $(1/\sigma, 1/\sigma)$ lies in the interior of the convex hull of cases (i) to (iv) (see Figure 4.2), Step 3 will follow from real interpolation (Theorem 3.3.5).

To prove (i), observe that (4.31) gives

$$\begin{aligned} |B_0(F,G)| &\le \iint_{t-2<s<t-1} |\langle U(s)^*F(s),\, U(t)^*G(t)\rangle| \,\mathrm{d}s\,\mathrm{d}t \\ &\lesssim \iint_{t-2<s<t-1} |t-s|^{-\sigma} \|F(s)\|_{\mathcal{B}_1} \|G(t)\|_{\mathcal{B}_1} \,\mathrm{d}s\,\mathrm{d}t \\ &\le \|F\|_{L^1(\mathbb{R};\mathcal{B}_1)} \|G\|_{L^1(\mathbb{R};\mathcal{B}_1)}. \end{aligned}$$

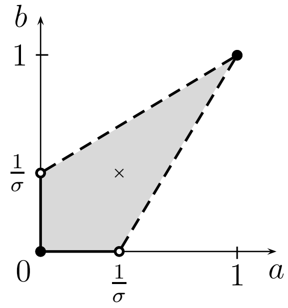


Figure 4.2: Interpolation region for Step 3.

We now move to the proof of (ii). By the triangle and Cauchy–Schwarz inequalities,

$$\begin{aligned}
|B_0(F,G)| &\leq \int_{\mathbb{R}} \left| \left\langle \int_{t-2<s<t-1} U(s)^* F(s)\,\mathrm{d}s,\, U(t)^* G(t) \right\rangle \right| \mathrm{d}t \\
&\leq \left(\sup_{t\in\mathbb{R}} \left\| \int_{t-2<s<t-1} U(s)^* F(s)\,\mathrm{d}s \right\|_{\mathcal{H}} \right) \int_{\mathbb{R}} \| U(t)^* G(t) \|_{\mathcal{H}} \,\mathrm{d}t. \quad (4.41)
\end{aligned}$$

The first term of the above product can be estimated using Lemma 4.3.5 and Theorem 4.4.1 applied to the sharp σ-admissible pair $(q(a), a)$, where $q(a) = 2(\sigma a)^{-1}$. The dual (4.32) of the energy estimate gives a bound for the second term. Thus

$$|B_0(F,G)| \lesssim \|F\|_{L^{q(a)}(\mathbb{R};\mathcal{B}_a)} \|G\|_{L^1(\mathbb{R};\mathcal{B}_0)}\,.$$

Since F has compact support we can write F as $F = 1_I F$, where 1_I is the characteristic function of some interval I of length 2. By Hölder's inequality,

$$\|F\|_{L^{q(a)'}(\mathbb{R};\mathcal{B}_a)} \leq \|1_I\|_{L^p(\mathbb{R})} \|F\|_{L^2(\mathbb{R};\mathcal{B}_a)} \leq 2\,\|F\|_{L^2(\mathbb{R};\mathcal{B}_a)}\,, \quad (4.42)$$

where $1/q(a)' = 1/p + 1/2$. We estimate $\|G\|_{L^1(\mathbb{R};\mathcal{B}_0)}$ similarly to obtain

$$|B_0(F,G)| \lesssim \|F\|_{L^2(\mathbb{R};\mathcal{B}_a)} \|G\|_{L^2(\mathbb{R};\mathcal{B}_0)}\,.$$

By symmetry (see Lemma 4.3.3), (iii) follows from (ii). To prove (iv), successive applications of the triangle inequality, Cauchy-Schwarz inequality and the dual (4.32) of the energy estimate give

$$|B_0(F,G)| \lesssim \|F\|_{L^1(\mathbb{R};\mathcal{B}_0)} \|G\|_{L^1(\mathbb{R};\mathcal{B}_0)}\,.$$

The technique used in (4.42) yields the result. This completes Step 3. $\square$

To prove Theorem 4.5.1 it suffices to show (4.25). One would hope that this could be achieved, via (4.39), by

$$\begin{aligned}|B(F,G)| &\le \sum_{j\in\mathbb{Z}} |B_j(F,G)| \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{-j\beta(1/\sigma,1/\sigma)} \|F\|_{L^2(\mathbb{R};\mathcal{B}_{1/\sigma})} \|G\|_{L^2(\mathbb{R};\mathcal{B}_{1/\sigma})} \\ &\lesssim \|F\|_{L^2(\mathbb{R};\mathcal{B}_{1/\sigma})} \|G\|_{L^2(\mathbb{R};\mathcal{B}_{1/\sigma})}.\end{aligned}$$

However, $\beta(1/\sigma, 1/\sigma) = 0$ so the sum diverges. Instead, we slightly perturb the exponent pair $(1/\sigma, 1/\sigma)$, obtaining three estimates of the form (4.39). An abstract real interpolation argument, applied to the three estimates, then gives the homogeneous Strichartz estimate for the endpoint.

Proof of Theorem 4.5.1. In light of Lemma 4.3.4 and the triangle inequality, it suffices to show that

$$\sum_{j\in\mathbb{Z}} |B_j(F,G)| \lesssim \|F\|_{L^2(\mathbb{R};\mathcal{B}_{1/\sigma})} \|G\|_{L^2(\mathbb{R};\mathcal{B}_{1/\sigma})}$$

whenever F and G belong to $L^2(\mathbb{R};\mathcal{B}_{1/\sigma}) \cap L^1(\mathbb{R};\mathcal{B}_0)$. Define a function $\widetilde{B}$ on $L^1(\mathbb{R};\mathcal{B}_0) \times L^1(\mathbb{R};\mathcal{B}_0)$ by $\widetilde{B}(F,G) = \{B_j(F,G)\}_{j\in\mathbb{Z}}$. Recall the definition of ℓ^s_q given by (3.6). If we can show that the map

$$\widetilde{B} : L^2(\mathbb{R};\mathcal{B}_{1/\sigma}) \times L^2(\mathbb{R};\mathcal{B}_{1/\sigma}) \to \ell^1_0 \tag{4.43}$$

is bounded then the proof will be complete.

By Lemma 4.5.2 there is a positive ϵ such that the map

$$\widetilde{B} : L^2(\mathbb{R};\mathcal{B}_a) \times L^2(\mathbb{R};\mathcal{B}_b) \to \ell^\infty_{\beta(a,b)} \tag{4.44}$$

is bounded for all (a,b) in the set Ψ_ϵ. We carefully choose three points (a,b) in Ψ_ϵ so that interpolating between (4.44) for these three points gives (4.43).

Suppose that $a_0 = b_0 = 1/\sigma + \epsilon/3$ and $a_1 = b_1 = 1/\sigma - 2\epsilon/3$. Then

$$\beta(a_0,b_1) = \beta(a_1,b_0) \ne \beta(a_0,b_0)$$

and the maps

$$\begin{aligned}\widetilde{B} :& L^2(\mathbb{R};\mathcal{B}_{a_0}) \times L^2(\mathbb{R};\mathcal{B}_{b_0}) \to \ell^\infty_{\beta(a_0,b_0)}\\ \widetilde{B} :& L^2(\mathbb{R};\mathcal{B}_{a_0}) \times L^2(\mathbb{R};\mathcal{B}_{b_1}) \to \ell^\infty_{\beta(a_0,b_1)}\\ \widetilde{B} :& L^2(\mathbb{R};\mathcal{B}_{a_1}) \times L^2(\mathbb{R};\mathcal{B}_{b_0}) \to \ell^\infty_{\beta(a_1,b_0)}\end{aligned}$$

are bounded by (4.44). From Theorem 3.2.4 we deduce that the map

$$\begin{aligned}\widetilde{B} : \big(L^2(\mathbb{R};\mathcal{B}_{a_0}), L^2(\mathbb{R};\mathcal{B}_{a_1})\big)_{\eta_0,2} \times \big(L^2(\mathbb{R};\mathcal{B}_{b_0}), L^2(\mathbb{R};\mathcal{B}_{b_1})\big)_{\eta_1,2}&\\ \to \big(\ell^\infty_{\beta(a_0,b_0)}, \ell^\infty_{\beta(a_0,b_1)}\big)_{\eta,1}&\end{aligned} \tag{4.45}$$

is bounded, where $\eta_0 = \eta_1 = \frac{1}{3}$ and $\eta = \eta_0 + \eta_1$. It is easy to check that

$$(1-\eta)\beta(a_0,b_0) + \eta\beta(a_0,b_1) = \beta(1/\sigma,1/\sigma) = 0.$$

If we combine this with (3.7) then (4.45) simplifies to

$$\widetilde{B} : L^2(\mathbb{R};\mathcal{B}_{1/\sigma}) \times L^2(\mathbb{R};\mathcal{B}_{1/\sigma}) \to \ell^1_0.$$

Hence (4.43) is bounded, as desired. □

4.6 Proof of the inhomogeneous estimates

In this section, we prove the inhomogeneous Strichartz estimate (4.16) for σ-admissible pairs. The problem of finding other exponent pairs for which (4.16) is valid will be examined in Chapter 5.

Theorem 4.6.1. *Suppose that $\sigma > 0$ and that $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate (4.10) and the untruncated decay estimate (4.11). If the pairs (q,θ) and $(\widetilde{q},\widetilde{\theta})$ are sharp σ-admissible then the retarded Strichartz estimate (4.16) and the bilinear estimate (4.21) hold.*

Proof. Suppose that (q,θ) and $(\widetilde{q},\widetilde{\theta})$ are sharp σ-admissible. By Lemma 4.3.1 it suffices to show (4.21). Observe that $(\infty,0)$ is sharp σ-admissible and that the three points $(1/q', \theta)$, $(1/\widetilde{q}', \widetilde{\theta})$ and $(1/\infty', 0)$ are therefore collinear. We break the proof into three cases.

Case 1. If $(q, \theta) = (\widetilde{q}, \widetilde{\theta})$ then Theorem 4.4.1 implies that

$$|B(F, G)| \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})} \|G\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})} \tag{4.46}$$

and the proof is complete.

Case 2. Suppose that $(1/q', \theta)$ lies closer to $(1/\infty', 0)$ than does $(1/\widetilde{q}', \widetilde{\theta})$. If $F \in L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}}) \cap L^1(\mathbb{R}; \mathcal{B}_0)$ and $G \in L^1(\mathbb{R}; \mathcal{B}_0)$ then the same argument used to derive (4.41) gives

$$|B(F, G)| \leq \left(\sup_{t \in \mathbb{R}} \left\| \int_{s<t} U(s)^* F(s) \, \mathrm{d}s \right\|_{\mathcal{H}} \right) \int_{\mathbb{R}} \|U(t)^* G(t)\|_{\mathcal{H}} \, \mathrm{d}t. \tag{4.47}$$

We use Lemma 4.3.5 to estimate the first term of the product and use the dual (4.32) of the energy estimate to estimate the second term. This gives

$$|B(F, G)| \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})} \|G\|_{L^{\infty'}(\mathbb{R};\mathcal{B}_0)} . \tag{4.48}$$

Complex interpolation (see Lemma 3.3.6) between (4.46) and (4.48) yields (4.21).

Case 3. Finally, if $(1/\widetilde{q}', \widetilde{\theta})$ lies closer to $(1/\infty', 0)$ than does $(1/q', \theta)$ then (4.21) follows from Case 2 and symmetry (see Lemma 4.3.2). $\square$

The proof of Theorem 4.2.2 is almost complete. We only need to show that (4.16) holds for σ-admissible pairs under the truncated decay hypothesis (4.12). The strategy will be to prove the bilinear estimate (4.21) for σ-admissible exponents which lie on the boundary of the admissibility region and then interpolate between them. We already have estimates for the boundary corresponding to sharp σ-admissible pairs. The other necessary boundary estimates are deduced in the following technical lemma, which is a crude variant of Lemma 4.5.2.

Lemma 4.6.2. *Suppose that $\sigma > 0$, (q, θ) is nonsharp σ-admissible and $(\widetilde{q}, 1)$ is σ-admissible. Then there exists a positive δ such that, for every j in $\mathbb{Z}$,*

$$|B_j(F, G)| \lesssim 2^{\alpha j} \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_1)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)}$$
$$\forall F \in L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_1) \cap L^1(\mathbb{R}; \mathcal{B}_0) \quad \forall G \in L^{q'}(\mathbb{R}; \mathcal{B}_\theta) \cap L^1(\mathbb{R}; \mathcal{B}_0) \tag{4.49}$$

whenever $\alpha \in (-\delta, \delta)$.

Proof. We prove the lemma in three steps.

Step 1. Suppose that $\alpha \in \mathbb{R}$ and $j \in \mathbb{Z}$. We observe that if (4.49) holds with the modification that every F and G is supported on an interval of length 2^j, then (4.49) holds for any F and G. To establish this we adapt the approach of Step 2 in the proof of Lemma 4.5.2. In this case, $\phi_0 = 0$ outside the interval $[-2^{j-2}, 2^{j-1} + 2^{j-2}]$ and $\phi_m(t) = \phi_0(t - m2^{j-1})$ whenever $m \in \mathbb{Z}$ and $t \in \mathbb{R}$.

Step 2. Suppose that $j \in \mathbb{Z}$ and consider the inequality

$$\begin{aligned} &|B_j(F, G)| \lesssim 2^{\alpha j} \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_1)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_a)} \\ &\qquad \forall F \in L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_1) \cap L^1(\mathbb{R};\mathcal{B}_0) \quad \forall G \in L^{q'}(\mathbb{R};\mathcal{B}_a) \cap L^1(\mathbb{R};\mathcal{B}_0) \end{aligned} \tag{4.50}$$

We aim to show that (4.50) holds in the following cases:

(i) $a = 0$ and $\alpha = 1/q$,

(ii) $a = 1$ and $\alpha = -\sigma + 1/\widetilde{q} + 1/q$, and

(iii) $a = 1$ and $\alpha = 1/\widetilde{q} + 1/q$.

In light of Step 1, we may assume in the following calculations that F and G are supported on an interval of length 2^j.

We establish (i) using the same argument employed to derive (4.41), followed by an application of the dual (4.32) of the energy estimate. This yields

$$|B_j(F, G)| \lesssim \left(\sup_{t\in\mathbb{R}} \left\| \int_{t-2^{j+1}<s<t-2^j} U(s)^* F(s)\, \mathrm{d}s \right\|_{\mathcal{H}} \right) \|G\|_{L^1(\mathbb{R};\mathcal{B}_0)} .$$

Since $(\widetilde{q}, 1)$ is σ-admissible, Theorem 4.4.2 gives

$$|B_j(F, G)| \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_1)} \|G\|_{L^1(\mathbb{R};\mathcal{B}_0)} .$$

Now G is supported on an interval I of length 2^j, so

$$\|G\|_{L^1(\mathbb{R};\mathcal{B}_0)} = \|1_I G\|_{L^1(\mathbb{R};\mathcal{B}_0)} \le \|1_I\|_{L^q(\mathbb{R})} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_0)} \le 2^{j/q} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_0)} , \tag{4.51}$$

where the first inequality is justified by Hölder's inequality. This proves (i).

To prove (ii), we integrate (4.31) to get

$$|B_j(F,G)| \lesssim 2^{-j\sigma} \|F\|_{L^1(\mathbb{R};\mathcal{B}_1)} \|G\|_{L^1(\mathbb{R};\mathcal{B}_1)} .$$

Using Hölder's inequality as in (4.51) gives (ii).

Case (iii) may be proved by integrating (4.36) when $\theta = 1$ to get

$$|B_j(F,G)| \lesssim \|F\|_{L^1(\mathbb{R};\mathcal{B}_1)} \|G\|_{L^1(\mathbb{R};\mathcal{B}_1)}$$

Once again, Hölder's inequality is applied to yield the result.

Step 3. Interpolating between (i) and (iii) gives (4.50) when $a = \theta$ and $\alpha = 1/q + \theta/\widetilde{q} > 0$. On the other hand, interpolating between (i) and (ii) gives (4.50) when $a = \theta$ and

$$\begin{aligned} \alpha &= (1-\theta)\frac{1}{q} + \theta\left(-\sigma + \frac{1}{\widetilde{q}} + \frac{1}{q}\right) \\ &= \left(\frac{1}{q} - \frac{\sigma\theta}{2}\right) + \theta\left(\frac{1}{\widetilde{q}} - \frac{\sigma}{2}\right). \end{aligned} \tag{4.52}$$

From the hypothesis, $1/q - \sigma\theta/2 < 0$ and $1/\widetilde{q} - \sigma/2 \leq 0$. Hence the α in (4.52) is negative. The lemma now follows by interpolating between the case when $\alpha > 0$ and the case when $\alpha < 0$. □

Theorem 4.6.3. *Suppose that $\sigma > 0$ and that $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate (4.10) and the truncated decay estimate (4.12). Then the retarded Strichartz estimate (4.16) holds for all σ-admissible pairs (q,θ) and $(\widetilde{q},\widetilde{\theta})$.*

Proof. We will establish (4.21) for σ-admissible pairs (q,θ) and $(\widetilde{q},\widetilde{\theta})$. Every σ-admissible pair is an interpolant between a sharp σ-admissible pair and a σ-admissible pair $(q,1)$. Hence it suffices to show (4.21) when

(i) (q,θ) and $(\widetilde{q},\widetilde{\theta})$ are sharp σ-admissible,

(ii) (q,θ) and $(\widetilde{q},\widetilde{\theta})$ are σ-admissible and $\widetilde{\theta} = 1$ and

(iii) (q,θ) and $(\widetilde{q},\widetilde{\theta})$ are σ-admissible and $\theta = 1$.

Case (i) is immediate from Theorem 4.6.1. For case (ii) there is, by Lemma 4.6.2, a negative α_0 and a positive α_1 such that

$$\begin{aligned}
|B(F,G)| &\leq \sum_{j=-\infty}^{0} |B_j(F,G)| + \sum_{j=1}^{\infty} |B_j(F,G)| \\
&\lesssim \sum_{j=-\infty}^{0} 2^{\alpha_1 j} \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_1)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \\
&\qquad + \sum_{j=1}^{\infty} 2^{\alpha_0 j} \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_1)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \\
&\lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_1)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} .
\end{aligned}$$

By symmetry (Lemma 4.3.2), case (iii) may be deduced from case (ii). □

4.7 Application to the Schrödinger equation

To illustrate how Theorem 4.2.2 is applied to a concrete setting, we first examine the inhomogeneous Schrödinger equation in the Euclidean space $\mathbb{R}^n$ with initial data. The results stated in this section are relatively well known (see, for example, [42]).

Our strategy is to show that the evolution group $\{U(t) : t \geq 0\}$ on $L^2(\mathbb{R}^n)$ associated to the Schrödinger equation satisfies the energy estimate and untruncated decay estimate when $\sigma = n/2$, $\mathcal{B}_0 = \mathcal{H} = L^2(\mathbb{R}^n)$ and $\mathcal{B}_1 = L^1(\mathbb{R}^n)$. In these circumstances, the sharp σ-admissibility criteria correspond to the following conditions on the time exponent q and the spatial exponent r.

Definition 4.7.1. Suppose that $n \geq 1$. We say that a pair (q,r) of Lebesgue exponents are *Schrödinger n-admissible* if $q \in [2,\infty]$,

$$\frac{1}{q} + \frac{n}{2r} = \frac{n}{4} \tag{4.53}$$

and $(q,r,n) \neq (2,\infty,2)$.

Corollary 4.7.2. *Suppose that* $n \geq 1$ *and that* (q,r) *and* $(\widetilde{q},\widetilde{r})$ *are Schrödinger n-admissible pairs. If* $f \in L^2(\mathbb{R}^n)$, $F \in L^{\widetilde{q}'}(\mathbb{R}; L^{\widetilde{r}'}(\mathbb{R}^n))$ *and* u *is a (weak)* *solution to the problem*

$$\begin{cases} iu'(t) + \Delta u(t) = F(t) & \forall t \in \mathbb{R} \\ u(0) = f, \end{cases} \tag{4.54}$$

then

$$\|u\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\widetilde{q}'}(\mathbb{R};L^{\widetilde{r}'}(\mathbb{R}^n))} . \tag{4.55}$$

Conversely, if u is a weak solution to (4.54) and the estimate (4.55) holds for all f in $L^2(\mathbb{R}^n)$ and F in $L^{\widetilde{q}'}(\mathbb{R};L^{\widetilde{r}'}(\mathbb{R}^n))$, then (q,r) and $(\widetilde{q},\widetilde{r})$ are necessarily Schrödinger n-admissible pairs.

