Dennis The – Research Statement

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My research is in differential geometry, and symmetry is a prevalent theme in my work. My current research focuses on studying geometric structures that fall under the broad class of *parabolic geometries* [13]. This area combines differential geometry, analysis, and algebraic and geometric tools from Lie theory — see §1 for a brief overview.

The study of such structures is often motivated by mathematical physics, e.g. conformal geometry. Distributions are of relevance to control theory, e.g. $(2, 3, 5)$-geometry. See also §3.3–3.4 for instances where these are interrelated.

Representation theory and exterior differential systems are recurring tools in my work. My recent results include: (i) a uniform solution to the symmetry gap problem for (complex) parabolic geometries — accepted in *J. reine angew. Math. (Crelle's Journal)*; and (ii) a uniform study of homological rigidity for Schubert varieties in compact Hermitian symmetric spaces — published in *Selecta Mathematica.*

1 Brief overview of parabolic geometry

A key advantage of the perspective of parabolic geometries is the ability to study a zoo of geometries in a *uniform* manner. Conformal geometry and projective geometry[1] are prototypical examples of parabolic geometries, which also include CR structures, systems of 2nd order ODE, and various bracket-generating distributions. Each type of geometry is to be thought of as a *curved version* of a *flat model.* Here, "curved version" refers to a *Cartan geometry* $(\mathcal{G} \to M, \omega)$ modelled on $(G \to G/P, \omega_G)$, where ω_G is the Maurer–Cartan form on G. For parabolic geometries, G is a semisimple Lie group and P is a parabolic subgroup.[2] (In the case of conformal geometry, $G = \mathrm{SO}(1, n+1)$ and G/P is the conformal sphere S^n embedded as the null cone in $\mathbb{R}^{1,n+1}$.) The Cartan picture gives an "upstairs" perspective on the original geometric structure classically defined "downstairs" on M. With suitable normalization conditions on the Cartan connection ω, this is indeed an equivalent viewpoint [32, 26, 10].

The flat model strongly influences the study of general curved geometries. In particular, common structural features inherent in the groups (G, P) (well-studied in representation theory) can be exploited. For instance, all parabolic geometries admit a fundamental curvature quantity called *harmonic curvature* κ_H, which is a complete obstruction to flatness. The Weyl tensor is the specific instance of κ_H in conformal geometry. More generally, κ_H is valued in a certain Lie algebra cohomology group $H^2_+(\mathfrak{g}_-, \mathfrak{g})$, whose structure can be immediately deduced from Kostant's version of the Bott–Borel–Weil theorem [4, 23]. A second instance is the uniform construction of a large class of invariant linear differential operators called Bernstein–Gelfand–Gelfand (BGG) operators [9, 14], for which the operators characterizing almost Einstein scales and conformal Killing tensors are specific examples.

2 Completed projects

My earlier work (≤ 2011; see my publication list for references) focused on:

- dynamical systems in general relativity (late-time asymptotics of Bianchi VIII cosmologies);
- integrable systems (invariants of Killing tensors);
- Lie symmetry analysis (nonlocal symmetries and conservations laws of Maxwell's equations);
- geometry of differential equations (generic hyperbolic PDE; surfaces in the Lagrangian–Grassmannian).

Here I will describe my more recent projects in parabolic geometry.

[1] Riemannian geometry is a Cartan geometry, but not a parabolic geometry.

[2] Parabolic subgroups P are the distinguished stabilizer subgroups in G that arise when letting G act on (generalized) flag varieties, e.g. projective space or Grassmannians in the case of $G = \mathrm{SL}(n, \mathbb{R})$.

2.1 Symmetry gaps for geometric structures

For structures admitting at most a finite-dimensional symmetry algebra, maximally symmetric models are often straightforward to construct and understand. The *symmetry gap problem* refers to the determination of the *submaximal* symmetry[3] dimension $\mathfrak{S}$. Gaps were first observed in the late 19th century:

- Surfaces with a Riemannian metric have at most a 3-dimensional isometry algebra, realized by the Euclidean plane 2-sphere, or hyperbolic plane, while $\mathfrak{S} = 1$ here, realized by a generic surface of revolution. (An infinite cylinder has 2 global symmetries, but is locally flat, so has 3 infinitesimal symmetries.)
- Scalar second order ODE $y'' = f(x, y, y')$ admit at most an 8-dimensional point symmetry algebra, realized by the flat model $y'' = 0$, while $\mathfrak{S} = 3$ here.

