Killing tensors in METRIC h-projective geometry

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1. Historical motivation
2. How they (= Killing tensors) appear and (metric) h-projective invariance
3. How they help
4. Projective geometry within h-projective geometry

1. This is a natural analog of projective geometry in the Kähler situation:
    1. Otsuki, Tashiro and Co since 1956,
    2. Sinjukov, Mikes and Co since 1968,
    3. Matveev and Co since 2009
    4. Eastwood and Co since 2012
2. The geodesic flows of $h-$projectively equivalent metrics admit canonical Killing vectors and Killing (0,2) tensors (Kiyohara since 1996, Topalov since 2000); in the most nondegenerate case we have Liouville integrability and this is generally interesting e.g. because there are (roughly speaking) only few Kähler examples.
3. H-projectively equivalent metrics provide interesting new examples of Kähler manifolds (Apostolov and Co since 2003)

Theorem 1. Suppose $g \stackrel{h.p.}{\sim} \bar{g}$ on (M, J). Then, for every Killing vector field K^i for g one can $\underbrace{\text{canonically construct}}_{\text{show later}}$ a Killing vector field $\bar{K}^i$ for $\bar{g}$.

THIS IS NATURAL AND EXPECTED.

Metaphysical argument that is actually a proof in the Riemannian situation:

- the condition that a metric has a genuine h-projective vector field is a very rare condition and a generic metric admitting an $h-$projectively equivalent one does not admit h-projective vector field.
- A Killing vector field for g is h-projective for $\bar{g}$, and since a generic $\bar{g}$ does not admit genuine h-projective vector field, it is a Killing vector field.
- Thus, there must be a formula of the type $K^i \mapsto \bar{K}^i$. The formula can not be though trivial $\bar{K}^i = K^i$ since it fails on CP^n.

Theorem (Knebelman 1930/ Eastwood 2006). Killing equation is projectively invariant.

Proof (version of Knebelman):
Recall: The Killing equation is

$$K_{i,j} + K_{j,i} = 0 \quad \text{e.g., } K_{i,j} \text{ is skew}$$

(has sense for any connection Γ; we assume that the Ricci-tensors of all connections are symmetric).

Consider a projectively equivalent connection $\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \delta^i_j \phi_{,k} + \delta^i_k \phi_{,j}$ and $\bar K_i := e^{2\phi} K_i$. Then,

$$(e^{2\phi} K_i)_{;j} = e^{2\phi} \left(2\phi_{,j} K_i + \underbrace{K_{i,j} - K_i \phi_{,j} - K_j \phi_{,i}}_{K_{i;j}} \right) = K_{i,j} + K_i \phi_{,j} - K_j \phi_{,i}.$$

We see that $\bar K_{i;j}$ is skew in i and j so that $\bar K_i$ is a Killing form.

Let K^i be a Killing vector field for (g, J). Then, it is locally Hamiltonian so there exists a function K such that $K_i = J^\alpha_{\ i} K_{,\alpha}$.

Suppose now $g \stackrel{h.p}{\sim} \bar{g}$. Then,
$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \phi_{,k} + \delta^i_k \phi_{,j} - J^i_{\ j} \phi_{,\alpha} J^\alpha_{\ k} + J^i_{\ k} \phi_{,\alpha} J^\alpha_{\ j}$ for a certain *function* ϕ.
Actually, the function ϕ can be found explicitly in terms of g and $\bar{g}$: we contract w.r.t. i, k and obtain

$$\underbrace{\bar{\Gamma}^i_{ji}}_{\frac{1}{2}\log(|det(\bar{g})|)} - \underbrace{\Gamma^i_{ji}}_{\frac{1}{2}\log(|det(g)|)} = 2(n+1)\phi_{,k}$$

so $\phi = \frac{1}{4(n+1)} \log \left| \frac{det(\bar{g})}{det(g)} \right|$. Consider the function $\bar{K} := e^{2\phi} K$.

Theorem 1. $\bar{K}_i = J^\alpha_{\ i} \bar{K}_{,\alpha}$ is a Killing form for $\bar{g}$.

Special case. If $K = \mathrm{const} = 1$, then $K_i = J^\alpha_{\ i} K_{,\alpha} = 0$ is a trivial Killing form. The corresponding $\bar{K}_i = 2 J^\alpha_{\ i} \phi_{,\alpha} e^{2\phi}$ is a nontrivial Killing form for $\bar{g}$. (This Killing form was almost known to Mikes-Domashev 1978; Apostolov and Co 2004, Topalov 20??, Matveev-Rosemann 2010)

Remark. The proof requires that both connections are metric and J is integrable.

We need to show that $(e^{2\phi}K)_{,\alpha j}J^{\alpha}_{i}$ is scew in i,j; i.e., that $(e^{2\phi}K)_{;ij}$ is Hermitian. We calculate:

$$(e^{2\phi}K)_{;ij} = (e^{2\phi})_{;ij}K + e^{2\phi}\Big(K_{i,j} - K_{,i}\phi_{,j} - K_{,j}\phi_{,i} + K_{,\alpha}\phi_{,\beta}J^{\alpha}_{i}J^{\beta}_{j} + K_{,\alpha}\phi_{,\beta}J^{\alpha}_{j}J^{\beta}_{i} + 2(K_{,i}\phi_{,j} + K_{,j}\phi_{,i})\Big)$$

$$= \underbrace{(e^{2\phi})_{;ij}}_{\text{Mikes}}K + \underbrace{K_{i,j}}_{\text{Killing for } g} + \underbrace{(K_{,i}\phi_{,j} + K_{,j}\phi_{,i} + K_{,\alpha}\phi_{,\beta}J^{\alpha}_{i}J^{\beta}_{j} + K_{,\alpha}\phi_{,\beta}J^{\alpha}_{j}J^{\beta}_{i})}_{\text{hermitian}}$$

Theorem 2 (Matveev-(Schöbel)/Calderbank) 201?. Let K_{ij} be a hermitian Killing $(0,2)$-tensor for Γ (with symmetric Ricci). Then, for any $\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \delta^i_j \phi_{,k} + \delta^i_k \phi_{,j} - J^i{}_j \phi_{,\alpha} J^\alpha_{\ k} - J^i{}_k \phi_{,\alpha} J^\alpha_{\ j}$
$e^{4\phi} K_{ij}$ is Killing for $\bar\Gamma$.