Proof. Fix the spatial dimension n and suppose that $f \in L^2(\mathbb{R}^n)$. Define the family $\{U(t) : t \in \mathbb{R}\}$ by $U(t)f = e^{it\Delta}f$ and note that if $u(t) = U(t)f$, then u is the solution to the homogeneous initial value problem

$$\begin{cases} iu'(t) + \Delta u(t) = 0 & \forall t \in \mathbb{R} \\ u(0) = f. \end{cases} \tag{4.56}$$

If $F \in L^{\widetilde{q}'}([0,T];L^{\widetilde{r}'}(\mathbb{R}^n))$ for some exponents $\widetilde{q}$ and $\widetilde{r}$, then by Duhamel's principle the solution u to the inhomogeneous initial value problem (4.54) may be formally written as

$$\begin{aligned} u(t) &= e^{it\Delta}f - i\int_{-\infty}^{t} e^{i(t-s)\Delta}F(s)\,\mathrm{d}s \\ &= Tf(t) - i(TT^*)_R F(t) \qquad \forall t \in \mathbb{R}, \end{aligned}$$

where T is the operator introduced in Section 4.2. The right-hand side will be estimated by Theorem 4.2.2 once we demonstrate that the hypotheses of the theorem are satisfied.

First, the Fourier transform $\mathcal{F}$ gives the identity

$$\mathcal{F}\big(U(t)g\big)(\xi) = e^{it|\xi|^2}\mathcal{F}g(\xi)$$

whenever $g \in L^2(\mathbb{R}^n)$ and $t \geq 0$, and Plancherel's theorem now gives

$$\|U(t)g\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F}(U(t)g)\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F}g\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} .$$

Hence $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate. Second, Plancherel's theorem and the above identity also shows that $\{U(t) : t \in \mathbb{R}\}$ satisfies the group property

$$U(t)U(s)^* = U(t-s) \qquad \forall s,t \in \mathbb{R}. \tag{4.57}$$

If we combine this with the explicit representation

$$e^{it\Delta}g(x) = (2\pi it)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2it}}g(y)\,\mathrm{d}y$$

of the Schrödinger evolution operator, then

$$\|U(t)U(s)^*g\|_{L^\infty(\mathbb{R}^n)} = \left\|e^{i(t-s)\Delta}g\right\|_{L^\infty(\mathbb{R}^n)} \lesssim |t-s|^{-n/2}\,\|g\|_{L^1(\mathbb{R}^n)} \quad \forall \text{ real } s \neq t.$$

Hence $\{U(t) : t \in \mathbb{R}\}$ satisfies the dispersive estimate (4.11). We may now apply Theorem 4.2.2 when $\sigma = n/2$, $\mathcal{H} = L^2(\mathbb{R}^n)$, $\mathcal{B}_0 = L^2(\mathbb{R}^n)$ and $\mathcal{B}_1 = L^1(\mathbb{R}^n)$ so that

$$\begin{aligned}\|u\|_{L^q(\mathbb{R};\mathcal{B}_\theta^*)} &\le \|Tf\|_{L^q(\mathbb{R};\mathcal{B}_\theta^*)} + \|(TT^*)_R F\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta^*)}\\ &\lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})}\end{aligned}$$

whenever (q,θ) and $(\widetilde{q},\widetilde{\theta})$ are sharp $(n/2)$-admissible. Now

$$\mathcal{B}_\theta = \big(L^2(\mathbb{R}^n), L^1(\mathbb{R}^n)\big)_{\theta,2} = L^{r',2}(\mathbb{R}^n) \supset L^{r'}(\mathbb{R}^n),$$

where $1/r' = (1-\theta)/2 + \theta/1$ and the inclusion is continuous (see Section 3.3). It is not hard to show that (q,r) is Schrödinger n-admissible if and only if (q,θ) is sharp $(n/2)$-admissible and $1/r' = (1-\theta)/2+\theta/1$. This completes the proof of the first half of the corollary.

To prove the converse, suppose that u is a solution to the initial valuable problem (4.54) satisfying (4.55) for all forcing terms F in $L^{\widetilde{q}'}(\mathbb{R}; L^{\widetilde{r}'}(\mathbb{R}^n))$ and all initial data f in $L^2(\mathbb{R}^n)$. Since the solution u can be written in the form

$$u(t) = Tf(t) - i(TT^*)_R F(t),$$

the Strichartz estimate (4.55) implies that

$$\|Tf\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \tag{4.58}$$

and

$$\|(TT^*)_R F\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};L^{\widetilde{r}'}(\mathbb{R}^n))} \tag{4.59}$$

by taking F equal to 0 and f equal to 0 respectively.

We first show that (q,r) is Schrödinger n-admissible. By interpreting u as a function $(t,x) \mapsto u(t,x)$ of time t and spatial position x in $\mathbb{R}^n$, we see that the initial value problem

$$\begin{cases} i\dfrac{\partial u}{\partial t}(t,x) + \Delta u(t,x) = 0 & \forall t \in \mathbb{R} \quad \forall x \in \mathbb{R}^n \\ u(0,x) = f(x) \end{cases}$$

is invariant under the rescaling $t \leftarrow \lambda^2 t$ and $x \leftarrow \lambda x$. If this rescaling is applied to (4.58) then

$$\lambda^{-(n/r+2/q)} \|Tf\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \leq C\lambda^{-n/2} \|f\|_{L^2(\mathbb{R}^n)} .$$

Hence $n/r + 2/q = n/2$, which is equivalent to (4.53). The negative result of [51] shows that the estimate $\|Tf\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$ cannot hold when $(q,r,n) = (2,\infty,2)$. Finally, since the operator $(TT^*)_R$ is translation invariant (see Lemma 4.3.6) and satisfies (4.59), it follows from Lemma 3.5.2 that $q \geq q'$. This shows that $q \geq 2$. Hence (q,r) is Schrödinger n-admissible.

To show that $(\widetilde{q},\widetilde{r})$ is Schrödinger n-admissible, we use the duality and time reversing arguments of Section 4.3 (see especially Lemma 4.3.2 and its proof). Explicitly, the solution v to the initial value problem

$$\begin{cases} iv'(t) - \Delta v(t) = F(t) & \forall t \in \mathbb{R} \\ v(0) = f \end{cases}$$

is given by $v(t) = Sf(t) + (SS^*)_R F(t)$ where $V(t) = U(-t)$ and $Sf(t) = V(t)f$. The operator norms of S and T are equal while the operator norms of $(SS^*)_A$ and $(TT^*)_R$ are equal. The estimate (4.58) gives

$$\|Sf\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} = \|Tf\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \qquad \forall f \in L^2(\mathbb{R}^n),$$

while (4.59) yields

$$\|(SS^*)_A F\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} = \|(TT^*)_R F_0\|_{L^q(\mathbb{R};L^r(R^n))} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};L^{\widetilde{r}'}(\mathbb{R}^n))} \qquad \forall F \in L^{\widetilde{q}'}(\mathbb{R};L^{\widetilde{r}'}(\mathbb{R}^n)),$$

where $F_0(t) = F(-t)$. By duality,

$$\|(SS^*)_R F\|_{L^{\widetilde{q}'}(\mathbb{R};L^{\widetilde{r}'}(\mathbb{R}^n))} \lesssim \|F\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \qquad \forall F \in L^q(\mathbb{R};L^r(\mathbb{R}^n)).$$

Hence

$$\|v\|_{L^{\widetilde{q}}(\mathbb{R};L^{\widetilde{r}}(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^n))} \qquad \forall f \in L^2(\mathbb{R}^n) \quad \forall F \in L^q(\mathbb{R};L^r(\mathbb{R}^n))$$

and we may repeat the arguments in the preceding paragraph to show that $(\widetilde{q}, \widetilde{r})$ is Schrödinger n-admissible. □

Remark 4.7.3. The Strichartz estimates (4.5), (4.6) and (4.7) for the meson equation in $\mathbb{R}^2$ are obtained from Corollary 4.7.2 by noticing that the exponent pair $(4, 4)$ is Schrödinger 2-admissible.

4.8 Application to the wave equation

Strichartz estimates for the wave equation may be found using the truncated decay hypothesis (see the approach of Keel and Tao in [42, Section 8]). Instead, we show that they can be found using the untruncated decay hypothesis and Besov spaces. The results obtained are exactly those stated in [28], with the exception that we now also have the Strichartz estimate corresponding to the endpoint P. The material of this section will also lay the groundwork for Section 5.8, where new Strichartz estimates are obtained for the inhomogeneous wave equation with zero initial data. Before reading this section, it is vital that the reader is familiar with Besov spaces (see Section 3.4).

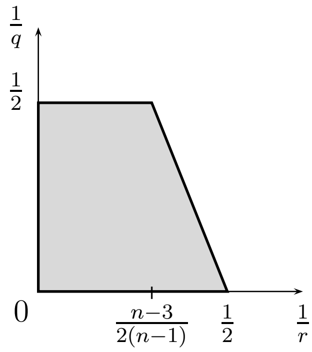


Figure 4.3: Wave admissible pairs (q, r) when $n > 3$.

Corollary 4.8.1. *Suppose that $n \geq 1$, that $\mu, \rho, \widetilde{\rho} \in \mathbb{R}$, that $q, \widetilde{q} \in [2, \infty]$ and that the following conditions are satisfied:*

$$\begin{aligned} q \geq 2, &\qquad \widetilde{q} \geq 2, \\ \frac{1}{q} \leq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right), &\qquad \frac{1}{\widetilde{q}} \leq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right), \\ (q, r, n) \neq (2, \infty, 3), &\qquad (\widetilde{q}, \widetilde{r}, n) \neq (2, \infty, 3), \\ \rho + n\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q} = \mu &= 1 - \left(\widetilde{\rho} + n\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right) - \frac{1}{\widetilde{q}}\right). \end{aligned} \tag{4.60}$$

Suppose also that $f \in \dot{H}^{\mu}$, $g \in \dot{H}^{\mu-1}$ and $F \in L^{\widetilde{q}'}(\mathbb{R}; B^{-\widetilde{\rho}}_{\widetilde{r}',2})$. If u is a (weak) solution to the initial value problem

$$\begin{cases} -u''(t) + \Delta u(t) = F(t) \\ u(0) = f \\ u'(0) = g \end{cases} \tag{4.61}$$

then

$$\|u\|_{L^q(\mathbb{R}; \dot{B}^{\rho}_{r,2})} \lesssim \|f\|_{\dot{H}^{\mu}} + \|g\|_{\dot{H}^{\mu-1}} + \|F\|_{L^{\widetilde{q}'}(\mathbb{R}; \dot{B}^{-\widetilde{\rho}}_{\widetilde{r}',2})}. \tag{4.62}$$

The closed region in Figure 4.3 represents the range of exponent pairs (q, r) and $(\widetilde{q}, \widetilde{r})$ such that the Strichartz estimate (4.62) is valid.

Remark 4.8.2. Corollary 4.8.1 implies Strichartz estimates for spaces more familiar than the Besov spaces. By Besov–Sobolev embedding, estimate (4.62) still holds when $\dot{B}^{\rho}_{r,2}$ is replaced everywhere by $\dot{H}^{\rho}_{r}$ under the additional assumption that $r < \infty$ and $\widetilde{r} < \infty$. In fact, using Sobolev embedding, one can deduce that

$$\|u\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{\dot{H}^\mu} + \|g\|_{\dot{H}^{\mu-1}} + \|F\|_{L^{\widetilde{q}'}(\mathbb{R};L^{\widetilde{r}'}(\mathbb{R}^n))}$$

under the additional assumption that $r < \infty$ and $\widetilde{r} < \infty$. One may also replace the infinite interval $\mathbb{R}$ by any finite time interval $[0, \tau]$ where $\tau > 0$. See [42, Corollary 1.3] and [28] for these variations.

We begin with a heuristic argument to indicate how Theorem 4.2.2 will be applied in this setting. For convenience, write ω for the operator $(-\Delta)^{1/2}$. The homogeneous problem may be written as

$$v''(t) + \omega^2 v(t) = 0, \qquad v(0) = f, \qquad v'(0) = g,$$

with solution v is given by

$$v(t) = \cos(\omega t)h_1 + \sin(\omega t)h_2$$

for some functions h_1 and h_2 determined by imposing initial conditions. Hence

$$v(t) = \cos(\omega t)f + \omega^{-1}\sin(\omega t)g.$$

The inhomogeneous problem

$$-w''(t) + \Delta w(t) = F(t), \qquad w(0) = 0, \qquad w'(0) = 0$$

may be solved by Duhamel's principle to give

$$w(t) = \int_{s<t} \omega^{-1}\sin\big(\omega(t-s)\big)F(s)\,\mathrm{d}s.$$

Define $\{U(t) : t \in \mathbb{R}\}$ by $U(t) = e^{i\omega t}$. Then the solution u to problem (4.61) can be written as

$$\begin{aligned} u(t) &= v(t) + w(t) \\ &= \frac{1}{2}\big(U(t) + U(-t)\big)f + \omega^{-1}\frac{1}{2i}\big(U(t) - U(-t)\big)g \\ &\quad + \int_{s<t} \omega^{-1}\frac{1}{2i}\big(U(t)U(s)^* - U(-t)U(-s)^*\big)F(s)\,\mathrm{d}s \end{aligned} \tag{4.63}$$

and it is clear that if we have appropriate Strichartz estimates for the group $\{U(t) : t \in \mathbb{R}\}$ then (4.62) will follow.

In what follows, let φ_j and $\widetilde{\varphi}_j$ denote the Littlewood–Paley functions of Section 3.4. We define the operator T by $Tf(t) = U(t)f$, whenever f belongs to the Hilbert space $\dot{B}^0_{2,2}$.

Lemma 4.8.3. *Suppose that $n \geq 1$ and that the triples (q, r, γ) and $(\widetilde{q}, \widetilde{r}, \widetilde{\gamma})$ satisfy the conditions*

$$q \geq 2, \qquad \widetilde{q} \geq 2, \tag{4.64}$$

$$\frac{1}{q} = \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right), \qquad \frac{1}{\widetilde{q}} = \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right), \tag{4.65}$$

$$\gamma = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{r}\right), \qquad \widetilde{\gamma} = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right), \tag{4.66}$$

$$(q, r, n) \neq (2, \infty, 3), \qquad (\widetilde{q}, \widetilde{r}, n) \neq (2, \infty, 3). \tag{4.67}$$

Then the operator T satisfies the Strichartz estimates

$$\|Tf\|_{L^q(\mathbb{R};\dot{B}^{-\gamma}_{r,2})} \lesssim \|f\|_{\dot{B}^0_{2,2}} \qquad \forall f \in \dot{B}^0_{2,2} \tag{4.68}$$

and

$$\|(TT^*)_R F\|_{L^q(\mathbb{R};\dot{B}^{-\gamma}_{r,2})} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\dot{B}^{\widetilde{\gamma}}_{\widetilde{r}',2})} \qquad \forall F \in L^{\widetilde{q}'}(\mathbb{R}; \dot{B}^{\widetilde{\gamma}}_{\widetilde{r}',2}). \tag{4.69}$$

Proof. We begin with the stationary phase estimate

$$\sup_{x\in\mathbb{R}^n} \left| \int_{\xi\in\mathbb{R}^n} \exp(it|\xi| + i\,\langle x,\, \xi\rangle)\hat{\varphi}_0(\xi)\,\mathrm{d}\xi \right| \leq C|t|^{-(n-1)/2},$$

where C is a positive constant (see, for example, [35, Section 7.7]). For j in $\mathbb{Z}$, apply the scaling $\xi \leftarrow 2^{-j}\xi$, $x \leftarrow 2^j x$, $t \leftarrow 2^j t$ to obtain

$$\sup_{x\in\mathbb{R}^n} 2^{-jn} \left| \int_{\xi\in\mathbb{R}^n} \exp(it|\xi| + i\,\langle x,\, \xi\rangle)\hat{\varphi}_j(\xi)\,\mathrm{d}\xi \right| \leq C|t|^{-(n-1)/2} 2^{-j(n-1)/2}.$$

The above estimate may be rewritten as

$$\|U(t)\varphi_j\|_{L^\infty(\mathbb{R}^n)} \lesssim |t|^{-(n-1)/2} 2^{j(n+1)/2}.$$

If f is a sufficiently regular function (or distribution) in the spatial variable then

$$\begin{aligned}\|\varphi_j * U(t)f\|_{L^\infty(\mathbb{R}^n)} &= \|\varphi_j * U(t)\widetilde{\varphi}_j * f\|_{L^\infty(\mathbb{R}^n)}\\ &\le \|U(t)\varphi_j\|_{L^\infty(\mathbb{R}^n)}\,\|\widetilde{\varphi}_j * f\|_{L^1(\mathbb{R}^n)}\\ &\lesssim |t|^{-(n-1)/2}2^{j(n+1)/2}\,\|\widetilde{\varphi}_j * f\|_{L^1(\mathbb{R}^n)}, \end{aligned} \tag{4.70}$$

by (3.9) and Young's inequality. Multiplying by $2^{j(n+1)/4}$ gives

$$\left\|2^{-j/2}\varphi_j * U(t)f\right\|_{L^\infty(\mathbb{R}^n)} \lesssim |t|^{-(n-1)/2}\left\|2^{jn/2}\widetilde{\varphi}_j * f\right\|_{L^1(\mathbb{R}^n)},$$

where the left- and right-hand sides define the jth term of two sequences. If we take the ℓ^2 norm of each sequence and apply (3.8), then the above inequality yields

$$\|U(t)f\|_{\dot{B}^{-(n+1)/4}_{\infty,2}} \lesssim |t|^{-(n-1)/2}\,\|f\|_{\dot{B}^{(n+1)/4}_{1,2}} \qquad \forall f \in \dot{B}^{(n+1)/4}_{1,2}. \tag{4.71}$$

This corresponds to the abstract untruncated decay estimate (4.11).