Classical studies by Tresse, Fubini, Cartan, Wang, Egorov, Kobayashi and others considered the gap problem for geometries on a case-by-case basis using a variety of different techniques. In joint work with B. Kruglikov [24], we presented a *uniform* algebraic approach to the symmetry gap problem for parabolic geometries. This uniformity manifests itself "upstairs" in terms of the Cartan geometry picture. (See §1.) Our approach was based on Tanaka theory [31, 32] and work of Čap and Neusser [12] that used Kostant's theorem [4]. We significantly generalized known results — an outline is given below. (See §3 for further extensions and applications.)

Geometry	Range	Max	$\mathfrak{S}$
Sig. (p, q) conformal geometry in dim. $n = p + q$	$p, q \geq 2$	$\binom{n+2}{2}$	$\binom{n-1}{2} + 6$
Systems of 2nd order ODE in m dependent variables	$m \geq 2$	$(m + 2)^2 - 1$	$m^2 + 5$
Exotic parabolic contact structure of type E_8	-	248	147

Table 1: Sample new symmetry gap results for parabolic geometries

Given (G, P), we have a Lie algebra grading $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$. The fundamental (harmonic) curvature κ_H for (regular, normal) parabolic geometries $(\mathcal{G} \to M, \omega)$ naturally takes values in a $\mathfrak{g}_0$-representation $H^2_+(\mathfrak{g}_-, \mathfrak{g})$. If the geometry is not locally flat, then $\kappa_H \not\equiv 0$, and we were led to study the algebraic quantity

$$\mathfrak{U} := \max\{\dim(\mathfrak{a}^\phi) : 0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})\},$$

where the *Tanaka prolongation* $\mathfrak{a}^\phi \subset \mathfrak{g}$ is the graded subalgebra with:

$$\text{(i) } \mathfrak{a}^\phi_- = \mathfrak{g}_-, \qquad \text{(ii) } \mathfrak{a}^\phi_0 = \mathfrak{ann}(\phi), \qquad \text{(iii) } \mathfrak{a}^\phi_i = \{X \in \mathfrak{g}_i : [X, \mathfrak{g}_{-1}] \subseteq \mathfrak{a}^\phi_{i-1}\}, \quad i > 0.$$

We established the following main results in [24]:

1. (*Universal upper bound*) $\mathfrak{S} \leq \mathfrak{U}$ always. An important technical advance was that our proof did not depend on any homogeneity or any curvature regularity assumptions, which are often used in the classical literature.[4]
2. (*Sharpness*) In the complex or split-real settings, we showed that $\mathfrak{S} = \mathfrak{U}$ almost always.[5] In these cases, we uniformly constructed a model realizing the upper bound via a (filtered) deformation of a (graded) Tanaka prolongation algebra. We gave a simple combinatorial (Dynkin diagram) recipe for explicitly computing $\mathfrak{U}$.
3. Our techniques extend beyond the submaximal context: e.g. for conformal Lorentzian 4-manifolds, simple computations led to the maximal symmetry dimension for each Petrov type.[6]

In a follow-up joint work with B. Doubrov [18], I settled the gap problem for conformal Riemannian and Lorentzian geometries. (All other signatures were settled in [24].) The representation theory is much more subtle for these (non-split) real forms, and Lie subalgebra classification techniques due to Dynkin played a key role.

[3] Henceforth, "symmetry" refers to *infinitesimal* symmetry.
[4] Homogeneity assumptions are unwarranted: outside the parabolic context, surfaces with Riemannian metric have $\mathfrak{S} = 1$.
[5] The few exceptions generally satisfy $\mathfrak{S} = \mathfrak{U} - 1$.
[6] We gave the first known Petrov type II metric with (the maximal) 4 symmetries. (The corresponding models in [30] are incorrect.)