On the other hand, each $U(t)$ is an isometry on the homogeneous Sobolev space $\dot{H}^0$ and hence we have the energy estimate

$$\|U(t)f\|_{\dot{B}^0_{2,2}} \lesssim \|f\|_{\dot{B}^0_{2,2}} \qquad \forall f \in \dot{B}^0_{2,2},$$

by (3.13). If $\mathcal{H} = \mathcal{B}_0 = \dot{B}^0_{2,2}$ and $\mathcal{B}_1 = \dot{B}^{(n+1)/4}_{1,2}$ then

$$\dot{B}^\gamma_{r',2} \subseteq \mathcal{B}_\theta = (\mathcal{B}_0, \mathcal{B}_1)_{\theta,2}$$

by (3.11), where $1/r' = (1-\theta)/2 + \theta$ and $\gamma = (n+1)\theta/4$. It is not hard to show from here that the sharp case of Theorem 4.2.2 proves the lemma. □

Proof of Corollary 4.8.1. It is well known that if $\mu \in \mathbb{R}$, then ω^μ is an isomorphism from $\dot{B}^\gamma_{r,2}$ to $\dot{B}^{\gamma-\mu}_{r,2}$. Hence replacing f with $\omega^\mu f$ in (4.68) gives

$$\|Tf\|_{L^q(\mathbb{R};\dot{B}^{-\gamma+\mu}_{r,2})} \lesssim \|f\|_{\dot{B}^\mu_{2,2}} \qquad \forall f \in \dot{B}^\mu_{2,2}.$$

The same trick applied to (4.69) yields

$$\|(TT^*)_R F\|_{L^q(\mathbb{R};\dot{B}^{-\gamma+\mu}_{r,2})} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\dot{B}^{\widetilde{\gamma}+\mu}_{\widetilde{r}',2})} \qquad \forall F \in L^{\widetilde{q}'}(\mathbb{R};\dot{B}^{\widetilde{\gamma}+\mu}_{\widetilde{r}',2}).$$

If $\rho = -\gamma + \mu$ then these estimates combine with (4.63) to give

$$\begin{aligned}
\|u\|_{L^q(\mathbb{R};\dot{B}^{\rho}_{r,2})} &\lesssim \|Tf\|_{L^q(\mathbb{R};\dot{B}^{\rho}_{r,2})} + \left\|\omega^{-1}Tg\right\|_{L^q(\mathbb{R};\dot{B}^{\rho}_{r,2})} + \left\|\omega^{-1}(TT^*)_R F\right\|_{L^q(\mathbb{R};\dot{B}^{\rho}_{r,2})} \\
&\lesssim \|f\|_{\dot{B}^{\mu}_{2,2}} + \left\|\omega^{-1}g\right\|_{\dot{B}^{\mu}_{2,2}} + \left\|\omega^{-1}F\right\|_{L^{\widetilde{q}'}(\mathbb{R};\dot{B}^{\widetilde{\gamma}+\mu}_{\widetilde{r}',2})} \\
&\lesssim \|f\|_{\dot{B}^{\mu}_{2,2}} + \|g\|_{\dot{B}^{\mu-1}_{2,2}} + \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\dot{B}^{\widetilde{\gamma}+\mu-1}_{\widetilde{r}',2})}.
\end{aligned}$$

If $\widetilde{\rho} = -(\widetilde{\gamma} + \mu - 1)$ then the estimate above becomes

$$\|u\|_{L^q(\mathbb{R};\dot{B}^{\rho}_{r,2})} \lesssim \|f\|_{\dot{H}^{\mu}} + \|g\|_{\dot{H}^{\mu-1}} + \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\dot{B}^{-\widetilde{\rho}}_{\widetilde{r}',2})}. \tag{4.72}$$

So far we have imposed the conditions $\mu \in \mathbb{R}$, (4.64), (4.65), (4.67) and

$$\rho + \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) = \mu = 1 - \widetilde{\rho} - \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right).$$

This last condition may be rewritten as

$$\rho + n\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q} = \mu = 1 - \widetilde{\rho} - n\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right) + \frac{1}{\widetilde{q}}.$$

Now if $r_1 \geq r$ and $\rho - n/r = \rho_1 - n/r_1$, then

$$\|u\|_{L^q(\mathbb{R};\dot{B}^{\rho_1}_{r_1,2})} \leq C\,\|u\|_{L^q(\mathbb{R};\dot{B}^{\rho}_{r,2})}$$

by Lemma 3.4.2. Similarly, if $\widetilde{r}_1 \geq \widetilde{r}$ and $\widetilde{\rho} - n/\widetilde{r} = \widetilde{\rho}_1 - n/\widetilde{r}_1$, then

$$\|F\|_{L^{\widetilde{q}'}(\mathbb{R};\dot{B}^{-\widetilde{\rho}}_{\widetilde{r}',2})} \leq C\,\|F\|_{L^{\widetilde{q}'}(\mathbb{R};\dot{B}^{\widetilde{\rho}_1}_{\widetilde{r}'_1,2})}.$$

Applying these estimates to (4.72) gives

$$\|u\|_{L^q(\mathbb{R};\dot{B}^{\rho_1}_{r_1,2})} \lesssim \|f\|_{\dot{H}^{\mu}} + \|g\|_{\dot{H}^{\mu-1}} + \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\dot{B}^{-\widetilde{\rho}_1}_{\widetilde{r}'_1,2})} \tag{4.73}$$

whenever the conditions

$$\begin{gathered}
q \geq 2, \qquad \widetilde{q} \geq 2, \\
\frac{1}{q} \leq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{r_1}\right), \qquad \frac{1}{\widetilde{q}} \leq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{\widetilde{r}_1}\right), \\
(q, r_1, n) \neq (2, \infty, 3), \qquad (\widetilde{q}, \widetilde{r}_1, n) \neq (2, \infty, 3), \\
\rho_1 + n\left(\frac{1}{2} - \frac{1}{r_1}\right) - \frac{1}{q} = \mu = 1 - \widetilde{\rho}_1 - n\left(\frac{1}{2} - \frac{1}{\widetilde{r}_1}\right) + \frac{1}{\widetilde{q}}
\end{gathered}$$

are satisfied. These conditions and the Strichartz estimate (4.73) coincide with those in the statement of Corollary 4.8.1. $\square$

Remark 4.8.4. One can see from (4.63) that the derivative u' can also be expressed in terms of T, $(TT^*)_R$ and ω. Thus we have the Strichartz estimate

$$\|u'\|_{L^q(\mathbb{R};\dot{B}^{\rho-1}_{r,2})} \lesssim \|f\|_{\dot{H}^\mu} + \|g\|_{\dot{H}^{\mu-1}} + \|F\|_{L^{\tilde{q}'}(\mathbb{R};\dot{B}^{-\tilde{\rho}}_{\tilde{r}',2})}\,.$$

whenever the exponents satisfy the conditions of Corollary 4.8.1.

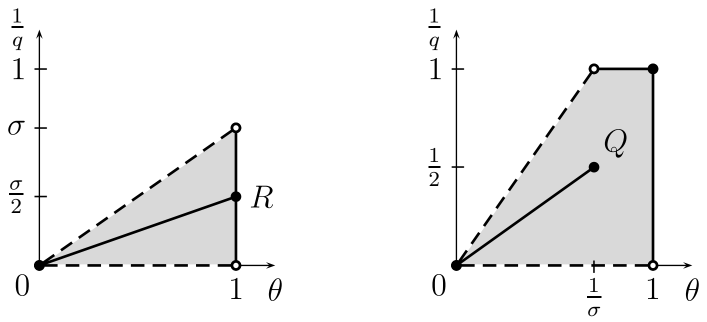
Chapter 5

Inhomogeneous Strichartz estimates

It was remarked by Keel and Tao in [42] that, assuming the energy and one of the dispersive estimates, the inhomogeneous Strichartz estimate (4.16) holds for exponent pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ other than those satisfying the admissibility criteria of Theorem 4.2.2. Suppose, in the notation of Section 4.2, that $\mathcal{B}_0 = L^2(X)$ and $\mathcal{B}_1 = L^1(X)$ for some measure space X. It was the aim of D. Foschi to find the largest range of pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ which guarantees the validity of the inhomogeneous Strichartz estimate (4.16), assuming only the energy estimate (4.10) and the untruncated decay estimate (4.11). He made substantial progress in this direction in [24] by using techniques introduced by Keel and Tao [42]. Independently of but simultaneously to Foschi, M. Vilela [75] also obtained similar results for solutions to the inhomogeneous Schrödinger equation.

In this chapter we show that much of the argument of [24] can be adapted to a more general setting where $(\mathcal{B}_0, \mathcal{B}_1)$ is a Banach couple. Where Foschi's argument cannot be generalised, we instead use abstract methods introduced in [42]. As a result, we are able to obtain new Strichartz estimates for the wave equation and a range of other equations.

The structure of the chapter is as follows. In Section 5.1, we announce



(a) $0 < \sigma < 1$ (b) $\sigma > 1$

Figure 5.1: σ-acceptable pairs (q, θ) for different values of σ.

the main result. This will be proved in Sections 5.2, 5.3 and 5.4. We present an alternate proof of part of the main theorem in Section 5.6 using p-atomic decompositions. The sharpness of the main theorem will be discussed in Section 5.7. Finally, in the last two sections we apply the result to the Schrödinger equation, wave equation, Klein–Gordon equation and Schrödinger equation with potential.

5.1 Global and local inhomogeneous Strichartz estimates

Our aim is to find exponent pairs (q, θ) and $(\widetilde{q}, \theta)$ other than those which are σ-admissible such that the inhomogeneous Strichartz estimate (4.16) holds. With this is mind we give the following definition.

Definition 5.1.1. Suppose that $\sigma > 0$. We say that a pair (q, θ) of exponents is *σ-acceptable* if either

$$1 \leq q < \infty, \quad 0 \leq \theta \leq 1, \quad \frac{1}{q} < \sigma\theta$$

or $(q, \theta) = (\infty, 0)$.

The shaded regions of Figure 5.1 represent the set of σ-acceptable pairs (q, θ) for different values of σ. The closed line segments OQ and OR correspond to the sharp σ-admissible pairs in each case.

If $(\mathcal{B}_0, \mathcal{B}_1)$ is a Banach interpolation couple then we write $\mathcal{B}_{\theta,q}$ for $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}$. As was the case in Chapter 4, we shall continue to denote $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,2}$ by $\mathcal{B}_\theta$. The main result of this chapter, given by the following theorem, extends the work of D. Foschi [24, Theorem 1.4].

Theorem 5.1.2. *Suppose that $\sigma > 0$ and that $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate (4.10) and the untruncated decay estimate (4.11). Suppose also that the exponents pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ are σ-acceptable and satisfy the scaling condition*

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{\sigma}{2}(\theta + \widetilde{\theta}). \tag{5.1}$$

(i) If

$$\frac{1}{q} + \frac{1}{\widetilde{q}} < 1, \tag{5.2}$$

$$(\sigma - 1)(1 - \theta) \leq \sigma(1 - \widetilde{\theta}), \qquad (\sigma - 1)(1 - \widetilde{\theta}) \leq \sigma(1 - \theta), \tag{5.3}$$

and, in the case when $\sigma = 1$, we have $\theta < 1$ and $\widetilde{\theta} < 1$, then the inhomogeneous Strichartz estimate (4.16) holds.

(ii) If $q, \widetilde{q} \in (1, \infty)$,

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = 1 \tag{5.4}$$

and

$$(\sigma - 1)(1 - \theta) < \sigma(1 - \widetilde{\theta}), \qquad (\sigma - 1)(1 - \widetilde{\theta}) < \sigma(1 - \theta) \tag{5.5}$$

then the inhomogeneous Strichartz estimate

$$\|(TT^*)_R F\|_{L^q(\mathbb{R};(\mathcal{B}_{\theta,q'})^*)} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta},\widetilde{q}'})} \qquad \forall F \in L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta},\widetilde{q}'}) \cap L^1(\mathbb{R}; \mathcal{B}_0) \tag{5.6}$$

holds.

Remark 5.1.3. Suppose that the scaling condition (5.1) holds. Then the exponent conditions appearing in (i) and (ii) above are always satisfied if $\sigma < 1$ or if $\sigma = 1$, $\theta < 1$ and $\widetilde{\theta} < 1$.

Remark 5.1.4. Condition (5.1) is a consequence of the invariance of (4.16) with respect to the rescaling (4.29). In fact, (5.1) and (5.4) are necessary conditions (see Section 5.7). The combination of these two conditions have the following geometric interpretation: if the points $(1/q, \theta)$ and $(1/\widetilde{q}, \widetilde{\theta})$ satisfy (5.1) and (5.4), then their midpoint is a sharp σ-admissible pair.

Remark 5.1.5. By specialising Theorem 5.1.2 to the case when $(\mathcal{B}_0, \mathcal{B}_1) = (L^2(X), L^1(X))$, where X is a measure space X, we recover the results [24, Theorem 1.4] of Foschi.

As in [24], our proof that global inhomogeneous Strichartz estimates of Theorem 5.1.2 exist is based on the existence of localised inhomogeneous estimates given by the theorem below. In Chapter 5, these localised estimates will play the same role as did those Strichartz estimates obtained in Chapter 4 for functions having compact support (see, for example, the proofs of Lemma 4.5.2 and Lemma 4.6.2).

Theorem 5.1.6. *Suppose that $\sigma > 0$ and that $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate (4.10) and the untruncated decay estimate (4.11). Assume also that I and J are two time intervals of unit length separated by a distance of scale* 1 *(that is, $|I| = |J| = 1$ and $\operatorname{dist}(I, J) \approx 1$). Then the local inhomogeneous Strichartz estimate*

$$\|TT^*F\|_{L^q(J;\mathcal{B}_\theta^*)} \lesssim \|F\|_{L^{\widetilde{q}'}(I;\mathcal{B}_{\widetilde{\theta}})} \qquad \forall F \in L^{\widetilde{q}'}(I;\mathcal{B}_{\widetilde{\theta}}) \cap L^1(I;\mathcal{B}_0) \tag{5.7}$$

holds whenever the pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ satisfy the conditions

$$q, \widetilde{q} \in [1, \infty], \qquad \theta, \widetilde{\theta} \in [0, 1], \tag{5.8}$$

$$(\sigma - 1)(1 - \theta) \leq \sigma(1 - \widetilde{\theta}), \qquad (\sigma - 1)(1 - \widetilde{\theta}) \leq \sigma(1 - \theta), \tag{5.9}$$

$$\frac{1}{q} \geq \frac{\sigma}{2}(\theta - \widetilde{\theta}), \qquad \frac{1}{\widetilde{q}} \geq \frac{\sigma}{2}(\widetilde{\theta} - \theta). \tag{5.10}$$

If $\sigma = 1$ then θ and $\widetilde{\theta}$ must be strictly less than 1.

The range of possible values of θ and $\widetilde{\theta}$ that give local and global inhomogeneous estimates is compared in Figure 5.2. Region $AOEC$ represents sharp

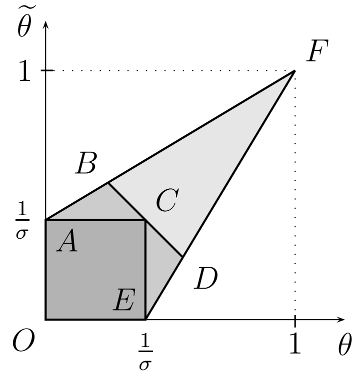


Figure 5.2: The range for exponents θ and $\widetilde{\theta}$ when $\sigma > 1$.

σ-admissible exponents, region $AOEDB$ represents exponents for the global estimates of Theorem 5.1.2 and region $AOEF$ represents exponents for the local estimates of Theorem 5.1.6. The boundaries of each region are included except at the points B and D for the global estimates.

Remark 5.1.7. The requirement in the hypothesis that $\operatorname{dist}(I, J) \approx 1$ is due to the lack of integrability of the dispersion estimate (4.11) when s is close to t (see (5.13)). This condition can be removed if $\sigma < 1$.

Remark 5.1.8. If one assumes the scaling condition (5.1) then strict inequalities in conditions (5.8) and (5.10) are equivalent to saying that the pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ are σ-acceptable.

We prove Theorem 5.1.2 over the next three sections. Section 5.2 gives a proof of Theorem 5.1.6 by interpolating between the local inhomogeneous estimate (5.7) when the exponents are σ-admissible and when $(q, \theta; \widetilde{q}, \widetilde{\theta}) = (\infty, 1; \infty, 1)$. In Section 5.3 we show how the global inhomogeneous estimate (4.16) can be decomposed as a sum of local estimates via a dyadic Whitney decomposition. These sections closely follow the approach of Foschi [24, Sections 2 and 3]. Section 5.4 marks a departure from Foschi's method. Here we show, using abstract real interpolation, how the local estimates and Whitney decomposition combine to prove Theorem 5.1.2. While we could stop there, we choose to present an alternate proof of some of the global estimates to illustrate other available techniques. The main technique, first introduced to the Strichartz community by Keel and Tao [42] and known as *p-atomic decomposition* of functions in L^p, is given in Section 5.5. This alternate proof of Theorem 5.1.2 (i) is presented in Section 5.6.

5.2 Proof of the local Strichartz estimates

Given two intervals I and J of $\mathbb{R}$, write $Q = I \times J$ and define B_Q by the formula

$$B_Q(F, G) = B(1_I F, 1_J G) = \iint_{(s,t)\in I\times J} \langle U(s)^* F(s),\, U(t)^* G(t)\rangle \,\mathrm{d}s\,\mathrm{d}t \quad (5.11)$$

whenever F and G belong to $L^1(\mathbb{R}; \mathcal{B}_0)$. One can easily show (using calculations similar to those of Section 4.3) that the local inhomogeneous estimate (5.7) is equivalent to the bilinear estimate

$$\begin{aligned} |B_Q(F, G)| &\lesssim \|F\|_{L^{\widetilde{q}'}(I;\mathcal{B}_{\widetilde{\theta}})} \|G\|_{L^{q'}(J;\mathcal{B}_\theta)} \\ &\qquad \forall F \in L^{\widetilde{q}'}(I; \mathcal{B}_{\widetilde{\theta}}) \cap L^1(I; \mathcal{B}_0) \quad \forall G \in L^{q'}(J; \mathcal{B}_\theta) \cap L^1(J; \mathcal{B}_0). \quad (5.12) \end{aligned}$$

We use this equivalence to prove results in this and subsequent sections.

Proof of Theorem 5.1.6. Suppose that I and J are two intervals satisfying the hypothesis of the theorem and write $Q = I \times J$. Let Ψ denote the set of points $(1/q, \theta; 1/\widetilde{q}, \widetilde{\theta})$ in $[0, 1]^4$ corresponding to the pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ for which estimate (5.7), or its bilinear equivalent (5.12), is valid.

If we apply the bilinear version (4.31) of the dispersive estimate then

$$\begin{aligned} |B_Q(F, G)| &\lesssim \int_J \int_I |t - s|^{-\sigma} \|F(s)\|_{\mathcal{B}_1} \|G(t)\|_{\mathcal{B}_1} \,\mathrm{d}s\,\mathrm{d}t \\ &\lesssim \|F\|_{L^1(I;\mathcal{B}_1)} \|G\|_{L^1(J;\mathcal{B}_1)}. \end{aligned} \quad (5.13)$$

Hence $(0, 1; 0, 1) \in \Psi$. On the other hand, if the homogeneous Strichartz estimate (4.15) of Theorem 4.2.2 is applied then

$$\begin{aligned} |B_Q(F, G)| &\le \left\| \int_I U(s)^* F(s) \,\mathrm{d}s \right\|_{\mathcal{H}} \left\| \int_J U(t)^* G(t) \,\mathrm{d}s \right\|_{\mathcal{H}} \\ &\lesssim \|F\|_{L^{\widetilde{q}'}(I;\mathcal{B}_{\widetilde{\theta}})} \|G\|_{L^{q'}(J;\mathcal{B}_\theta)} \end{aligned} \quad (5.14)$$

whenever (q,θ) and $(\widetilde{q},\widetilde{\theta})$ are sharp σ-admissible. Complex interpolation (see Lemma 3.3.6) between (5.13) and (5.14) shows that Ψ contains the convex hull of the set

$$(0,1;0,1) \cup \Big\{(1/q,\theta;1/\widetilde{q},\widetilde{\theta}) : (q,\theta) \text{ and } (\widetilde{q},\widetilde{\theta}) \text{ are } \sigma\text{-admissible pairs}\Big\}. \quad (5.15)$$

Since G is restricted to a unit time interval, Hölder's inequality gives

$$\|G\|_{L^{q'}(J;\mathcal{B}_\theta)} = \|1_J G\|_{L^{q'}(J;\mathcal{B}_\theta)} \le \|1_J\|_{L^{r'}(J)} \|G\|_{L^{p'}(J;\mathcal{B}_\theta)} \lesssim \|G\|_{L^{p'}(J;\mathcal{B}_\theta)}$$

whenever $1/q' = 1/r' + 1/p'$. We can always perform this calculation provided that $p \le q$. Similarly, if $\widetilde{p} \le \widetilde{q}$ then

$$\|F\|_{L^{\widetilde{q}'}(I;\mathcal{B}_{\widetilde{\theta}})} \lesssim \|F\|_{L^{\widetilde{p}'}(I;\mathcal{B}_{\widetilde{\theta}})}\,.$$

Hence if $(1/q,\theta;1/\widetilde{q},\widetilde{\theta}) \in \Psi$ then $(1/p,\theta;1/\widetilde{p},\widetilde{\theta}) \in \Psi$ whenever $p \le q$ and $\widetilde{p} \le \widetilde{q}$. If we apply this property to the points of the convex hull of (5.15) then we obtain a set Ψ_*, contained in Ψ, that is described precisely by the conditions appearing in Theorem 5.1.6. Details of this computation are given in the following lemma. □

Lemma 5.2.1. *The set Ψ_*, defined in the proof above, is precisely the set of all $(q,\theta;\widetilde{q},\widetilde{\theta})$ in $[0,1]^4$ which satisfy the conditions on q, $\widetilde{q}$, θ and $\widetilde{\theta}$ given in statement of Theorem 5.1.6.*

Proof. We will construct the set Ψ_* in three steps.

First, Ψ_* contains the point $(1/p,\phi;1/\widetilde{p},\widetilde{\phi})$ when the pairs (p,ϕ) and $(\widetilde{p},\widetilde{\phi})$ are sharp σ-admissible. The collection of such points is a square in $[0,1]^4$ defined by

$$\frac{1}{p} = \frac{\sigma\phi}{2}, \qquad \frac{1}{\widetilde{p}} = \frac{\sigma\widetilde{\phi}}{2}, \qquad \frac{1}{p}, \frac{1}{\widetilde{p}} \in \Big[0,\frac{1}{2}\Big], \qquad \phi,\widetilde{\phi} \in [0,1] \quad (5.16)$$

and if $\sigma = 1$ then we require that $(p,\phi) \ne (2,1)$ and $(\widetilde{p},\widetilde{\phi}) \ne (2,1)$.

Second, Ψ_* contains the convex hull of the above square with the point $(0,1;0,1)$. These are points of the form

$$\big(\alpha/p, 1+\alpha(\phi-1); \alpha/\widetilde{p}, 1+\alpha(\widetilde{\phi}-1)\big)$$

where (p,ϕ) and $(\widetilde{p},\widetilde{\phi})$ satisfy (5.16) and $0 \le \alpha \le 1$.

Third, Ψ_* contains points of the form $(1/q,\theta;1/\widetilde{q},\widetilde{\theta})$ where

$$\frac{1}{q} \ge \frac{\alpha}{p}, \qquad \theta = 1+\alpha(\phi-1), \qquad \frac{1}{\widetilde{q}} \ge \frac{\alpha}{\widetilde{p}}, \qquad \theta = 1+\alpha(\widetilde{\phi}-1).$$

Hence Ψ_* is the set of points $(1/q,\theta;1/\widetilde{q},\widetilde{\theta})$ in $[0,1]^4$ for which there exist p, $\widetilde{p}$, θ, $\widetilde{\theta}$ and α such that

$$\begin{aligned}
&\frac{1}{p} = \frac{\sigma\phi}{2}, && \frac{1}{\widetilde{p}} = \frac{\sigma\widetilde{\phi}}{2},\\
&\frac{1}{p},\frac{1}{\widetilde{p}} \in \left[0,\frac{1}{2}\right], && \phi,\widetilde{\phi},\alpha \in [0,1],\\
&\frac{1}{q} \ge \frac{\alpha}{p}, && \frac{1}{\widetilde{q}} \ge \frac{\alpha}{\widetilde{p}},\\
&\theta = 1+\alpha(\phi-1), && \theta = 1+\alpha(\widetilde{\phi}-1),
\end{aligned}$$

and if $\sigma = 1$ then we require that $\phi \ne 1$ and $\widetilde{\phi} \ne 1$. We will show that this description is identical to the one given by the conditions of Theorem 5.1.6.

The last two equalities can be used to eliminate ϕ and $\widetilde{\phi}$:

$$\begin{aligned}
&\frac{\alpha}{p} = \frac{\sigma}{2}(\theta+\alpha-1), && \frac{\alpha}{\widetilde{p}} = \frac{\sigma}{2}(\widetilde{\theta}+\alpha-1),\\
&\frac{1}{p},\frac{1}{\widetilde{p}} \in \left[0,\frac{1}{2}\right], && \theta,\widetilde{\theta} \in [1-\alpha,1],\\
&\frac{1}{q} \ge \frac{\alpha}{p}, && \frac{1}{\widetilde{q}} \ge \frac{\alpha}{\widetilde{p}},\\
&\alpha \in [0,1].
\end{aligned}$$

Eliminate p and $\widetilde{p}$ using the first two inequalities:

$$\begin{aligned}
&0 \le \frac{\sigma}{2}(\theta+\alpha-1) \le \frac{\alpha}{2}, && 0 \le \frac{\sigma}{2}(\widetilde{\theta}+\alpha-1) \le \frac{\alpha}{2},\\
&\alpha \in [0,1], && \theta,\widetilde{\theta} \in [1-\alpha,1],\\
&\frac{1}{q} \ge \frac{\sigma}{2}(\theta+\alpha-1), && \frac{1}{\widetilde{q}} \ge \frac{\sigma}{2}(\widetilde{\theta}+\alpha-1).
\end{aligned}$$

If we rearrange the inequalities we get

$$\begin{aligned}
&\alpha \in [0,1],\\
&1-\alpha \le \theta \le \alpha(1/\sigma-1)+1, && 1-\alpha \le \widetilde{\theta} \le \alpha(1/\sigma-1)+1,\\
&\frac{1}{q} + \frac{\sigma}{2}(1-\theta) \ge \frac{\sigma\alpha}{2}, && \frac{1}{\widetilde{q}} + \frac{\sigma}{2}(1-\widetilde{\theta}) \ge \frac{\sigma\alpha}{2}.
\end{aligned}$$

The quantity α is isolated:

$$
\begin{aligned}
&0 \leq \alpha \leq 1, \\
&1-\theta \leq \alpha \leq \frac{\sigma(1-\theta)}{\sigma-1}, && 1-\widetilde{\theta} \leq \alpha \leq \frac{\sigma(1-\widetilde{\theta})}{\sigma-1}, \\
&\alpha \leq \frac{2}{\sigma q}+1-\theta, && \alpha \leq \frac{2}{\sigma \widetilde{q}}+1-\widetilde{\theta}.
\end{aligned}
$$

Now there exists some α which satisfies the above system of inequalities if and only if any expression which appears on the left of α in these inequalities is less than or equal to any expression which appears on the right. This means that

$$
\begin{aligned}
&0 \leq \frac{\sigma(1-\theta)}{\sigma-1}, && 0 \leq \frac{\sigma(1-\widetilde{\theta})}{\sigma-1} \\
&1-\theta \leq \frac{\sigma(1-\widetilde{\theta})}{\sigma-1}, && 1-\widetilde{\theta} \leq \frac{\sigma(1-\theta)}{\sigma-1}, \\
&1-\widetilde{\theta} \leq \frac{2}{\sigma q}+1-\theta, && 1-\theta \leq \frac{2}{\sigma \widetilde{q}}+1-\widetilde{\theta}.
\end{aligned}
$$

We rearrange these inequalities into their final form:

$$
\begin{aligned}
&\theta \leq 1, && \widetilde{\theta} \leq 1, \\
&(\sigma-1)(1-\theta) \leq \sigma(1-\widetilde{\theta}), && (\sigma-1)(1-\widetilde{\theta}) \leq \sigma(1-\theta), \\
&\frac{1}{q} \geq \frac{\sigma}{2}(\theta-\widetilde{\theta}), && \frac{1}{\widetilde{q}} \geq \frac{\sigma}{2}(\widetilde{\theta}-\theta).
\end{aligned}
$$

We recall that if $\sigma = 1$ then $\theta \neq 1$ and $\widetilde{\theta} \neq 1$. It is now clear that these conditions coincide with those given in the statement of Theorem 5.1.6. $\square$

Recall (see Proposition 4.3.7) that the energy estimate (4.10) and untruncated decay estimate (4.11) are invariant with respect to the rescaling (4.29). We shall apply this scaling to the local inhomogeneous estimate (5.7) to obtain a version of Theorem 5.1.6 for intervals I and J that don't have unit length.

When scaling (4.29) is applied, (5.7) becomes

$$\lambda^{-\sigma\theta/2}\left(\int_J \left\| \int_{\mathbb{R}} U(t/\lambda)U(s/\lambda)^* F(s)\,\mathrm{d}s \right\|_{\mathcal{B}_\theta^*}^q \mathrm{d}t\right)^{1/q} \leq C\lambda^{\sigma\widetilde{\theta}/2} \|F\|_{L^{\widetilde{q}'}(I;\mathcal{B}_{\widetilde{\theta}})}.$$

The substitution $s \mapsto \lambda s$ gives

$$\lambda^{-\sigma\theta/2+1}\left(\int_J \|(TT^*F_0)(t/\lambda)\|_{\mathcal{B}_\theta^*}^q \,\mathrm{d}t\right)^{1/q} \leq C\lambda^{\sigma\widetilde{\theta}/2+1/\widetilde{q}'} \|F_0\|_{L^{\widetilde{q}'}(\lambda^{-1}I;\mathcal{B}_{\widetilde{\theta}})}$$

where $F_0(s) = F(\lambda s)$. A further substitution $t \mapsto \lambda t$ yields

$$\lambda^{-\sigma\theta/2+1+1/q} \|TT^*F_0\|_{L^q(\lambda^{-1}J;\mathcal{B}_\theta^*)} \leq C\lambda^{\sigma\widetilde{\theta}/2+1-1/\widetilde{q}} \|F_0\|_{L^{\widetilde{q}'}(\lambda^{-1}I;\mathcal{B}_{\widetilde{\theta}})}.$$

Hence

$$\|TT^*F_0\|_{L^q(\lambda^{-1}J;\mathcal{B}_\theta^*)} \leq C\lambda^{-\beta(q,\theta;\widetilde{q},\widetilde{\theta})} \|F_0\|_{L^{\widetilde{q}'}(\lambda^{-1}I;\mathcal{B}_{\widetilde{\theta}})}$$

where

$$\beta(q,\theta;\widetilde{q},\widetilde{\theta}) = \frac{1}{q} + \frac{1}{\widetilde{q}} - \frac{\sigma}{2}(\theta + \widetilde{\theta}). \tag{5.17}$$

If we replace λ with λ^{-1} in the last inequality then we obtain the following proposition.

Proposition 5.2.2. *Suppose that $\sigma > 0$, $\lambda > 0$ and $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate (4.10) and the untruncated decay estimate (4.11). Assume also that I and J are two time intervals of length λ separated by a distance of scale λ (that is, $|I| = |J| = \lambda$ and $\operatorname{dist}(I, J) \approx \lambda$). Then the local inhomogeneous Strichartz estimate*

$$\|TT^*F\|_{L^q(J;\mathcal{B}_\theta^*)} \lesssim \lambda^{\beta(q,\theta;\widetilde{q},\widetilde{\theta})} \|F\|_{L^{\widetilde{q}'}(I;\mathcal{B}_{\widetilde{\theta}})} \qquad \forall F \in L^{\widetilde{q}'}(I;\mathcal{B}_{\widetilde{\theta}}) \cap L^1(I;\mathcal{B}_0) \tag{5.18}$$

holds whenever the pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ satisfy the conditions appearing in Theorem 5.1.6.

5.3 Dyadic decompositions

Recall from Lemma 4.3.1 that the bilinear estimate (4.21) is equivalent to the operator estimate (4.16). In Sections 4.5 and 4.6, we decomposed the bilinear form B dyadically as a sum $\sum_{j\in\mathbb{Z}} B_j$ to prove Strichartz estimates. Theorem 5.1.2 will be similarly proved by decomposing B into a sum of localised operators to which Proposition 5.2.2 is applied. What was achieved in Chapter 4 by taking a simple dyadic decomposition of B combined with the localisation of the functions F and G can also be achieved by taking a *Whitney decomposition* of the domain of integration $\{(s,t) \in \mathbb{R}^2 : s < t\}$ in (4.21). We adopt the latter approach in Chapter 5.

We begin with a few preliminaries. We say that λ is a dyadic number if $\lambda = 2^k$ for some integer k. The set $2^{\mathbb{Z}}$ of all dyadic numbers is a multiplicative Abelian group. In this and the following three sections, λ always denotes a dyadic number.

We say that a square in $\mathbb{R}^2$ is a *dyadic square* if its side length λ is a dyadic number and if the all the coordinates if its vertices are integer multiples of λ. Any open set in $\mathbb{R}^2$ can be decomposed as the union of essentially disjoint cubes whose lengths are proportional to their distance from the boundary of the open set. In fact, this is true of $\mathbb{R}^n$ in general. If Q is a cube in $\mathbb{R}^n$ then let $\ell(Q)$ denote its length.

Lemma 5.3.1 (Dyadic Whitney decomposition). *[29, Appendix J] If Ω is a proper open subset of $\mathbb{R}^n$ then there exists a countable family $\mathcal{Q}$ of closed dyadic cubes such that*

(a) $\bigcup_{Q\in\mathcal{Q}} Q = \Omega$ and the interiors of the cubes in $\mathcal{Q}$ are pairwise disjoint,

(b) $\sqrt{n}\,\ell(Q) \le \operatorname{dist}(Q,\Omega^c) \le 4\sqrt{n}\,\ell(Q)$ for every cube Q in $\mathcal{Q}$,

(c) if the boundaries of two cubes Q and Q' in $\mathcal{Q}$ touch then

$$\frac{1}{4} \le \frac{\ell(Q)}{\ell(Q')} \le 4$$

and

(d) for any given cube Q in $\mathcal{Q}$ there are at most 12^n other cubes in $\mathcal{Q}$ that touch it.

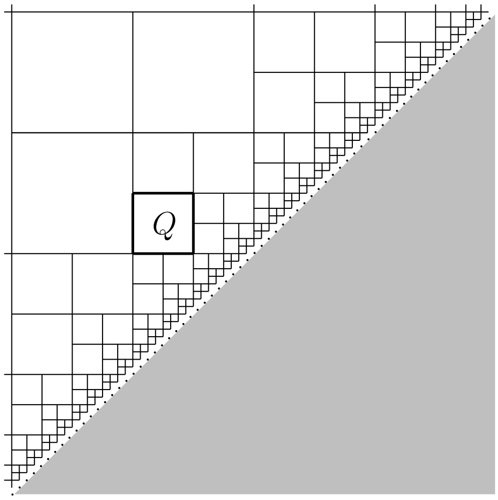


Figure 5.3: Whitney's decomposition for the region $s < t$.

From now on, let $\mathcal{Q}$ denote a Dyadic Whitney decomposition, given by Lemma 5.3.1 and illustrated in Figure 5.3, for the domain Ω, where

$$\Omega = \{(s,t) \in \mathbb{R}^2 : s < t\}.$$

For each dyadic number λ, let $\mathcal{Q}_\lambda$ denote the family contained in $\mathcal{Q}$ consisting of squares with side length λ. Each square Q in $\mathcal{Q}_\lambda$ is the Cartesian product $I \times J$ of two intervals of $\mathbb{R}$ and has the property that

$$\lambda = |I| = |J| \approx \operatorname{dist}(Q, \partial\Omega) \approx \operatorname{dist}(I, J). \tag{5.19}$$

Since

$$\Omega = \sum_{\lambda \in 2^{\mathbb{Z}}} \sum_{Q \in \mathcal{Q}_\lambda} Q$$

and the squares Q in the decomposition are essentially disjoint, we have the decomposition

$$B = \sum_{\lambda \in 2^{\mathbb{Z}}} \sum_{Q \in \mathcal{Q}_\lambda} B_Q, \tag{5.20}$$

where B_Q is given by (5.11) whenever $Q = I \times J$. The scaled version

$$|B_Q(F,G)| \lesssim \lambda^{\beta(q,\theta;\widetilde{q},\widetilde{\theta})} \|F\|_{L^{\widetilde{q}'}(I;\mathcal{B}_{\widetilde{\theta}})} \|G\|_{L^{q'}(J;\mathcal{B}_\theta)}$$
$$\forall F \in L^{\widetilde{q}'}(I;\mathcal{B}_{\widetilde{\theta}}) \cap L^1(I;\mathcal{B}_0) \quad \forall G \in L^{q'}(J;\mathcal{B}_\theta) \cap L^1(J;\mathcal{B}_0) \tag{5.21}$$

of (5.12) is equivalent to the scaled local inhomogeneous Strichartz estimate (5.18). The next proposition will enable us to replace the spaces $L^{\widetilde{q}'}(I;\mathcal{B}_{\widetilde{\theta}})$ and $L^{q'}(J;\mathcal{B}_\theta)$ with $L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})$ and $L^{q'}(\mathbb{R};\mathcal{B}_\theta)$ at the cost of imposing another condition on $\widetilde{q}$ and q.

Proposition 5.3.2. *Suppose that $\sigma > 0$, $1/q + 1/\widetilde{q} \leq 1$, λ is a dyadic number and $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate (4.10) and untruncated decay (4.11). If the pairs (q,θ) and $(\widetilde{q},\widetilde{\theta})$ satisfy the conditions appearing in Theorem 5.1.6 then*

$$\sum_{Q\in\mathcal{Q}_\lambda} |B_Q(F,G)| \lesssim \lambda^{\beta(q,\theta;\widetilde{q},\widetilde{\theta})} \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)}$$

$$\forall F \in L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}}) \cap L^1(\mathbb{R};\mathcal{B}_0) \quad \forall G \in L^{q'}(\mathbb{R};\mathcal{B}_\theta) \cap L^1(\mathbb{R};\mathcal{B}_0). \tag{5.22}$$

The proposition is an immediate consequence of Proposition 5.2.2, the equivalence of (5.18) and (5.21), and the following lemma.

Lemma 5.3.3. *Suppose that $1/p + 1/\widetilde{p} \geq 1$. If λ is a dyadic number then*

$$\sum_{I\times J\in\mathcal{Q}_\lambda} \|f\|_{L^{\widetilde{p}}(I)} \|g\|_{L^p(J)} \leq 4 \|f\|_{L^{\widetilde{p}}(\mathbb{R})} \|g\|_{L^p(\mathbb{R})}$$

whenever $f \in L^{\widetilde{p}}(\mathbb{R})$ and $g \in L^p(\mathbb{R})$.

Proof. Suppose that $f \in L^{\widetilde{p}}(\mathbb{R})$ and $g \in L^p(\mathbb{R})$. The inequality

$$\sum_{n\in\mathbb{Z}} |a_n b_n| \leq \Big(\sum_{n\in\mathbb{Z}} |a_n|^{\widetilde{p}}\Big)^{1/\widetilde{p}} \Big(\sum_{n\in\mathbb{Z}} |b_n|^p\Big)^{1/p},$$

valid whenever $1/p + 1/\widetilde{p} \geq 1$, gives

$$\sum_{I\times J\in\mathcal{Q}_\lambda} \|f\|_{L^{\widetilde{p}}(I)} \|g\|_{L^p(J)} \leq \Big(\sum_{I\times J\in\mathcal{Q}_\lambda} \|f\|^{\widetilde{p}}_{L^{\widetilde{p}}(I)}\Big)^{1/\widetilde{p}} \Big(\sum_{I\times J\in\mathcal{Q}_\lambda} \|g\|^p_{L^p(J)}\Big)^{1/p}.$$

Note that for each dyadic interval J there are at most two dyadic intervals I such that $I \times J \in \mathcal{Q}_\lambda$ (see (5.19) and Figure 5.3). Also each such interval J has the form $[m\lambda, (m+1)\lambda]$ where $m \in \mathbb{Z}$. Hence

$$\Big(\sum_{I\times J\in\mathcal{Q}_\lambda} \|g\|^p_{L^p(J)}\Big)^{1/p} \leq \Big(2\sum_{m\in\mathbb{Z}} \int_{m\lambda}^{(m+1)\lambda} |g(t)|^p \,\mathrm{d}t\Big)^{1/p} \leq 2 \|g\|_{L^p(\mathbb{R})}.$$

A similar argument applied to the sum involving f completes the proof. ∎

5.4 Proof of Theorem 5.1.2

In this section we shall prove Theorem 5.1.2. Our proof marks a total departure from the approach of Foschi [24, Sections 4 and 5], whose chief technical tool is p-atomic decomposition of L^p functions. In our abstract setting, the luxury of such decompositions for elements of the Banach space $\mathcal{B}_\theta$ is not present (but see Remark 5.5.3). Instead we prefer to use an abstract argument that appeals to real interpolation theory in much the same way as [42, Section 6] (see Section 4.5). The advantage of this approach is twofold. First, the proofs are shorter than Foschi's proofs. Second, it admits function spaces other than the Lebesgue spaces. It must be conceded that some transparency may lost by using the abstract interpolation argument; as such we present in Section 5.6 an alternate proof of Theorem 5.4.1 imitating Foschi's approach.