2.2 Homogeneous integrable Legendrian contact structures in dimension five

A longstanding problem in the CR community is the classification of all real homogeneous (Levi-nondegenerate) hypersurfaces in $\mathbb{C}^3$ having at least 6 symmetries. The corresponding study in $\mathbb{C}^2$ was carried out in 1932 by Cartan [16] as part of his study of bounded homogeneous domains. In $\mathbb{C}^3$, the maximum is 15, and there is a drop to 8 in the Levi-indefinite case and 7 in the Levi-definite case. These models are known [25], but the 6-dimensional case remains incomplete. Recent attempts were analytic in flavour, focusing on Chern–Moser normal form techniques.

In joint work [5] with B. Doubrov and A. Medvedev, we adopt a Cartan-style (algebraic) approach by considering the corresponding classification problem for ILC structures, which can be regarded as "complexified" CR structures. (Real forms of the algebraic models appearing in our classification will yield the desired CR classification.)

An ILC structure (in dimension 5) is a contact 5-manifold $(M, \mathcal{C})$ with contact distribution $\mathcal{C}$ split into a pair of rank 2 integrable Legendrian distributions E and V. Locally, there exist coordinates (x, y, u, p, q) such that

$$\mathcal{C} = \mathfrak{ann}(du - pdx - qdy), \quad V = \text{span}\{\partial_p, \partial_q\}, \quad E = \text{span}\{\partial_x + p\partial_u + F\partial_p + G\partial_q,\ \partial_y + q\partial_u + G\partial_p + H\partial_q\}.$$

Thus, an equivalent local description of an ILC structure is as a *compatible*[7] overdetermined planar PDE system

$$u_{xx} = F(x, y, u, u_x, u_y), \quad u_{xy} = G(x, y, u, u_x, u_y), \quad u_{yy} = H(x, y, u, u_x, u_y)$$

considered up to *point transformations*, i.e. prolongations of diffeomorphisms in the (x, y, u) variables. In this way, ILC structures are a natural generalization of the classical geometry of scalar 2nd order ODE.

An ILC structure can be viewed as a parabolic geometry. Its harmonic curvature κ_H is a binary quartic polynomial, so there is a Petrov classification analogous to that for the Weyl tensor in 4-dimensional Lorentzian (conformal) geometry. My work on symmetry gaps (see §2.1) yields immediate bounds for each Petrov type. We have carried out a *systematic* classification of homogeneous models (having at least 6 symmetries) using Cartan's reduction algorithm (used first in [15]). Despite the conceptual simplicity of this method, its implementation by-hand is very tedious. Symbolic computation tools from the `DifferentialGeometry` and `Cartan` packages in Maple have greatly facilitated the analysis. Our ILC classification is now complete and we are currently classifying real forms.

2.3 Rigidity of Schubert varieties in compact Hermitian symmetric spaces

Let G be a complex simple Lie group and $P \subset G$ a parabolic subgroup such that $X = G/P$ (in its minimal homogeneous embedding) is an irreducible compact Hermitian symmetric space (CHSS), e.g. Grassmannians, quadrics, and the Cayley plane. The integral homology of X is generated by Schubert varieties $\xi_w \subset X$. (Here, the parameter w ranges over a distinguished subset of the Weyl group called the Hasse diagram.) A natural rigidity question arises: *If a complex subvariety $Y \subset X$ has homology $[Y]$ an integer multiple of $[\xi_w]$, must Y be a G-translate $g \cdot \xi_w$?* If so, ξ_w is said to be *Schur rigid*; if not, it is *Schur flexible*, i.e. there are imposters. Understanding Schur rigidity yields insight on the following longstanding question: *Can a* **singular** *Schubert variety admit a* **smooth** *complex representative in its homology class?* If ξ_w is singular and Schur rigid, such a phenomenon is impossible.