Theorem 5.4.1. *Suppose that $\sigma > 0$ and that $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate (4.10) and the untruncated decay estimate (4.11). Then the inhomogeneous Strichartz estimate (4.16) holds whenever the exponent pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ satisfy the conditions*

$$\begin{gathered}
q, \widetilde{q} \in (1, \infty), \qquad \theta, \widetilde{\theta} \in [0, 1], \\
(\sigma - 1)(1 - \theta) \leq \sigma(1 - \widetilde{\theta}), \qquad (\sigma - 1)(1 - \widetilde{\theta}) \leq \sigma(1 - \theta), \\
\frac{1}{q} > \frac{\sigma}{2}(\theta - \widetilde{\theta}), \qquad \frac{1}{\widetilde{q}} > \frac{\sigma}{2}(\widetilde{\theta} - \theta), \\
\frac{1}{q} + \frac{1}{\widetilde{q}} < 1
\end{gathered}$$

and

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{\sigma}{2}(\theta + \widetilde{\theta}). \tag{5.23}$$

If $\sigma = 1$ then we also require that $\theta < 1$ and $\widetilde{\theta} < 1$.

Proof. Suppose that the exponent pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ satisfy the conditions appearing in the statement of the theorem. Then there is a positive ϵ such that the pairs (q_0, θ) and $(\widetilde{q}_0, \widetilde{\theta})$ and the pairs (q_1, θ) and $(\widetilde{q}_1, \widetilde{\theta})$, defined by

$$\begin{aligned} \frac{1}{q_0} &= \frac{1}{q} - \epsilon, \qquad & \frac{1}{\widetilde{q}_0} &= \frac{1}{\widetilde{q}} - \epsilon, \\ \frac{1}{q_1} &= \frac{1}{q} + 2\epsilon, \qquad & \frac{1}{\widetilde{q}_1} &= \frac{1}{\widetilde{q}} + 2\epsilon, \end{aligned}$$

also satisfy all the conditions appearing in the statement of the theorem except for (5.23).

Define a function $\widetilde{B}$ on $L^1(\mathbb{R}; \mathcal{B}_0) \times L^1(\mathbb{R}; \mathcal{B}_0)$ by

$$\widetilde{B}(F, G) = \left\{ \sum_{Q \in \mathcal{Q}_{2^{-j}}} B_Q(F, G) \right\}_{j \in \mathbb{Z}}$$

Recall once again the definition of ℓ^p_s given by (3.6). Proposition 5.3.2 implies that the maps

$$\begin{aligned} \widetilde{B} &: L^{\widetilde{q}_0'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}}) \times L^{q_0'}(\mathbb{R}; \mathcal{B}_\theta) \to \ell^\infty_{\beta(q_0,\theta;\widetilde{q}_0,\widetilde{\theta})} \\ \widetilde{B} &: L^{\widetilde{q}_0'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}}) \times L^{q_1'}(\mathbb{R}; \mathcal{B}_\theta) \to \ell^\infty_{\beta(q_1,\theta;\widetilde{q}_0,\widetilde{\theta})} \\ \widetilde{B} &: L^{\widetilde{q}_1'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}}) \times L^{q_0'}(\mathbb{R}; \mathcal{B}_\theta) \to \ell^\infty_{\beta(q_0,\theta;\widetilde{q}_1,\widetilde{\theta})} \end{aligned}$$

are bounded. Note that $\beta(q_1, \theta; \widetilde{q}_0, \widetilde{\theta}) = \beta(q_0, \theta; \widetilde{q}_1, \widetilde{\theta})$. So we may apply Lemma 3.2.4 to obtain the bounded map

$$\begin{aligned} \widetilde{B} : \left(L^{\widetilde{q}_0'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}}), L^{\widetilde{q}_1'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}})\right)_{\eta_0, \widetilde{q}'} \times \left(L^{q_0'}(\mathbb{R}; \mathcal{B}_\theta), L^{q_1'}(\mathbb{R}; \mathcal{B}_\theta)\right)_{\eta_1, q'} \\ \to \left(\ell^\infty_{\beta(q_0,\theta;\widetilde{q}_0,\widetilde{\theta})}, \ell^\infty_{\beta(q_1,\theta;\widetilde{q}_0,\widetilde{\theta})}\right)_{\eta, 1} \end{aligned} \tag{5.24}$$

where $\eta_0 = \eta_1 = \frac{1}{3}$ and $\eta = \eta_0 + \eta_1$. It is easy to check that

$$(1 - \eta)\beta(q_0, \theta; \widetilde{q}_0, \widetilde{\theta}) + \eta\beta(q_1, \theta; \widetilde{q}_0, \widetilde{\theta}) = \beta(q, \theta; \widetilde{q}, \widetilde{\theta}) = 0.$$

If we combine this with (3.7) then (5.24) simplifies to

$$\widetilde{B} : L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}}) \times L^{q'}(\mathbb{R}; \mathcal{B}_\theta) \to \ell^1_0.$$

By (5.20) this is equivalent to the bilinear estimate (4.21) and hence the theorem is proved. $\square$

The above theorem perturbed the time exponents q and $\widetilde{q}$ in estimate (5.22) and then interpolated. For the perturbation to work we required strict inequalities in most of the conditions appearing in Theorem (5.1.6) that involved q and $\widetilde{q}$. To prove the next theorem, we instead perturb the spatial exponents θ and $\widetilde{\theta}$. This allows us to recover some boundary cases that the previous theorem excludes.

Theorem 5.4.2. *Suppose that $\sigma > 0$ and that $\{U(t) : t \in \mathbb{R}\}$ satisfies the energy estimate (4.10) and the untruncated decay estimate (4.11). Then the inhomogeneous Strichartz estimate (5.6) holds whenever the exponent pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ satisfy the conditions*

$$q, \widetilde{q} \in (1, \infty], \qquad \theta, \widetilde{\theta} \in (0, 1),$$

$$(\sigma - 1)(1 - \theta) < \sigma(1 - \widetilde{\theta}), \qquad (\sigma - 1)(1 - \widetilde{\theta}) < \sigma(1 - \theta), \tag{5.25}$$

$$\frac{1}{q} > \frac{\sigma}{2}(\theta - \widetilde{\theta}), \qquad \frac{1}{\widetilde{q}} > \frac{\sigma}{2}(\widetilde{\theta} - \theta), \tag{5.26}$$

$$\frac{1}{q} + \frac{1}{\widetilde{q}} \leq 1 \tag{5.27}$$

and

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{\sigma}{2}(\theta + \widetilde{\theta}). \tag{5.28}$$

Proof. Suppose that the exponent pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ satisfy the conditions appearing in the statement of the theorem. Then there is a positive ϵ such that the pairs (q, θ_0) and $(\widetilde{q}, \widetilde{\theta}_0)$ and the pairs (q, θ_1) and $(\widetilde{q}, \widetilde{\theta}_1)$, defined by

$$\theta_0 = \theta - \epsilon, \qquad \widetilde{\theta}_0 = \widetilde{\theta} - \epsilon,$$

$$\theta_1 = \theta + 2\epsilon, \qquad \widetilde{\theta}_1 = \widetilde{\theta} + 2\epsilon,$$

also satisfy all the conditions appearing in the statement of the theorem except (5.28).

Define a function $\widetilde{B}$ on $L^1(\mathbb{R}; \mathcal{B}_0) \times L^1(\mathbb{R}; \mathcal{B}_0)$ by

$$\widetilde{B}(F, G) = \left\{ \sum_{Q \in \mathcal{Q}_{2^{-j}}} B_Q(F, G) \right\}_{j \in \mathbb{Z}}.$$

Proposition 5.3.2 implies that the maps

$$\widetilde{B} : L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}_0}) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_0}) \to \ell^{\infty}_{\beta(q,\theta_0;\widetilde{q},\widetilde{\theta}_0)}$$
$$\widetilde{B} : L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}_0}) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_1}) \to \ell^{\infty}_{\beta(q,\theta_1;\widetilde{q},\widetilde{\theta}_0)}$$
$$\widetilde{B} : L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}_1}) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_0}) \to \ell^{\infty}_{\beta(q,\theta_0;\widetilde{q},\widetilde{\theta}_1)}$$

are bounded. Note that $\beta(q, \theta_1; \widetilde{q}, \widetilde{\theta}_0) = \beta(q, \theta_0; \widetilde{q}, \widetilde{\theta}_1)$. So we may apply Theorem 3.2.4 to obtain the bounded map

$$\widetilde{B} : \big(L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}_0}), L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}_1})\big)_{\eta_0,\widetilde{q}'} \times \big(L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_0}), L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_1})\big)_{\eta_1,q'} \to \big(\ell^{\infty}_{\beta(q,\theta_0;\widetilde{q},\widetilde{\theta}_0)}, \ell^{\infty}_{\beta(q,\theta_1;\widetilde{q},\widetilde{\theta}_0)}\big)_{\eta,1} \tag{5.29}$$

where $\eta_0 = \eta_1 = \frac{1}{3}$ and $\eta = \eta_0 + \eta_1$. It is easy to check that

$$(1 - \eta)\beta(q, \theta_0; \widetilde{q}, \widetilde{\theta}_0) + \eta\beta(q, \theta_1; \widetilde{q}, \widetilde{\theta}_0) = \beta(q, \theta; \widetilde{q}, \widetilde{\theta}) = 0.$$

If we combine this with (3.7) then (5.29) simplifies to

$$\widetilde{B} : L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta},\widetilde{q}'}) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta,q'}) \to \ell^1_0.$$

By (5.20) this is equivalent to the bilinear estimate

$$|B(F, G)| \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta},\widetilde{q}'})} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_{\theta,q'})}$$

which in turn implies (5.6). □

The two theorems of this section combine to give Theorem 5.1.2. For example, suppose that (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ satisfy the conditions appearing in Theorem 5.1.2 case (ii). If $\theta > 0$ and $\widetilde{\theta} > 0$ then σ-acceptability is equivalent to (5.26) by the scaling condition (5.1). In this case Theorem 5.4.2 shows that the retarded Strichartz estimate (5.6) holds. On the other hand, if either $\theta = 0$ or $\widetilde{\theta} = 0$ then σ-acceptability, (5.1) and (5.5) imply that both (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ are sharp σ-admissible. Hence the Strichartz estimate (4.16) holds by Theorem 4.2.2. But since $q \geq 2$ and $\widetilde{q} \geq 2$, Theorem 3.2.3 gives the continuous embeddings $\mathcal{B}_{\theta,q'} \subseteq \mathcal{B}_\theta$ and $\mathcal{B}_{\widetilde{\theta},\widetilde{q}'} \subseteq \mathcal{B}_{\widetilde{\theta}}$ and (4.16) implies (5.6).

5.5 Atomic decompositions of functions in L^p spaces

As mentioned in the previous section, we shall present an alternate proof of Theorem 5.4.1 by adapting the approach of Foschi [24, Section 4]. The main technical tool of employed, which is of interest in its own right, is a special decomposition of functions belonging to Lebesgue spaces. We remind readers that for this section and the next, λ is always a dyadic number.

Definition 5.5.1. Suppose that $1 \leq p \leq \infty$, X is a measure space and $\mathcal{B}$ is a Banach space. A *p-atom in $L^p(X;\mathcal{B})$ of size λ* is a measurable function $\varphi : X \to \mathcal{B}$ such that

(i) $x \mapsto \varphi(x)$ is supported on a set of measure less than 2λ and

(ii) $\|\varphi\|_{L^\infty(X;\mathcal{B})} \leq \lambda^{-1/p}$.

It follows from the definition that, for any p-atom of size λ and any exponent q in $[1,\infty]$,

$$\|\varphi\|_{L^q(X;\mathcal{B})} \lesssim \lambda^{1/q-1/p}. \tag{5.30}$$

The following lemma says that any function in L^p can be decomposed into a dyadic sum of p-atoms. A sketch proof was indicated by Keel and Tao [42, Section 5] in the scalar-valued case, but these kinds of results appear to have a longer history. In [24] it was observed, without proof, that the natural vector-valued analogue presented below is also true.

Lemma 5.5.2. *If $1 \leq p \leq \infty$, (X,μ) is a σ-finite measure space, $\mathcal{B}$ is a Banach space and $F \in L^p(X;\mathcal{B})$ then F can be decomposed as*

$$F = \sum_{\lambda \in 2^{\mathbb{Z}}} a_\lambda \varphi_\lambda$$

where

(i) each φ_λ is a p-atom in $L^p(X;\mathcal{B})$ of size λ,

(ii) the atoms φ_λ have disjoint supports, and

(iii) each a_λ is a nonnegative constant and if $a = \{a_\lambda\}_{\lambda\in 2^{\mathbb{Z}}}$ then $\|F\|_{L^p(X;\mathcal{B})} \approx \|a\|_{\ell^p}$.

Proof. Suppose that $F \in L^p(X;\mathcal{B})$ and define the distribution function F_* by

$$F_*(\alpha) = \mu(\{x \in X : \|F(x)\|_{\mathcal{B}} > \alpha\})$$

whenever $\alpha > 0$. We note for future reference that the function $\alpha \mapsto F_*(\alpha)$ is nonincreasing and right-continuous (see, for example, [61, p. 166]). For each $\lambda \in 2^{\mathbb{Z}}$ define α_λ, a_λ and φ_λ by the formulae

$$\begin{aligned}
\alpha_\lambda &= \inf\{\alpha > 0 : F_*(\alpha) < \lambda\},\\
a_\lambda &= \lambda^{1/p}\alpha_\lambda,\\
\varphi_\lambda &= \frac{1}{a_\lambda}\, 1_{(\alpha_{2\lambda},\alpha_\lambda]}(\|F\|_{\mathcal{B}})F \qquad \text{for almost every } x \in X.
\end{aligned}$$

(Note that the last equality is interpreted pointwise as

$$\varphi_\lambda(x) = \frac{1}{a_\lambda}\, 1_{(\alpha_{2\lambda},\alpha_\lambda]}\big(\, \|F(x)\|_{\mathcal{B}} \,\big) F(x)$$

for almost every x in X.) Now the function $\lambda \mapsto \alpha_\lambda$ is nonincreasing, so the set $\{(\alpha_{2\lambda},\alpha_\lambda] : \lambda \in 2^{\mathbb{Z}}\}$ consists of pairwise disjoint intervals whose union is $(0,\infty)$. Hence we have property (ii) and the pointwise identities

$$1 = \sum_{\lambda\in 2^{\mathbb{Z}}} 1_{(\alpha_{2\lambda},\alpha_\lambda]}\big(\, \|F\|_{\mathcal{B}} \,\big)$$

and

$$F = \sum_{\lambda\in 2^{\mathbb{Z}}} a_\lambda\varphi_\lambda.$$

We will now show that each φ_λ is a p-atom of length λ. First, if

$$1_{(\alpha_{2\lambda},\alpha_\lambda]}\big(\, \|F(x)\|_{\mathcal{B}} \,\big) = 1$$

then $\|F(x)\|_{\mathcal{B}} \le \alpha_\lambda$ and hence

$$\|\varphi\|_{L^\infty(X;\mathcal{B})} \le \frac{1}{a_\lambda}\,\alpha_\lambda = \lambda^{1/p}.$$

Second, by the right-continuity of F_* and the definition of α_λ,

$$F_*(\alpha_\lambda) \leq \lambda \tag{5.31}$$

for any dyadic number λ. Hence

$$\begin{aligned}
\mu(\{x \in X : \varphi_\lambda(x) \neq 0\}) &\leq \mu(\{x \in X : \alpha_{2\lambda} < \|F(x)\|_{\mathcal{B}} \leq \alpha_\lambda\}) \\
&= \mu(\{x \in X : \|F(x)\|_{\mathcal{B}} > \alpha_{2\lambda}\}) \\
&\quad - \mu(\{x \in X : \|F(x)\|_{\mathcal{B}} > \alpha_\lambda\}) \\
&= F_*(\alpha_{2\lambda}) - F_*(\alpha_\lambda) \\
&\leq F_*(\alpha_{2\lambda}) \\
&\leq 2\lambda
\end{aligned}$$

and each φ_λ is a p-atom of length λ.

It remains to show property (iii). Formally,

$$\begin{aligned}
\sum_{\lambda \in 2^{\mathbb{Z}}} a_\lambda^p &= \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda \alpha_\lambda^p \\
&= \int_0^\infty \alpha^p \Big(\sum_{\lambda \in 2^{\mathbb{Z}}} \lambda \delta(\alpha_\lambda - \alpha) \Big) \,\mathrm{d}\alpha \\
&= \int_0^\infty \alpha^p \big(-G'(\alpha) \big) \,\mathrm{d}\alpha
\end{aligned} \tag{5.32}$$

where δ is the Dirac delta function,

$$G(\alpha) = \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda H(\alpha_\lambda - \alpha)$$

whenever $\alpha > 0$ and H is the Heaviside step function. By the definition of H,

$$G(\alpha) = \sum_{\alpha_\lambda > \alpha} \lambda$$

whenever α is positive and does not belong to the discrete set Λ given by

$$\Lambda = \{\alpha_\lambda : \lambda \in 2^{\mathbb{Z}}\}.$$

Our ultimate goal will be to integrate (5.32) by parts and show that the resulting expression is no bigger than $2\,\|F\|_{L^p(X;\mathcal{B})}^p$. The formal manipulation above can then be justified by working backwards. To fulfil these aims we shall establish the bound

$$F_*(\alpha) \leq G(\alpha) \leq 2F_*(\alpha) \tag{5.33}$$

whenever $\alpha > 0$ and $\alpha \notin \Lambda$.

Suppose that $\alpha > 0$ and $\alpha \notin \Lambda$. If $\{\lambda \in 2^{\mathbb{Z}} : \alpha_\lambda > \alpha\}$ is empty then $F_*(\alpha) = 0 = G(\alpha)$ and we are done. So suppose otherwise. Since $\alpha_\lambda \to 0$ as $\lambda \to \infty$, we can define the dyadic number ν by

$$\nu = \max\{\lambda \in 2^{\mathbb{Z}} : \alpha_\lambda > \alpha\}.$$

Hence

$$G(\alpha) = \sum_{\alpha_\lambda > \alpha} \lambda = \nu + \nu/2 + \nu/2^2 + \cdots = 2\nu.$$

By the definition of ν we have the ordering $\alpha_{2\nu} \leq \alpha < \alpha_\nu$. Hence

$$\nu \leq F_*(\alpha) \leq F_*(\alpha_{2\nu}) \leq 2\nu = G(\alpha) \leq 2F_*(\alpha),$$

where the first inequality is justified by the definition of α_ν, the second by the fact that F_* is nonincreasing, the third by (5.31) and the fourth by the first inequality. This proves (5.33).

Now

$$\int_X \|F(x)\|_{\mathcal{B}}^p \,\mathrm{d}\mu(x) = p \int_0^\infty \alpha^{p-1} F_*(\alpha)\,\mathrm{d}\alpha \tag{5.34}$$

(see [61, p. 163]) and since $F \in L^p(X;\mathcal{B})$ we consequently have $\alpha^p F_*(\alpha) \to 0$ as $\alpha \to 0$ and as $\alpha \to \infty$ (see [61, p. 162]). We use these limits in conjunction with (5.33) to integrate (5.32) by parts. Hence

$$\sum_{\lambda \in 2^{\mathbb{Z}}} a_\lambda^p = p \int_0^\infty \alpha^{p-1} G(\alpha)\,\mathrm{d}\alpha.$$

If we apply the two-sided estimate (5.33) followed by the identity (5.34) then

$$\|F\|_{L^p(X;\mathcal{B})}^p \leq \sum_{\lambda \in 2^{\mathbb{Z}}} a_\lambda^p \leq 2\,\|F\|_{L^p(X;\mathcal{B})}^p$$

and (iii) is proved. □

Remark 5.5.3. Similar atomic compositions exist for functions belonging to spaces other than the Lebesgue spaces L^p. See, for example, the atomic decompositions of the Besov spaces $B^{\rho}_{r,p}$ in [74, Section 1.5]. It is likely that the arguments of the next section can be adapted to these spaces.

5.6 Alternate proof of Theorem 5.1.2

Before presenting an alternate proof of Theorem 5.4.1, we begin with an exposition of the basic strategy. This will point out a technical difficulty which is resolved with a lemma. The proof of the theorem will follow after we establish the lemma. Throughout this section λ, μ and ν denote dyadic numbers.

To establish the global inhomogeneous Strichartz estimate (4.16), we prove the equivalent bilinear estimate (4.21). Suppose that $F \in L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}}) \cap L^1(\mathbb{R};\mathcal{B}_1)$ and $G \in L^{q'}(\mathbb{R};\mathcal{B}_\theta) \cap L^1(\mathbb{R};\mathcal{B}_1)$. By Lemma 5.5.2, we obtain the decompositions

$$F(t) = \sum_{\mu \in 2^{\mathbb{Z}}} a_\mu \varphi_\mu(t), \qquad G(t) = \sum_{\nu \in 2^{\mathbb{Z}}} b_\nu \psi_\nu(t),$$

where φ_μ is a $\widetilde{q}'$-atom in $L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})$ of size μ, ψ_ν is a q'-atom in $L^q(\mathbb{R};\mathcal{B}_\theta)$ of size ν and

$$\|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})} \approx \left\|\{a_\mu\}_{\mu\in 2^{\mathbb{Z}}}\right\|_{\ell^{\widetilde{q}'}}, \qquad \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)} \approx \left\|\{b_\nu\}_{\nu\in 2^{\mathbb{Z}}}\right\|_{\ell^{q'}}. \tag{5.35}$$

If we substitute these function decompositions into the Whitney decomposition (5.20) of B, then

$$B(F,G) = \sum_{\lambda,\mu,\nu\in 2^{\mathbb{Z}}} a_\mu b_\nu \sum_{Q\in\mathcal{Q}_\lambda} B_Q(\varphi_\mu,\psi_\nu). \tag{5.36}$$

By assuming the hypothesis of Proposition 5.3.2 we have

$$\sum_{Q\in\mathcal{Q}_\lambda} |B_Q(\varphi_\mu,\psi_\nu)| \lesssim \lambda^{\beta(q,\theta;\widetilde{q},\widetilde{\theta})} \|\varphi_\mu\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})} \|\psi_\nu\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta)}.$$

However, if we sum the above over all λ in $2^{\mathbb{Z}}$ then the right-hand side will diverge and we make no progress towards estimating the right-hand side of

(5.36). A similar situation was encountered in Section 4.5. There we perturbed the exponents θ and $\widetilde{\theta}$ slightly to gain some summability; here instead we will perturb q and $\widetilde{q}$.

In what follows it will simplify notation if we introduce the function $[\cdot] : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$[\lambda] = \max\{\lambda, 1/\lambda\}.$$

This function plays a role in the multiplicative group $\mathbb{R}^+$ similar to the role played by the absolute values function in the additive group $\mathbb{R}$. In particular, $[2^k] = 2^{|k|}$.

Lemma 5.6.1. *Suppose that $\sigma > 0$, that $\{U(t) : t \in \mathbb{R}\}$ satisfies the estimates (4.10) and (4.11) and that the exponent pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ satisfy the conditions*

$$\begin{aligned} q, \widetilde{q} &\in (1, \infty), & \theta, \widetilde{\theta} &\in [0, 1], \\ (\sigma - 1)(1 - \theta) &\le \sigma(1 - \widetilde{\theta}), & (\sigma - 1)(1 - \widetilde{\theta}) &\le \sigma(1 - \theta), \\ \frac{1}{q} &> \frac{\sigma}{2}(\theta - \widetilde{\theta}), & \frac{1}{\widetilde{q}} &> \frac{\sigma}{2}(\widetilde{\theta} - \theta), \end{aligned} \tag{5.37}$$

and

$$1/q + 1/\widetilde{q} < 1.$$

Then there exists a positive ϵ such that for all dyadic numbers λ, μ and ν,

$$\sum_{Q \in \mathcal{Q}_\lambda} |B_Q(\varphi_\mu, \psi_\nu)| \lesssim \lambda^{\beta(q,\theta;\widetilde{q},\widetilde{\theta})} \left[\frac{\mu}{\lambda}\right]^{-\epsilon} \left[\frac{\nu}{\lambda}\right]^{-\epsilon} \tag{5.38}$$

whenever φ_μ is a $\widetilde{q}'$-atom in $L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}})$ of size μ and ψ_ν is a q'-atom in $L^q(\mathbb{R}; \mathcal{B}_\theta)$ of size ν.

Proof. Since the conditions imposed on q and $\widetilde{q}$ in the hypothesis are given by strict inequalities (in contrast to the conditions of Theorem 5.1.6), there is a full neighbourhood $\mathcal{N}$ of points $(q_0, \widetilde{q}_0)$ about $(q, \widetilde{q})$ such that the exponent pairs (q_0, θ) and $(\widetilde{q}_0, \widetilde{\theta})$ also satisfy the hypothesis of the lemma. Hence the pairs

(q_0, θ) and $(\widetilde{q}_0, \widetilde{\theta})$ also satisfy the hypothesis of Proposition 5.3.2. Therefore, by the proposition and property (5.30) of dyadic atoms,

$$\begin{aligned}\sum_{Q \in \mathcal{Q}_\lambda} |B_Q(\varphi_\mu, \psi_\nu)| &\lesssim \lambda^{\beta(q_0,\theta;\widetilde{q}_0,\widetilde{\theta})} \|\varphi_\mu\|_{L^{\widetilde{q}_0'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})} \|\psi_\nu\|_{L^{q_0}(\mathbb{R};\mathcal{B}_\theta)} \\ &\lesssim \lambda^{\beta(q_0,\theta;\widetilde{q}_0,\widetilde{\theta})} \mu^{1/\widetilde{q}-1/\widetilde{q}_0} \nu^{1/q-1/q_0} \\ &\lesssim \lambda^{\beta(q,\theta;\widetilde{q},\widetilde{\theta})} \left(\frac{\mu}{\lambda}\right)^{1/\widetilde{q}-1/\widetilde{q}_0} \left(\frac{\nu}{\lambda}\right)^{1/q-1/q_0}\end{aligned}$$

whenever $(q_0, \widetilde{q}_0) \in \mathcal{N}$. Now choose $(q_0, \widetilde{q}_0)$ in $\mathcal{N}$ such that

$$\frac{1}{\widetilde{q}} - \frac{1}{\widetilde{q}_0} = \begin{cases} +\epsilon & \text{if } \mu \leq \lambda \\ -\epsilon & \text{if } \mu > \lambda \end{cases}$$

and

$$\frac{1}{q} - \frac{1}{q_0} = \begin{cases} +\epsilon & \text{if } \nu \leq \lambda \\ -\epsilon & \text{if } \nu > \lambda \end{cases}$$

where ϵ is a small positive number depending only on the neighbourhood $\mathcal{N}$. With this choice of $(q_0, \widetilde{q}_0)$ we have

$$\left(\frac{\mu}{\lambda}\right)^{1/\widetilde{q}-1/\widetilde{q}_0} = \left[\frac{\mu}{\lambda}\right]^{-\epsilon}$$

and

$$\left(\frac{\nu}{\lambda}\right)^{1/q-1/q_0} = \left[\frac{\nu}{\lambda}\right]^{-\epsilon},$$

completing the proof. □

Alternate proof of Theorem 5.4.1. Suppose that the exponent pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ satisfy the conditions appearing in Theorem 5.4.1. The decomposition (5.36) of B combines with Lemma 5.6.1 to give

$$|B(F,G)| \lesssim \sum_{\mu,\nu \in 2^{\mathbb{Z}}} a_\mu b_\nu \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{\beta(q,\theta;\widetilde{q},\widetilde{\theta})} \left[\frac{\mu}{\lambda}\right]^{-\epsilon} \left[\frac{\nu}{\lambda}\right]^{-\epsilon} \tag{5.39}$$

for some positive ϵ independent of the dyadic numbers λ, μ and ν. The scaling condition $1/q + 1/\widetilde{q} = \sigma(\theta + \widetilde{\theta})/2$ implies that $\beta(q, \theta; \widetilde{q}, \widetilde{\theta}) = 0$. (We remark that if this were not the case then the sum in λ would diverge). We now compute an upper bound for the sum in λ. If $\lambda = 2^l$, $\mu = 2^m$, $\nu = 2^n$ and $j = m - n$ then

$$\begin{aligned}
\sum_{\lambda \in 2^{\mathbb{Z}}} \left[\frac{\mu}{\lambda}\right]^{-\epsilon} \left[\frac{\nu}{\lambda}\right]^{-\epsilon} &= \sum_{l \in \mathbb{Z}} \left[\frac{2^m}{2^l}\right]^{-\epsilon} \left[\frac{2^n}{2^l}\right]^{-\epsilon} \\
&= \sum_{l \in \mathbb{Z}} 2^{-\epsilon|m-l|} 2^{-\epsilon|n-l|} \\
&= \sum_{l \in \mathbb{Z}} 2^{-\epsilon|j+l|} 2^{-\epsilon|l|}.
\end{aligned}$$

If we consider the case when $j \geq 0$, then

$$\begin{aligned}
\sum_{\lambda \in 2^{\mathbb{Z}}} \left[\frac{\mu}{\lambda}\right]^{-\epsilon} \left[\frac{\nu}{\lambda}\right]^{-\epsilon} &= \sum_{l < -j} 2^{\epsilon(j+l)} 2^{\epsilon l} + \sum_{-j \leq l < 0} 2^{-\epsilon(j+l)} 2^{\epsilon l} + \sum_{l \geq 0} 2^{-\epsilon(j+l)} 2^{-\epsilon l} \\
&= 2^{\epsilon j} \sum_{l > j} \left(2^{-2\epsilon}\right)^l + j 2^{\epsilon j} + 2^{-\epsilon j} \sum_{l \geq 0} \left(2^{-2\epsilon}\right)^l \\
&\leq C(1+j) 2^{-\epsilon j},
\end{aligned}$$

where C is a positive constant independent of j. We combine this with a similar calculation for the case when $j < 0$ and obtain

$$\sum_{\lambda \in 2^{\mathbb{Z}}} \left[\frac{\mu}{\lambda}\right]^{-\epsilon} \left[\frac{\nu}{\lambda}\right]^{-\epsilon} \lesssim c_{m-n},$$

where $c_j = (1 + |j|) 2^{-\epsilon|j|}$ for all integers j. Note that if $c = \{c_j\}_{j \in \mathbb{Z}}$ then $\|c\|_{\ell^1} < \infty$. Continuing from (5.39),

$$\begin{aligned}
|B(F,G)| &\lesssim \sum_{m,n \in \mathbb{Z}} a_{2^m} b_{2^n} c_{m-n} \\
&\leq \|a\|_{\ell^{\widetilde{q}'}} \|b\|_{\ell^{q'}} \|c\|_{\ell^1} \\
&\lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R}; \mathcal{B}_{\widetilde{\theta}})} \|G\|_{L^{q'}(\mathbb{R}; \mathcal{B}_\theta)},
\end{aligned}$$

where the final two estimates are justified by Young's inequality for sequences (see Corollary 3.1.3) and the norm approximation (5.35). We remark that our use of Young's inequality is valid because

$$1/\widetilde{q}' + 1/q' + 1 = 3 - 1/\widetilde{q} + 1/q > 2$$

by the hypothesis that $1/\widetilde{q} + 1/q < 1$. Since we have proved (4.21), the proof of the theorem is complete. □

Remark 5.6.2. This method of proof may be adapted to prove Theorem 5.4.2 in the case when $\mathcal{B}_{\widetilde{\theta}}$ and $\mathcal{B}_\theta$ are the Lebesgue spaces $L^{\widetilde{r}'}$ and $L^{r'}$. Instead of decomposing F and G atomically in $L^{\widetilde{q}'}(\mathbb{R}; L^{\widetilde{r}'})$ and $L^{q'}(\mathbb{R}; L^{r'})$, one decomposes $F(s)$ and $G(t)$ atomically in $L^{\widetilde{r}'}$ and $L^{r'}$ at each point s and t in $\mathbb{R}$. One then perturb the space exponents $\widetilde{r}$ and r to obtain an estimate similar to (5.38). However, when the decomposition is reassembled, the technical condition

$$\widetilde{q} \leq \widetilde{r}, \qquad q \leq r \tag{5.40}$$

is needed to recover sequence estimates in the desired norms (see [24, Section 5]). If instead one applies the abstract result of Theorem 5.4.2 in this setting then condition (5.40) is precisely that needed to extract the $L^{r'}$ and $L^{\widetilde{r}'}$ norms from the $\mathcal{B}_{\theta,q'}$ and $\mathcal{B}_{\widetilde{\theta},\widetilde{q}'}$ norms in (5.6). Thus both methods yield the same result in this context.

5.7 The sharpness of Theorem 5.1.2

In this section we discuss the sharpness of the exponent conditions appearing in Theorem 5.1.2.

Proposition 5.7.1. *Suppose that $\sigma > 0$ and that the global inhomogeneous Strichartz estimate (4.16) holds for any $\{U(t) : t \geq 0\}$ satisfying the energy estimate (4.10) and the untruncated decay estimate (4.11). Then (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ must be σ-admissible pairs which satisfy the following conditions:*

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{\sigma}{2}(\theta + \widetilde{\theta}), \tag{5.41}$$

$$\frac{1}{q} + \frac{1}{\widetilde{q}} \leq 1, \tag{5.42}$$

$$|\theta - \widetilde{\theta}| \leq \frac{1}{\sigma} \tag{5.43}$$

and

$$(\sigma - 1)(1 - \theta) - \frac{2}{q} \leq \sigma(1 - \widetilde{\theta}), \qquad (\sigma - 1)(1 - \widetilde{\theta}) - \frac{2}{q} \leq \sigma(1 - \theta). \tag{5.44}$$

Moreover, if $\sigma = 1$ then the inhomogeneous estimate is false when $\theta = \widetilde{\theta} = 1$.

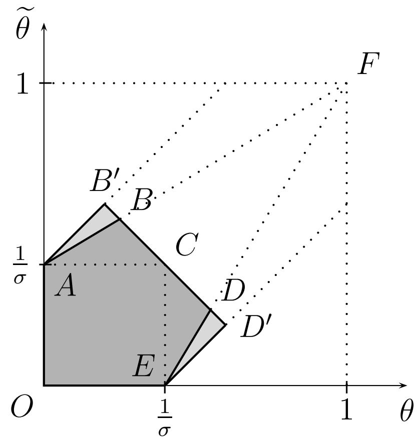


Figure 5.4: Necessary and sufficient conditions on the exponents θ and $\widetilde{\theta}$ for global inhomogeneous Strichartz estimates.

Before proving the proposition, we note that the difference in the necessary and sufficient conditions for the validity of the inhomogeneous Strichartz estimate (4.16) lies in three places. First there is the gap between (5.44) and (5.3). Second, there is the gap between (5.43) and the range of values for θ and $\widetilde{\theta}$ shown in Figure 5.2. This difference is shown by the triangles ABB' and EDD' in Figure 5.4. (More precisely, the region $AOEDB$ in Figure 5.4 corresponds to sufficient conditions for θ and $\widetilde{\theta}$ while the region $AOED'B'$ corresponds to necessary conditions. The boundaries of each region are included except at the points B and D for the sufficient conditions.) Third, there is the difference between (5.42) and the strict inequality of (5.2). This discrepancy is muted somewhat by the validity of the inhomogeneous estimate (5.6) when $1/q + 1/\widetilde{q} = 1$.

Proof. Suppose that $\sigma > 0$ and that the global inhomogeneous Strichartz estimate (4.16) holds for any $\{U(t) : t \geq 0\}$ satisfying the energy estimate (4.10) and the untruncated decay estimate (4.11). We we systemically establish the necessity of each of the conditions above.

Recall that (4.10) and (4.11) are invariant with respect to scaling (4.29).

When the same scaling is applied to (4.16), we obtain

$$\lambda^{\sigma\theta/2+1+1/q} \|(TT^*)_R F\|_{L^q(\mathbb{R};\mathcal{B}^*_\theta)} \lesssim \lambda^{\sigma\widetilde{\theta}/2+1/\widetilde{q}'} \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}})}$$
$$\forall F \in L^{\widetilde{q}'}(\mathbb{R};\mathcal{B}_{\widetilde{\theta}}) \cap L^1(\mathbb{R};\mathcal{B}_0)$$

(see the discussion prior to Proposition 5.2.2 for a similar calculation). Invariance with respect to scaling requires that

$$\frac{\sigma\theta}{2} + 1 + \frac{1}{q} = \frac{\sigma\widetilde{\theta}}{2} + \frac{1}{\widetilde{q}'},$$

which is equivalent to (5.41).

To show the necessity of condition (5.42), consider any family $\{U(t) : t \in \mathbb{R}\}$ which possesses the group property $U(t)U(s)^* = U(t-s)$ whenever s and t are real numbers (the Schrödinger group given by $U(t) = e^{-it\Delta}$ will suffice). Under such circumstances, $(TT^*)_R$ is translation invariant by Lemma 4.3.6 and hence Lemma 3.5.2 implies that $\widetilde{q}' \leq q$. This last inequality is equivalent to (5.42).

The necessity of (5.43) and (5.44) is the result of two particular forcing terms F constructed for the Schrödinger group (see [24, Examples 6.9 and 6.10] for details).

Finally, the exclusion of the case $(\theta, \widetilde{\theta}, \sigma) = (1, 1, 1)$ follows from the negative result of T. Tao [69] for the Schrödinger equation in two spatial dimensions. □

5.8 Applications to the wave, Schrödinger and Klein–Gordon equations

We illustrate how the abstract results of Theorem 5.1.2 give Strichartz estimates for inhomogeneous Schrödinger, wave and Klein–Gordon equations. For the first equation, the results aren't new (see Foschi [24, Section 6] and similar results by Vilela [75]). However, in the case of the wave equation, many of the Strichartz estimates are new. The estimates we give for the Klein–Gordon equation improves slightly the results of M. Nakamura and T. Ozawa [53] by admitting some boundary exponents.

Suppose that n is a positive integer. We say that a pair (q, r) of Lebesgue exponents are *Schrödinger n-acceptable* if either

$$1 \leq q < \infty, \qquad 2 \leq r \leq \infty, \qquad \frac{1}{q} < n\left(\frac{1}{2} - \frac{1}{r}\right)$$

or $(q, r) = (\infty, 2)$.

Corollary 5.8.1 (Foschi–Vilela). *Suppose that n is a positive integer and that the exponent pairs (q, r) and $(\widetilde{q}, \widetilde{r})$ are Schrödinger n-acceptable, satisfy the scaling condition*

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{n}{2}\left(1 - \frac{1}{r} - \frac{1}{\widetilde{r}}\right)$$

and either the conditions

$$\frac{1}{q} + \frac{1}{\widetilde{q}} < 1, \qquad \frac{n-2}{r} \leq \frac{n}{\widetilde{r}}, \qquad \frac{n-2}{\widetilde{r}} \leq \frac{n}{r}$$

or the conditions

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = 1, \qquad \frac{n-2}{r} < \frac{n}{\widetilde{r}}, \qquad \frac{n-2}{\widetilde{r}} < \frac{n}{r}, \qquad \frac{1}{r} \leq \frac{1}{q}, \qquad \frac{1}{\widetilde{r}} \leq \frac{1}{\widetilde{q}}.$$

When $n = 2$ we also require that $r < \infty$ and $\widetilde{r} < \infty$. If $F \in L^{\widetilde{q}'}(\mathbb{R}; L^{\widetilde{r}'}(\mathbb{R}^n))$ and u is a weak solution of the inhomogeneous Schrödinger equation

$$iu'(t) + \Delta u(t) = F(t), \qquad u(0) = 0$$

then

$$\|u\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^n))} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R}; L^{\widetilde{r}'}(\mathbb{R}^n))}. \tag{5.45}$$

Proof. In light of the work done in Section 4.7, this is a simple application of Theorem 5.1.2 when $(\mathcal{B}_0, \mathcal{B}_1) = (L^2(\mathbb{R}^n), L^1(\mathbb{R}^n))$, $\sigma = n/2$. To obtain (5.45) from (5.6), we use the embedding $L^{r'}(\mathbb{R}^n) \subseteq L^{r', q'}(\mathbb{R}^n)$ whenever $r' \leq q'$ (see Section 3.3). $\square$

Suppose that n is a positive integer. We say that a pair (q, r) of Lebesgue exponents are *wave n-acceptable* if either

$$1 \le q < \infty, \qquad 2 \le r \le \infty, \qquad \frac{1}{q} < (n-1)\left(\frac{1}{2} - \frac{1}{r}\right)$$

or $(q, r) = (\infty, 2)$.