In [28], C. Robles and I established first-order obstructions $\mathsf{H}(w)$ to the flexibility of a given Schubert variety ξ_w, and classified all ξ_w satisfying $\mathsf{H}(w) = 0$. (In particular, these are Schur rigid.) Robles [29] recently showed that all ξ_w with $\mathsf{H}(w) \neq 0$ are indeed flexible. Thus, our list is the complete list of ξ_w that are Schur rigid.

Our work extended earlier work of Walters [34], Bryant [7], and Hong [22, 21]. Walters and Bryant independently gave a differential-geometric reformulation to this algebro-topological problem of Schur rigidity. Namely, the condition $[Y] \in \mathbb{Z}[\xi_w]$ was reformulated as Y being an integral variety of a differential system $\mathcal{R}_w \subset \text{Gr}(|w|, TX)$, i.e. at any smooth point $y \in Y$, the tangent space T_yY is constrained to lie in a certain family of $|w|$-planes $(\mathcal{R}_w)|_y$ of T_yX. Via this approach, Hong made a detailed study of ξ_w in Grassmannians, obtaining similar first-order obstructions. In our work, we analyzed $\mathcal{R}_w$ in all CHSS in a uniform way via moving frame techniques and showed how the aforementioned first-order obstructions $\mathsf{H}(w)$ arise (and expressed them in terms of certain Lie algebra cohomology groups). If ξ_w is smooth, the relevant cohomology groups can be easily computed via Kostant's theorem. However, most ξ_w are singular, and one of our technical contributions was to adapt several key ingredients from Kostant's

[7] Compatibility is equivalent to the subbundle E being integrable.

setting to our setting. In particular, we defined a natural algebraic Laplacian $\Box$ (whose kernel corresponds to cohomology), and carried out a similar spectral analysis of $\Box$. This led to a characterization of the vanishing of $\mathsf{H}(w)$ in terms of representation-theoretic / combinatorial data associated with $(w, \mathfrak{g}, \mathfrak{p})$, which we were able to resolve.

3 Current research

3.1 Symmetry gaps for almost c-projective structures

Given a manifold M with a torsion-free affine connection ∇, an unparameterized geodesic γ is a curve in M such that $\nabla_{\dot\gamma}\dot\gamma = \lambda\dot\gamma$ for some function λ. A projective structure is an equivalence class $[\nabla]$ of such connections all sharing the same unparametrized geodesics. An *almost c-projective structure* $(M, J, [\nabla])$ is an almost complex manifold (M, J) with an equivalence class of *minimal complex connections*, i.e. $\nabla J = 0$ and the torsion of ∇ is a multiple of the Nijenhuis tensor of J. (Thus, ∇ is not torsion-free when J is not integrable.) Here, $\nabla \sim \nabla'$ if they share the same J-planar curves γ, i.e. curves that satisfy $\nabla_{\dot\gamma}\dot\gamma \in \mathrm{span}\{\dot\gamma, J(\dot\gamma)\}$. Such structures have attracted increasing interest [17, 2] and it was only recently discovered that they are in fact parabolic geometries.

In joint work with Boris Kruglikov and Vladimir Matveev, we are investigating the symmetry gap problem for almost c-projective structures. While the modelling groups in the parabolic geometry description are $(\mathrm{SL}(n+1, \mathbb{C}), P)$, where P is the subgroup stabilizing a complex line in $\mathbb{C}^{n+1}$, the subtlety here is that these are to be considered as *real* Lie groups, so some care is needed in applying the results of §2.1. Nevertheless, we have classified gaps and identified models, but we are also considering the gap problem restricted to the class of *metrisable* c-projective structures, i.e. where $[\nabla]$ contains the Levi-Civita connection of some Kähler metric.