Corollary 5.8.2. *Suppose that n is a positive integer and that the exponent pairs (q, r_1) and $(\widetilde{q}, \widetilde{r}_1)$ are wave n-acceptable, satisfy the scaling condition*

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{n-1}{2}\left(1 - \frac{1}{r_1} - \frac{1}{\widetilde{r}_1}\right)$$

and the conditions

$$\frac{1}{q} + \frac{1}{\widetilde{q}} < 1, \qquad \frac{n-3}{r_1} \le \frac{n-1}{\widetilde{r}_1}, \qquad \frac{n-3}{\widetilde{r}_1} \le \frac{n-1}{r_1}$$

When $n = 3$ we also require that $r_1 < \infty$ and $\widetilde{r}_1 < \infty$. If $r \ge r_1$, $\widetilde{r} \ge \widetilde{r}_1$, $\rho \in \mathbb{R}$,

$$\rho + n\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q} = 1 - \left(\widetilde{\rho} + n\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right) - \frac{1}{\widetilde{q}}\right),$$

$F \in L^{\widetilde{q}'}(\mathbb{R}; \dot{B}^{-\widetilde{\rho}}_{\widetilde{r}',2})$ and u is a weak solution of the inhomogeneous wave equation

$$-u''(t) + \Delta u(t) = F(t), \qquad u(0) = 0, \qquad u'(0) = 0,$$

then

$$\|u\|_{L^q(\mathbb{R};\dot{B}^{\rho}_{r,2})} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\dot{B}^{-\widetilde{\rho}}_{\widetilde{r}',2})}. \tag{5.46}$$

Figure 5.5 shows the range for various exponents appearing in Corollary 5.8.2. In the first diagram, the dark region represents the range for the homogeneous Strichartz estimate while the union of light and dark regions represents the range for the inhomogeneous Strichartz estimate. In the second diagram, the coordinates of C and D are given by

$$\left(\frac{(n-3)^2}{2(n-2)(n-1)}, \frac{n-3}{2(n-2)}\right)$$

and

$$\left(\frac{n-3}{2(n-2)}, \frac{(n-3)^2}{2(n-2)(n-1)}\right)$$

respectively.

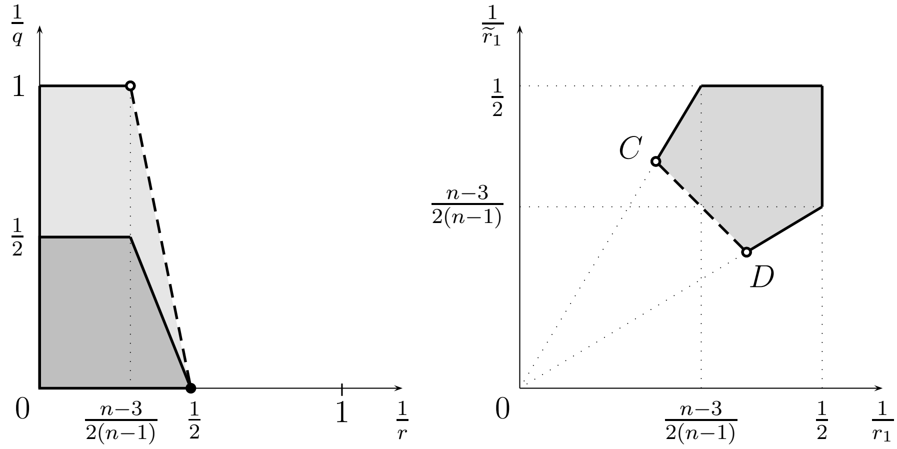


Figure 5.5: Range of exponents for Corollary 5.8.2 when $n > 3$.

Proof. In light of the work done in Section 4.8, the case when $r = r_1$ and $\widetilde{r} = \widetilde{r}_1$ is a simple application of Theorem 5.1.2 when $(\mathcal{B}_0, \mathcal{B}_1) = (\dot{B}^0_{2,2}, \dot{B}^{(n+1)/4}_{1,2})$ and $\sigma = (n-1)/2$. The case when $r > r_1$ and $\widetilde{r} > \widetilde{r}_1$ is obtained using the Besov embedding result of Lemma 3.4.2. $\square$

The case when $1/q + 1/\widetilde{q} = 1$ cannot be simply integrated into the above result via Besov embedding, except when $q = \widetilde{q} = 2$ (which corresponds to a sharp admissible estimate). We therefore state this case separately, using the notation $a \vee b$ and $a \wedge b$ for $\max\{a, b\}$ and $\min\{a, b\}$ respectively.

Corollary 5.8.3. *Suppose that n is a positive integer not equal to 3 and that the exponent pairs (q, r_1) and $(\widetilde{q}, \widetilde{r}_1)$ are wave n-acceptable, satisfy the scaling condition*

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{n-1}{2}\left(1 - \frac{1}{r_1} - \frac{1}{\widetilde{r}_1}\right)$$

and the conditions

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = 1, \qquad \frac{n-3}{r_1} < \frac{n-1}{\widetilde{r}_1}, \qquad \frac{n-3}{\widetilde{r}_1} < \frac{n-1}{r_1}, \qquad \frac{1}{r_1} \leq \frac{1}{q}, \qquad \frac{1}{\widetilde{r}_1} \leq \frac{1}{\widetilde{q}}.$$

If $r \geq r_1$, $\widetilde{r} \geq \widetilde{r}_1$, $\rho \in \mathbb{R}$,

$$\rho + n\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q} = 1 - \left(\widetilde{\rho}_1 + n\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right) - \frac{1}{\widetilde{q}}\right),$$

$F \in L^{\widetilde{q}'}(\mathbb{R}; \dot{B}^{-\widetilde{\rho}}_{\widetilde{r}',2})$ *and u is a weak solution of the inhomogeneous wave equation*

$$-u''(t) + \Delta u(t) = F(t), \qquad u(0) = 0, \qquad u'(0) = 0,$$

then

$$\|u\|_{L^q(\mathbb{R};\dot{B}^{\rho}_{r,2\vee q})} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};\dot{B}^{-\widetilde{\rho}}_{\widetilde{r}',2\wedge q})}. \tag{5.47}$$

Proof. We apply Theorem 5.1.2 (ii) when $\sigma = (n-1)/2$ and $(\mathcal{B}_0, \mathcal{B}_1) = (\dot{B}^0_{2,2}, \dot{B}^{(n+1)/4}_{1,2})$.

First suppose that $r = r_1$ and $\widetilde{r} = \widetilde{r}_1$. To obtain (5.47) from the abstract Strichartz estimate (5.6), we apply the embeddings

$$\mathcal{B}_{\widetilde{\theta},\widetilde{q}'} \supseteq \dot{B}^{(n+1)\widetilde{\theta}/4}_{\widetilde{r}',2\vee\widetilde{q}',(\widetilde{q}')} \supseteq \dot{B}^{(n+1)\widetilde{\theta}/4}_{\widetilde{r}',2\vee\widetilde{q}'}$$

and

$$(\mathcal{B}_{\theta,q'})^* = (\dot{B}^0_{2,2}, \dot{B}^{-(n+1)/4}_{\infty,2})_{\theta,q} \subseteq \dot{B}^{-(n+1)\theta/4}_{r,2\wedge q,(q)} \subseteq \dot{B}^{-(n+1)\theta/4}_{r,2\wedge q}$$

(see [72, p. 183], [2, Theorem 3.7.1] and Lemma 3.3.4) and follow the general approach of the proofs in Section 4.8. Here we have taken $1/\widetilde{r}' = (1-\widetilde{\theta})/2 + \widetilde{\theta}/1$, $1/r = (1-\theta)/2 + \theta/\infty$, imposed the restrictions $\widetilde{r}' \le \widetilde{q}'$ and $q \le r$ and used the fact that $\widetilde{q}' = q$.

Suppose now that $r > r_1$ and $\widetilde{r} > \widetilde{r}_1$. To obtain (5.47), simply apply Besov embedding (Lemma 3.4.2) to the result obtained for the case when $r = r_1$ and $\widetilde{r} = \widetilde{r}_1$. $\square$

Remark 5.8.4. The generalised Strichartz estimates given by the two corollaries above appear to be new except when the conditions coincide with those in Corollary 4.8.1. The results of Foschi [24] may be applied to the wave equation to obtain the inhomogeneous Strichartz estimate

$$\|u\|_{L^q(\mathbb{R};L^{r_1}(\mathbb{R}^n))} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};L^{\widetilde{r}_1'}(\mathbb{R}^n))},$$

provided that r_1 and $\widetilde{r}_1$ are strictly finite, and that (q, r_1) and $(\widetilde{q}, \widetilde{r}_1)$ satisfy either the conditions of Corollary 5.8.2 or the conditions of Corollary 5.8.3 with the additional assumption that $r_1 \ge q$ and $\widetilde{r}_1 \ge \widetilde{q}$. These results of Foschi extend previous results by D. Oberlin [54] and J. Harmse [31], who proved similar results in the case when $q = r_1$ and $\widetilde{q} = \widetilde{r}_1$.

Finally, we show that Theorem 5.1.2 (i), when applied to the inhomogeneous Klein–Gordon equation

$$-u''(t) + \Delta u(t) - u = F(t), \qquad u(0) = u''(0) = 0, \qquad t \geq 0, \tag{5.48}$$

slightly improves the range of inhomogeneous Strichartz estimates given by Nakamura and Ozawa [53, Proposition 2.1]. In a manner analogous to the wave equation, one can show that the weak solution u of (5.48) is given by

$$u(t) = \frac{1}{2i} \int_0^t \omega^{-1} \big(U(t)U(s)^* - U(-t)U(-s)^* \big) F(s) \, \mathrm{d}s,$$

where $\omega = (1 - \Delta)^{1/2}$ and the evolution group $\{U(t) : t \in \mathbb{R}\}$ is given by $U(t) = e^{it\omega}$. The resulting Strichartz inequalities are naturally expressed in Besov space norms, rather than homogeneous Besov space norms. We use the fact that the operator ω^{μ} is an isomorphism from $B^{\rho}_{r,2}$ to $B^{\rho-\mu}_{r,2}$ whenever $\mu \in \mathbb{R}$.

Corollary 5.8.5 (Nakamura–Ozawa). *Suppose that n is a positive integer, $0 \leq \eta \leq 1$ and the real numbers λ and σ satisfy*

$$2\lambda = n + 1 + \eta, \qquad n - 1 - \eta \leq 2\sigma \leq n - 1 + \eta, \qquad \sigma > 0.$$

Suppose also that the exponent pairs (q, r_1) and $(\widetilde{q}, \widetilde{r}_1)$ satisfy the acceptability condition

$$\begin{aligned} &1 \leq q < \infty, \quad 2 \leq r_1 \leq \infty, \quad \frac{1}{q} < 2\sigma \left(\frac{1}{2} - \frac{1}{r_1} \right); \quad \text{or } (q, r_1) = (\infty, 2); \\ &1 \leq \widetilde{q} < \infty, \quad 2 \leq \widetilde{r}_1 \leq \infty, \quad \frac{1}{\widetilde{q}} < 2\sigma \left(\frac{1}{2} - \frac{1}{\widetilde{r}_1} \right); \quad \text{or } (\widetilde{q}, \widetilde{r}_1) = (\infty, 2); \end{aligned}$$

the scaling condition

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = \sigma \left(1 - \frac{1}{r_1} - \frac{1}{\widetilde{r}_1} \right),$$

and the conditions

$$\begin{gathered} \frac{1}{q} + \frac{1}{\widetilde{q}} < 1, \\ \frac{\sigma - 1}{r_1} \leq \frac{\sigma}{\widetilde{r}_1}, \qquad \frac{\sigma - 1}{\widetilde{r}_1} \leq \frac{\sigma}{r_1}. \end{gathered} \tag{5.49}$$

When $\sigma = 1$ we also require that $r_1 < \infty$ and $\widetilde{r}_1 < \infty$. If $r \geq r_1$, $\widetilde{r} \geq \widetilde{r}_1$, $\rho \in \mathbb{R}$,

$$\rho + n\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{\lambda - n}{\sigma q} = 1 - \left(\widetilde{\rho} + n\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right) - \frac{\lambda - n}{\sigma \widetilde{q}}\right),$$

$F \in L^{\widetilde{q}'}(\mathbb{R}; B^{-\widetilde{\rho}}_{\widetilde{r}',2})$ and u is a weak solution of the inhomogeneous Klein–Gordon equation (5.48) then

$$\|u\|_{L^q(\mathbb{R};B^{\rho}_{r,2})} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};B^{-\widetilde{\rho}}_{\widetilde{r}',2})}. \tag{5.50}$$

Before proving the corollary, we note that our application of Theorem 5.1.2 improves [53, Proposition 2.1] by removing strict inequalities in (5.49).

Proof. As with the wave equation (see Section 4.8), we begin with a stationary phase estimate to derive the dispersive estimate

$$\|U(t)f\|_{B^{-\lambda/2}_{\infty,2}} \lesssim |t|^{-\sigma} \|f\|_{B^{\lambda/2}_{1,2}} \qquad \forall f \in B^{\lambda/2}_{1,2}$$

(see [53, pp. 261–262] for details). The corresponding energy estimate follows from the unitarity of $U(t)$ on $B^0_{2,2}$. Hence we may apply Theorem 5.1.2 (i) when $\mathcal{H} = \mathcal{B}_0 = B^0_{2,2}$, $\mathcal{B}_1 = B^{\lambda/2}_{1,2}$, $1/r' = (1-\theta)/2 - \theta/1$, $1/\widetilde{r}' = (1-\widetilde{\theta})/2 - \widetilde{\theta}/1$ and the pairs (q, θ) and $(\widetilde{q}, \widetilde{\theta})$ satisfy the hypothesis of Theorem 5.1.2 (i). This gives

$$\|(TT^*)_R F\|_{L^q(\mathbb{R};B^{-\lambda(1/2-1/r)}_{r,2})} \lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};B^{\lambda(1/2-1/\widetilde{r})}_{\widetilde{r}',2})}.$$

Suppose now that $\mu \in \mathbb{R}$ and

$$\rho = \mu + \lambda\left(\frac{1}{2} - \frac{1}{r}\right), \qquad \rho = 1 - \mu - \lambda\left(\frac{1}{2} - \frac{1}{r}\right).$$

We thus obtain

$$\begin{aligned}
\|u\|_{L^q(\mathbb{R};B^{\rho}_{r,2})} &\approx \|\omega^{\mu} u\|_{L^q(\mathbb{R};B^{-\lambda(1/2-1/r)}_{r,2})} \\
&\lesssim \left\|\omega^{\mu-1}(TT^*)_R F\right\|_{L^q(\mathbb{R};B^{-\lambda(1/2-1/r)}_{r,2})} \\
&\lesssim \left\|\omega^{\mu-1} F\right\|_{L^{\widetilde{q}'}(\mathbb{R};B^{-\lambda(1/2-1/\widetilde{r})}_{\widetilde{r}',2})} \\
&\lesssim \|F\|_{L^{\widetilde{q}'}(\mathbb{R};B^{-\widetilde{\rho}}_{\widetilde{r}',2})}.
\end{aligned}$$

This proves the corollary in the case when $r_1 = r$ and $\widetilde{r}_1 = \widetilde{r}$. The case when $r_1 < r$ and $\widetilde{r}_1 < \widetilde{r}$ can be proved by Besov embedding. ∎

Remark 5.8.6. In all the Strichartz estimates given in this section, one may exchange the infinite time interval $\mathbb{R}$ appearing in the spacetime norms with a finite time interval I or J. This is done by redefining each $U(t)$ as $1_I(t)U(t)$, where 1_I is the characteristic function of I on $\mathbb{R}$, and by redefining F as $1_J F$.

5.9 Applications to the Schrödinger equation with potential

The goal of this section is to demonstrate that our generalisation of Foschi's work [24] allows one to obtain Strichartz estimates for Schrödinger equations involving certain potentials. The potentials we consider introduce the difficulty that the dispersive estimate (4.11) does not hold when $(\mathcal{B}_0, \mathcal{B}_1) = (L^2(X), L^1(X))$ (see Remark 5.9.3 for further details).

Suppose that $V : \mathbb{R}^3 \to \mathbb{R}$ is a real-valued potential on $\mathbb{R}^3$ with decay

$$|V(x)| \leq C \langle x \rangle^{-\beta} \qquad \forall x \in \mathbb{R}^3, \tag{5.51}$$

where $\beta > 5/2$ and $\langle x \rangle = (1 + |x|^2)^{1/2}$. Consider the Hamiltonian operator H, given by $H = -\Delta + V$, on the Hilbert space $L^2(\mathbb{R}^3)$ with domain $W^{2,2}(\mathbb{R}^3)$, where $W^{k,p}(X)$ denotes the Sobolev space of order k in $L^p(X)$. Our goal is to obtain spacetime estimates for the solution u of the inhomogeneous initial value problem

$$\begin{cases} \left(i\frac{\partial}{\partial t} + H\right) u(t) = F(t) & \forall t \in [0, \tau], \\ u(0) = f, \end{cases} \tag{5.52}$$

where $\tau > 0$ and, for each time t in $\mathbb{R}$, f and $F(t)$ are complex-valued functions on $\mathbb{R}^3$.

Hamiltonians that satisfy the above conditions are considered by K. Yajima in [79]. There it mentions that H is self-adjoint on $L^2(\mathbb{R}^3)$ with a spectrum consisting of a finite number of nonpositive eigenvalues, each of finite multiplicity, and the absolutely continuous part $[0, \infty)$. Denote by P_c the orthogonal projection from $L^2(\mathbb{R}^3)$ onto the continuous spectral subspace for H. Under the general assumption (5.51), it is known that P_c, when viewed as an operator on $L^p(\mathbb{R}^3)$, is bounded only when $2/3 < p < 3$.

We denote by $\mathcal{H}_\gamma$ the weighted Lebesgue space $L^2(\mathbb{R}^3, \langle x \rangle^{2\gamma}\, \mathrm{d}x)$. When $\gamma \in (1/2, \beta - 1/2)$, define the null space $\mathcal{N}$ by

$$\mathcal{N} = \left\{ \phi \in \mathcal{H}_{-\gamma} : \phi(x) + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{V(y)\phi(y)}{|x-y|}\, \mathrm{d}y = 0 \right\}.$$

As noted in [79], the space $\mathcal{N}$ is finite dimensional and is independent of the choice of γ in the interval $(1/2, \beta - 1/2)$. All ϕ belonging to $\mathcal{N}$ satisfy the stationary Schrödinger equation

$$-\Delta\phi(x) + V(x)\phi(x) = 0, \tag{5.53}$$

where (5.53) is to be interpreted in the distributional sense. Conversely, any function $\phi \in \mathcal{H}_{-3/2}$ which satisfies (5.53) belongs to $\mathcal{N}$. Hence, if 0 is an eigenvalue of H, and $\mathcal{E}$ denotes the associated eigenspace, then $\mathcal{E}$ is a subspace of $\mathcal{N}$.

Definition 5.9.1. We say that H or V is of *generic type* if $\mathcal{N} = \{0\}$ and is of *exceptional type* otherwise. The Hamiltonian H is of *exceptional type of the first kind* if $\mathcal{N} \neq \{0\}$ and 0 is not an eigenvalue of H. It is of *exceptional type of the second kind* if $\mathcal{E} = \mathcal{N} \neq \{0\}$. Finally, we say that H is of *exceptional type of the third kind* if $\{0\} \subset \mathcal{E} \subset \mathcal{N}$ with strict inclusions.

While most V are of generic type, examples that are of exceptional type are interesting from a physical point of view. In particular, if V is of exceptional of the third kind then any function ϕ in $\mathcal{N} \backslash \mathcal{E}$ is called a *resonance* of H.

We would like to apply Theorems 4.2.2 and 5.1.2 to the case where $U(t)$ is the operator e^{-itH}, defined by the functional calculus for self-adjoint operators. However, if g is an eigenfunction of H with corresponding eigenvalue λ, then

$$U(s)U(t)^* g = e^{i(s-t)H} g = e^{i(s-t)\lambda} g \tag{5.54}$$

and therefore $U(s)U(t)^*g$ is stationary. Consequently, none of the decay hypotheses (4.11) or (4.12) of Theorem 4.2.2 are satisfied. Fortunately, this is not the case if g lies in the continuous spectral subspace of H.

Theorem 5.9.2 (K. Yajima [79]). *There exists a positive constant C_p such that the dispersive estimate*

$$\left\|e^{itH}P_c g\right\|_{p'} \leq C_p|t|^{-3(1/p-1/2)}\left\|g\right\|_p \qquad \forall g \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \quad \forall \text{ real } t \neq 0 \tag{5.55}$$

is satisfied in the following two cases:

(i) if H is of generic type, $\beta > 5/2$ and $1 \leq p \leq 2$; and

(ii) if H is of exceptional type, $\beta > 11/2$ and $3/2 < p \leq 2$.

Remark 5.9.3. If H is of exceptional type then (5.55) *cannot* hold when $p = 1$, otherwise it would contradict the local decay estimate of Jensen–Kato [37] or Murata [52]. Hence one cannot apply the results of Foschi [24] to this situation.

If u is a solution to (5.52), define u_c by

$$u_c(t) = P_c u(t) \qquad \forall t \in [0, \tau].$$

Similarly, let P_{pp} denote the orthogonal projection onto the pure-point spectral subspace of H and define u_{pp} by

$$u_{pp}(t) = P_{pp}u(t) \qquad \forall t \in [0, \tau].$$

It is clear that $u = u_{pp} + u_c$.

The dispersive estimate (5.55) gives rise to the admissibility conditions

$$\frac{1}{q} + \frac{3}{2r} = \frac{3}{4}, \quad 4 < q \leq \infty; \qquad \frac{1}{\widetilde{q}} + \frac{3}{2\widetilde{r}} = \frac{3}{4}, \quad 4 < \widetilde{q} \leq \infty \tag{5.56}$$

sketched in Figure 5.6. These correspond to the sharp σ-admissibility conditions in the case when $\sigma = 3(1/p - 1/2)$, $\mathcal{H} = \mathcal{B}_0 = L^2(\mathbb{R}^3)$, $\mathcal{B}_1 = L^p(\mathbb{R}^3)$ and $p \to 3/2$ from the right. Note that they also correspond to the Schrödinger 3-admissibility conditions (see Section 4.7) with restricted range.

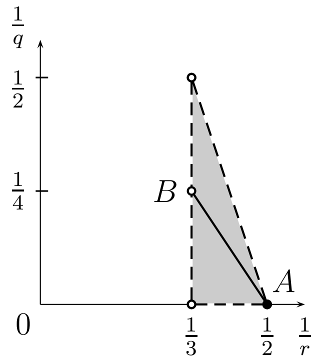


Figure 5.6: The line segment AB and the shaded region respectively give admissible and acceptable exponents for Strichartz estimates associated to the inhomogeneous initial value problem (5.64).

When considering the inhomogeneous problem with zero initial data, the exponent conditions of Theorem 5.1.2 reduce to the scaling condition

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{3}{2}\left(1 - \frac{1}{r} - \frac{1}{\widetilde{r}}\right) \tag{5.57}$$

and the acceptability conditions

$$1 \le q < \infty, \quad 2 \le r < 3, \quad \frac{1}{q} < 3\left(\frac{1}{2} - \frac{1}{r}\right), \quad \text{or } (q,r) = (\infty, 2); \tag{5.58}$$

$$1 \le \widetilde{q} < \infty, \quad 2 \le \widetilde{r} < 3, \quad \frac{1}{\widetilde{q}} < 3\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right), \quad \text{or } (\widetilde{q},\widetilde{r}) = (\infty, 2). \tag{5.59}$$

This is because $\sigma = 3(1/p - 1/2) < 1$.

Corollary 5.9.4. *Suppose that u is a (weak) solution to problem (5.52) for some data f in $L^2(\mathbb{R}^3)$, some source F and for some time τ in $(0,\infty)$.*

(i) If (q,r) and $(\widetilde{q},\widetilde{r})$ satisfy the admissibility condition (5.56) and F belongs to $L^{\widetilde{q}'}([0,\tau]; L^{\widetilde{r}'}(\mathbb{R}^3))$, then

$$\|u_c\|_{L^q([0,\tau],L^r(\mathbb{R}^3))} \lesssim \|f\|_{L^2(\mathbb{R}^3)} + \|F\|_{L^{\widetilde{q}'}([0,\tau],L^{\widetilde{r}'}(\mathbb{R}^3))}\,. \tag{5.60}$$

(ii) If the exponent pairs (q,r) and $(\widetilde{q},\widetilde{r})$ satisfy conditions (5.57), (5.58) and (5.59), $f = 0$ and $F \in L^{\widetilde{q}'}([0,\tau]; L^{\widetilde{r}'}(\mathbb{R}^3))$, then

$$\|u_c\|_{L^q([0,\tau],L^r(\mathbb{R}^3))} \lesssim \|F\|_{L^{\widetilde{q}'}([0,\tau],L^{\widetilde{r}'}(\mathbb{R}^3))}\,.$$

Proof. Fix p such that

$$3/2 < p < \min\{r', \widetilde{r}'\}.$$

For t in $\mathbb{R}$ define $U(t)$ on $L^2(\mathbb{R}^3)$ by $U(t) = 1_{[0,\tau]}(t)e^{itH}P_c$. If g belongs to $L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, then

$$\begin{aligned}
\|U(s)U(t)^* g\|_{p'} &= \left\|1_{[0,\tau]}(s)1_{[0,\tau]}(t)e^{i(s-t)H}P_c g\right\|_{p'} \\
&\leq \left\|e^{i(s-t)H}P_c g\right\|_{p'} \\
&\leq |s-t|^{-3(1/p-1/2)} \|g\|_p \,.
\end{aligned}$$

by Theorem 5.9.2. Therefore $\{U(t) : t \in \mathbb{R}\}$ satisfies the untruncated decay estimate (4.11) when

$$\sigma = 3(1/p - 1/2),$$

$\mathcal{B}_0 = \mathcal{H} = L^2(\mathbb{R}^3)$ and $\mathcal{B}_1 = L^p(\mathbb{R}^3)$. Moreover, since each operator e^{-itH} on $L^2(\mathbb{R}^3)$ is unitary and P_c is an orthogonal projection, $\{U(t) : t \in \mathbb{R}\}$ also satisfies the energy estimate (4.10). Now if u is a weak solution to (5.52) then

$$u(t) = e^{-itH}f - i\int_0^t e^{-i(t-s)H}F(s)\,\mathrm{d}s. \tag{5.61}$$

by Duhamel's principle and the functional calculus for self-adjoint operators. Hence

$$\begin{aligned}
u_c(t) &= e^{itH}P_c f - i\int_0^t e^{i(t-s)H}P_c F(s)\,\mathrm{d}s \\
&= Tf(t) - i(TT^*)_R F(t).
\end{aligned}$$

An application of Theorem 4.2.2 and Theorem 5.1.2 gives the required spacetime estimates for u_c once we observe that

$$L^{r'}(\mathbb{R}^3) \subset L^{r',2}(\mathbb{R}^3) = \mathcal{B}_\theta,$$

where $1/r' = (1-\theta)/2 + \theta/p$ and the inclusion is continuous. $\square$

To find a spacetime estimate for the solution u of (5.52), we now need only analyse the projection of each $u(t)$ onto the pure point spectral subspace of H.

It is known (see [79, p. 477]) that eigenfunctions of H with negative eigenvalues decay at least exponentially. Since such eigenfunctions belong to the domain of Δ, they are necessarily continuous and consequently also belong to $L^r(\mathbb{R}^3)$ whenever $1 \leq r \leq \infty$ by Sobolev embedding. However, if 0 is an eigenvalue then a corresponding eigenfunction ϕ may decay as slowly as $C\langle x\rangle^{-2}$ when $|x| \to \infty$. Hence, in general, ϕ is a member of $L^p(\mathbb{R}^3)$ only when $p > 3/2$.

Except in the case when the time exponent is ∞, one cannot hope for a spacetime estimate for u_{pp} which is global in time due to (5.54). However, one can still obtain spacetime estimates on finite time intervals. For illustrative purposes, the next lemma gives a crude spacetime estimate for u_{pp} when H is of exceptional type. No further analysis on H is needed.

Lemma 5.9.5. *Suppose that $\tau > 0$, that $q, \widetilde{q} \in [1, \infty]$ and that $r, \widetilde{r} \in (3/2, 3)$. Suppose also that $f \in L^2(\mathbb{R}^3)$, $F \in L^{\widetilde{q}'}([0,\tau], L^{\widetilde{r}'}(\mathbb{R}^3))$ and H is of exceptional type. If u is a (weak) solution to problem (5.52) then*

$$\|u_{pp}\|_{L^q([0,\tau],L^r(\mathbb{R}^3))} \leq C_{r,H} \left(\|P_{pp}f\|_2 + \tau^{1/q+1/\widetilde{q}} \|P_{pp}F\|_{L^{\widetilde{q}'}([0,\tau],L^{\widetilde{r}'}(\mathbb{R}^3))} \right)$$

where the positive constant $C_{r,H}$ depends on r and H only. If $q = \widetilde{q} = \infty$ then $\tau^{1/q+1/\widetilde{q}}$ is interpreted as 1.

Proof. Suppose that $\{\phi_j : j = 1, \ldots, n\}$ is a complete orthonormal set of eigenfunctions for H on $L^2(\mathbb{R}^3)$ corresponding to the set $\{\lambda_j : j = 1, \ldots, n\}$ of eigenvalues (counting multiplicities). Write

$$P_{pp}f = \sum_{i=1}^{n} \alpha_j \phi_j$$

and

$$P_{pp}F(s) = \sum_{j=1}^{n} \beta_j(s)\phi_j,$$

where each α_j and $\beta_j(s)$ is a complex scalar. By orthogonality and the equivalence of norms in finite dimensional normed spaces (see [14, p. 69]), there are positive constants C and C' (both independent of f, $F(s)$, $\{\alpha_j\}$ and $\{\beta_j(s)\}$)

such that

$$\sum_{j=1}^{n}|\alpha_j| \leq C\Big(\sum_{j=1}^{n}|\alpha_j|^2\Big)^{1/2} = C\left\|P_{pp}f\right\|_2$$

and

$$\sum_{j=1}^{n}|\beta_j(s)| \leq C\Big(\sum_{j=1}^{n}|\beta_j(s)|^2\Big)^{1/2} = C\left\|P_{pp}F(s)\right\|_2 \leq C'\left\|P_{pp}F(s)\right\|_{\widetilde{r}'}.$$

Following from (5.61),

$$\begin{aligned}
u_{pp}(t) &= e^{itH}P_{pp}f - i\int_0^t e^{i(t-s)H}P_{pp}F(s)\,\mathrm{d}s \\
&= \sum_{j=1}^{n}\alpha_j e^{it\lambda_j}\phi_j - i\int_0^t\sum_{j=1}^{n}\beta_j(s)e^{i(t-s)\lambda_j}\phi_j\,\mathrm{d}s.
\end{aligned}$$

By taking the $L^q([0,\tau],L^r(\mathbb{R}^3))$ norm and applying Hölder's inequality,

$$\begin{aligned}
\left\|u_{pp}\right\|_{L^q([0,\tau],L^r(\mathbb{R}^3))} &\leq \sum_{i=j}^{n}|\alpha_j|\left\|\phi_j\right\|_r + \left\|\int_0^t\sum_{j=1}^{n}|\beta_j(s)|\left\|\phi_j\right\|_r\,\mathrm{d}s\right\|_{L^q([0,\tau])} \\
&\leq C''\max_{1\leq j\leq n}\left\|\phi_j\right\|_r\left(\left\|P_{pp}f\right\|_2 + \left\|\int_0^t\left\|P_{pp}F(s)\right\|_{\widetilde{r}'}\,\mathrm{d}s\right\|_{L^q([0,\tau])}\right) \\
&\leq C_{r,H}\left(\left\|P_{pp}f\right\|_2 + \tau^{1/q}\left\|P_{pp}F\right\|_{L^1([0,\tau];L^{\widetilde{r}'}(\mathbb{R}^3))}\right) \\
&\leq C_{r,H}\left(\left\|P_{pp}f\right\|_2 + \tau^{1/q+1/\widetilde{q}}\left\|P_{pp}F\right\|_{L^{\widetilde{q}'}([0,\tau];L^{\widetilde{r}'}(\mathbb{R}^3))}\right)
\end{aligned}$$

where

$$C_{r,H} = C''\max_{1\leq j\leq n}\left\|\phi_j\right\|_r.$$

This completes the proof. □

Combining the lemma with Corollary 5.9.4 and the fact that $u = u_c + u_{pp}$ gives the following result.

Corollary 5.9.6. *Suppose that H is of exceptional type, that $\tau > 0$ and that (q,r) and $(\widetilde{q},\widetilde{r})$ satisfy the admissibility conditions (5.56). If $f \in L^2(\mathbb{R}^3)$ and $F \in L^{\widetilde{q}'}([0,\tau],L^{\widetilde{r}'}(\mathbb{R}^3))$ and u is a (weak) solution to problem (5.52) then*

$$\left\|u\right\|_{L_t^q([0,\tau],L^r(\mathbb{R}^3))} \lesssim \left\|f\right\|_{L^2(\mathbb{R}^3)} + \left(1+\tau^{1/q+1/\widetilde{q}}\right)\left\|F\right\|_{L_t^{\widetilde{q}'}([0,\tau],L^{\widetilde{r}'}(\mathbb{R}^3))}. \tag{5.62}$$

If $q = \widetilde{q} = \infty$ then $\tau^{1/q+1/\widetilde{q}}$ is interpreted as 1. If $f = 0$ then the conditions on (q,r) and $(\widetilde{q},\widetilde{r})$ may be relaxed to satisfying (5.57), (5.58) and (5.59).

While the Strichartz estimate (5.62) is not of the usual form, the next proposition shows that it can still answer questions about well-possedness.

Proposition 5.9.7. *Suppose that for each real number k in the interval $[1,2)$, there is a transformation F_k of functions satisfying*

$$\begin{cases} |F_k(u)| \lesssim |u|^k \\ |u||F_k'(u)| \approx |F_k(u)|. \end{cases} \tag{5.63}$$

Suppose also that $f \in L^2(\mathbb{R}^3)$, $3k/2 < r \leq 2k$ and (q,r) satisfies the admissibility conditions (5.56). Then there is a positive τ, depending only on f, and a unique weak solution u to the initial value problem

$$\begin{cases} \left(i\frac{\partial}{\partial t} + H\right) u(t) = F_k(u)(t), \qquad 0 < t < \tau, \\ u(0) = f \end{cases} \tag{5.64}$$

such that $u \in L^q([0,\tau]; L^r(\mathbb{R}^3))$ and u depends continuously on the initial data.

The proof of the proposition follows standard arguments which appeal to the contraction mapping theorem (see for example Section 4.1, [42, Section 9] and particularly [9], where (5.64) was studied in the nonpotential case when $H = \Delta$).

Proof. Suppose that $1 \leq k < 2$ and that (q,r) satisfies the hypotheses of the proposition. Then there exists an exponent $\widetilde{r}$ in $[2,3)$ such that

$$r = \widetilde{r}'k. \tag{5.65}$$

Now choose $\widetilde{q}$ such that $(\widetilde{q},\widetilde{r})$ satisfies the admissibility condition (5.56). One can easily verify that

$$q > \widetilde{q}'k. \tag{5.66}$$

In a manner similar to the proof of Theorem 4.1.1, we will show that the map $u \mapsto N_f(u)$, given by

$$N_f(u)(t) = e^{itH}f - i\int_0^t e^{i(t-s)H}F_k(u)(s)\,\mathrm{d}s,$$

is a contraction map on metric space X given by

$$X = \{u \in L^q([0,\tau]; L^r(\mathbb{R}^3)) : \|u\|_{L^q([0,\tau];L^r(\mathbb{R}^3))} \leq M\},$$

where M is chosen such that

$$\|N_f(0)\|_{L^q([0,\tau];L^r(\mathbb{R}^3))} \leq \frac{M}{2} \tag{5.67}$$

and τ is a positive constant to be determined. (Note that M is well-defined by the homogeneous part of the Strichartz estimate (5.62) and that M is independent of τ. Also, the metric on X is that induced by the $L^q([0,\tau]; L^r(\mathbb{R}^3))$ norm.)

First we will show that

$$\|N_f(u) - N_f(v)\|_X \leq \frac{1}{2}\|u - v\|_X \qquad \forall u, v \in X. \tag{5.68}$$

The assumptions (5.63) on F_k show that

$$\begin{aligned}
|F_k(u) - F_k(v)| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda} F_k\big(\lambda u + (1-\lambda)v\big)\, \mathrm{d}\lambda \right| \\
&= \left| \int_0^1 (u - v) F_k'\big(\lambda u + (1-\lambda)v\big)\, \mathrm{d}\lambda \right| \\
&\lesssim |u - v| \int_0^1 \big|\lambda u + (1-\lambda)v\big|^{k-1}\, \mathrm{d}\lambda \\
&\leq |u - v| \big(|u| + |v|\big)^{k-1}.
\end{aligned}$$

By the Strichartz estimate (5.62), we obtain

$$\begin{aligned}
\|N_f(u) - N_f(v)\|_X &\lesssim \big(1 + \tau^{1/q + 1/\widetilde{q}}\big) \|F_k(u) - F_k(v)\|_{L^{\widetilde{q}'}([0,\tau];L^{\widetilde{r}'}(\mathbb{R}^3))} \\
&\lesssim \big(1 + \tau^{1/q + 1/\widetilde{q}}\big) \left\| |u - v| \big(|u| + |v|\big)^{k-1} \right\|_{L^{\widetilde{q}'}([0,\tau];L^{\widetilde{r}'}(\mathbb{R}^3))}.
\end{aligned}$$

Two applications of Hölder's inequality give

$$\begin{aligned}
&\left\| |u - v| \big(|u| + |v|\big)^{k-1} \right\|_{L^{\widetilde{q}'}([0,\tau];L^{\widetilde{r}'}(\mathbb{R}^3))} \\
&\qquad \leq \|u - v\|_{L^q([0,\tau];L^r(\mathbb{R}^3))} \left\| \big(|u| + |v|\big)^{k-1} \right\|_{L^{q/(k-1)}([0,\tau];L^{r/(k-1)}(\mathbb{R}^3))} \\
&\qquad\qquad \times \left\| 1_{[0,\tau]} \right\|_{L^p([0,\tau];L^\infty(\mathbb{R}^3))},
\end{aligned}$$

where p is chosen such that

$$\frac{1}{\widetilde{q}'} = \frac{1}{q} + \frac{1}{q/(k-1)} + \frac{1}{p}.$$

Note that the above applications of Hölder's inequality are valid by (5.65) and (5.66) and that $1 < p < \infty$. Hence

$$\|N_f(u) - N_f(v)\|_X \leq C\tau^{1/p} M^{k-1}(1 + \tau^{1/q+1/\widetilde{q}}) \|u - v\|_X .$$

Now choose τ such that

$$C\tau^{1/p} M^{k-1}(1 + \tau^{1/q+1/\widetilde{q}}) \leq \frac{1}{2}$$

to obtain (5.68).

Note that since

$$\begin{aligned} \|N_f(u)\|_X &\leq \|N_f(0)\|_X + \|N_f(u) - N_f(0)\|_X \\ &\leq \frac{M}{2} + \frac{1}{2} \|u - 0\|_X \\ &\leq M \end{aligned} \tag{5.69}$$

whenever $u \in X$, we have $N_f : X \to X$ and hence N_f is a contraction on X. Hence the contraction mapping theorem implies that there is a unique solution u to the initial value problem (5.64). Following arguments similar to that contained in the proof of Theorem 4.1.1, one can show that the solution u in $L^q([0,\tau]; L^r(\mathbb{R}^3))$ depends continuously on the initial data f in $L^2(\mathbb{R}^3)$. $\square$

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