3.2 Jet-determination of symmetries of parabolic geometries

A classical problem, having origin in the works of É. Cartan, H. Cartan, Tanaka, Chern and Moser (see [33] for references), is to determine which *jet* of a symmetry is sufficient for its unique determination. A vector field $\mathbf{X}$ on M is k-jet determined at $p \in M$ if $j^k_p(\mathbf{X}) \neq 0$, i.e. in any local coordinate system centred at p, the k-th order Taylor polynomials of the coefficients of $\mathbf{X}$ in the coordinate basis do not all vanish at p. (Note k may not be minimal.) For example, the flat conformal structure $[g]$, with $g = dx^2 + dy^2 + dz^2$, is 2-jet determined: translation symmetries $\partial_x, \partial_y, \partial_z$ are 0-jet determined, the dilation symmetry $x\partial_x + y\partial_y + z\partial_z$ and rotational symmetries such as $x\partial_y - y\partial_x$ are 1-jet determined, while the inversion symmetries such as $xz\partial_x + yz\partial_y - \frac{1}{2}(x^2 + y^2 - z^2)\partial_z$ are 2-jet determined. For (Levi-non-degenerate, hypersurface) CR structures, 2-jet determinacy of the symmetry algebra follows from the work of the above authors (also in the non-flat case). Boris Kruglikov and I are studying the analogous question for general parabolic geometries.

It is not too difficult to show that the symmetries of the underlying geometry structure of any locally *flat* parabolic geometry (e.g. G/P itself) are 2-jet (and not 1-jet) determined. Indeed, if the $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ is the induced grading, then $\dim(\mathfrak{g}_k)$ is precisely the dimension of the subspace of symmetries which are 2-jet, but not 1-jet determined. This suggests a strategy in the non-flat case: Given $u \in \mathcal{G}$ with $\kappa_H(u) \neq 0$, does the Tanaka prolongation algebra $\mathfrak{a}^{\kappa_H(u)}$ (see §2.1) have non-trivial intersection with $\mathfrak{g}_k$, i.e. does it reach the last level? In fact, when G is simple, it does not, based on classification results of [24]. Thus, we obtain 1-jet determinacy at a non-flat (regular) point.

The situation at a flat point, i.e. $u \in \mathcal{G}$ such that $\kappa_H(u) = 0$, is much more subtle and $\mathfrak{a}^{\kappa_H(u)} = \mathfrak{g}$ gives no information. A natural way forward is to investigate derivatives of harmonic curvature. Using the *fundamental derivative* operator D in parabolic geometry theory, if we suppose that $(D^j\kappa_H)(u) = 0$, for $j = 0, ..., k-1$ and $(D^k\kappa_H)(u) \neq 0$, in what manner is the symmetry algebra constrained?

A related subject of intense interest (see [11] and references therein) is the study of symmetries with higher order fixed points, i.e. *strongly essential symmetries*, and the (simultaneous) linearizability of symmetries that fix a given point. In projective geometry and in conformal geometry in Riemannian or Lorentzian signature, there is a well-known rigidity result for symmetries about a fixed point: either the vector field is linearizable, or the structure is flat. However, this dichotomy does not persist in all parabolic geometries: the classifications in my gap paper [24] have led to the first examples of non-flat parabolic geometries with non-linearizable isotropy near a non-flat point.

3.3 Conformal geometry in dimension four and a special geometry in five

A $(2,3,5)$-distribution is a rank 2 distribution $D \subset TM$ on a 5-manifold M, such that $[[D,D],D] = TM$. Such structures date back to Cartan's famous 1910 paper [15]. In modern terms, $(2,3,5)$-geometry is equivalent to a parabolic geometry of type (G_2, P), where G_2 is the (split-form of the) 14-dimensional exceptional simple Lie group, and P is a certain parabolic subgroup. Bryant gave a fascinating realization of the flat model as the configuration space for a pair of 2-spheres rolling on each other without twisting or slipping and having $3:1$ radius ratio[2]. Other ratios (except the $1:1$ case which is not $(2,3,5)$) only yield $\mathrm{SO}(3) \times \mathrm{SO}(3)$ symmetry, i.e. rotations of each ball.

Many insights arise when two apparently different structures are shown to be concretely related. To any signature $(2,2)$ conformal structure $[g]$ on a 4-manifold M, there is an induced rank 2 *twistor distribution* $D \subset T(\mathbb{T})$ on the (5-dimensional) *twistor circle bundle* $\mathbb{T} \to M$, realized as the bundle of self-dual totally null 2-planes. For 2-spheres with metrics g_1, g_2, consider $\mathbb{T}$ for $g = g_1 - g_2$, which is the aforementioned configuration space and D encodes the twisting / no-slipping condition. Recently, An and Nurowski [1] studied the case of non-integrable D. Indeed, if the self-dual part of the Weyl tensor of g is not identically zero, then D is generically a $(2,3,5)$-distribution. Moreover, they discovered three new pairs of rolling surfaces with G_2 twistor distributions.

Mike Eastwood, Katja Sagerschnig, and I have observed that the An–Nurowski construction can be formulated in parabolic geometry terms. While conformal geometry and $(2,3,5)$-geometry are parabolic geometries, the An–Nurowski circle twistor bundle is itself a parabolic geometry that we refer to as "XXO-geometry" (because of its Dynkin diagram representation). This mediating geometry has a rich twistor theory: it contains the geometry of pairs of 2nd order ODE, 3-dimensional projective structures, and 4-dimensional conformal structures. We have exhibited new examples of conformal structures with G_2 twistor distributions, and are investigating whether all $(2,3,5)$-geometries arise via the An–Nurowski construction. (They do not — higher order obstructions exist.)

3.4 Generic 3-plane fields on 6-manifolds and special holonomy

A $(3,6)$-distribution is a rank 3 distribution D on a 6-manifold M such that $[D,D] = TM$. For such structures, Bryant [8] solved the equivalence problem and showed that a conformal structure $\mathbf{c}_D$ on M of signature $(3,3)$ is canonically determined. T. Willse and I are using Bryant's work to classify highly symmetric models and to investigate the holonomy of the Fefferman–Graham ambient metrics of the corresponding conformal structures.

Our study is motivated by the similar appearance of a canonically determined signature $(2,3)$ conformal structure for $(2,3,5)$-distributions discovered by Pawel Nurowski [27]. The geometry of such distributions is very rich, with connections to the split-octonions and (the split form of) the 14-dimensional exceptional simple Lie group G_2. Indeed, Graham and Willse [19] showed that the ambient metric for such conformal structures has holonomy always contained in G_2 and is generically equal to it. This exceptional holonomy group appeared in Berger's classification of holonomy groups. In the $(3,6)$-case, our aim is to carry out a similar programme and exhibit new examples of metrics with exceptional (and rarely exhibited) $\mathrm{Spin}_0(3,4)$ holonomy.

3.5 Solution space gaps for BGG operators

Associated to any parabolic geometry $(\mathcal{G} \to M, \omega)$ is a large distinguished class of linear differential operators called (first) Bernstein–Gelfand–Gelfand (BGG) operators [9, 14]. For example, in projective geometry, the Killing tensor and metrisability operators are important BGG operators, while in conformal geometry the almost Einstein and conformal Killing tensor operators are prototypical examples. There is a recent theory of prolongation [6, 20] which connects such BGG operators D to naturally associated vector bundles $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$ (where $\mathbb{V}$ is a $\mathfrak{g}$-representation) called *tractor bundles* endowed with a vector bundle connection $\nabla^{\mathcal{V}}$ called the *prolongation connection*. (In general, this is not the same as the canonical tractor connection determined from the Cartan connection ω.) Specifically, solutions to the BGG operator, i.e. elements of $\ker(D)$, correspond to sections of $\mathcal{V}$ that are parallel with respect to $\nabla^{\mathcal{V}}$. Since parallel sections are determined by their value at a point, it follows that $\dim(\ker(D)) \leq \dim(\mathbb{V})$ and this upper bound is realized on the flat model. A natural question is to determine the *submaximal solution space dimension*. Very little is known in the literature on this subject. For example, for valence two Killing tensors in

[2] See [3] for more history and an explanation of this model in terms of the split-octonions.

the plane, the maximum is 6 in the constant curvature case, while 4 is submaximal and realized by the Darboux super-integrable metrics. However, for higher dimensions and valence, the problem is open. Nevertheless, the *infinitesimal symmetry* equation of any parabolic geometry is also a BGG operator, so the success of my project on symmetry gaps [24] is compelling evidence that similar techniques can be applied for this more general problem.

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