Proof. We calculate (similar to the Knebelman's proof in the projective geometry)

The Killing equation (in $\bar\Gamma$): $K_{ij;k} V^i V^j V^k = 0$ for all V^i.

$$V^i V^j V^k (e^{4\phi} K_{ij})_{;k} = e^{4\phi} V^i V^j V^k \Big(K_{ij,k} - K_{ik}\phi_{,j} - K_{jk}\phi_{,i} - 2K_{ij}\phi_k$$

$$+ K_{\beta i}\phi_{,\alpha}(J^\alpha_{\ k} J^\beta_{\ j} + J^\alpha_{\ j} J^\beta_{\ k}) + K_{\beta j}\phi_{,\alpha}(J^\alpha_{\ k} J^\beta_{\ i} + J^\alpha_{\ i} J^\beta_{\ k}) + 4K_{ij}\phi_{,k}\Big) = 0.$$

- $V^i V^j V^k K_{ij,k} = 0$ since K_{ij} is Killing.
- $V^i V^j V^k K_{\alpha i}\phi_{,\beta} J^\alpha_{\ k} = 0$ because $K_{i\alpha} J^\alpha_{\ k}$ is skew in i, j.
- Otherwise we have four $V^i V^j V^k K_{ik}\phi_{,j}$ with “minus” sign which are cancelled by and four with “plus” sign.

Corollary 1 (Matveev-Rosemann 2009; this is a special case of Apostolov et al 2004 to be explained later.) Suppose $g \overset{h.p.}{\sim} \bar{g}$ on M. Now, assume $\Gamma = \bar{\Gamma}$ on open $U \subseteq M$. Then, $\Gamma = \bar{\Gamma}$ on the whole $U \subseteq M$.

Proof. We have that $K_i = e^{2\phi}_{,\alpha} J^{\alpha}_{i}$ is a Killing form. If $\Gamma = \bar{\Gamma}$ on $U \subseteq M$, then $\phi = \mathrm{const}$ on U implying $K^i = 0$ on U. It is known that if K^i is Killing on M and vanishes on an open subset, then $K^i \equiv 0$ on the whole M implying $\phi = \mathrm{const}$ on M^n, $\square$.

Contrast with projective geometry. There are many examples of metrics which are projectively equivalent on the whole manifold and affinely equivalent on an certain (open) subset only.

Fakt (Dini 1869): *The metric* $(X(x) - Y(y))(dx^2 + dy^2)$ *is geodesically equivalent to* $\left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right)$.

The metrics are not affinely equivalent at the points where $dX = dY = 0$. Taking an example of X and Y such that they a constant in a neighbborhood and not constant otherwise, we obtain metrics which are projectively equivalent on the whole manifold and affinely equivalent on an certain (open) subset only

We consider the $(1,1)$-tensor $A = a^i_j$ given by $a = \bar{g}^{-1} g \cdot \left(\frac{det(\bar{g})}{det(g)}\right)^{\frac{1}{2(n+1)}}$.
The condition that the metrics are h-projectively equivalent is a linear (Mikes-Domashev 1978) equation on a^i_j:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_\alpha J^\alpha_{\ i} J_{jk} + \lambda_\alpha J^\alpha_{\ j} J_{ik}. \quad (*)$$

(We have seen in the talk of Calderbank the $h-$projectively invariant version of this equation under the name "metrisability equation".)
It is easy to show that $e^{-2\phi} = \sqrt{det(A)}$, so $\sqrt{det(A)}_{,\alpha} J^\alpha_{\ i}$ is a Killing form.
Since for any solution A of $(*)$ we have that $A + t \cdot Id = a^i_j + t \cdot \delta^i_j$ is also a solution of $(*)$, we have the following 1-parameter family of Killing vector fields: $\left(\sqrt{det(A + t \cdot Id)}\right)_{,\alpha} J^{\alpha i}$:
It is an easy linear algebra to see that the number of independent Killing vector fields of such form is the number of nonconstant eigenvalues.

Corollary 2 (Apostolov et al 2004; Topalov-Kiyohara 2011). The number of nonconstant eigenvalues of a^i_j is the same at all generic points of M.

Let $g \overset{h.p.}{\sim} \bar{g}$. Since $\bar{g}$ is a Killing $(0,2)$-tensor for $\bar{g}$, our Theorem 2 gives us the following $(0,2)$ Killing tensor for g:
$K(\ ,\ ) = e^{-4\phi}\bar{g}_{ij} = \sqrt{det(A)}g(A^{-1}\ ,\ )$.
As we have seen before, if $A = a^i_j$ is a solution of Mikes-Domashev equation $(*)$, $A_t := A + t \cdot Id$ is also a solution. We obtain a (polynomial) one-parameter family of Killing tensor
$K_t(\ ,\ ) = \sqrt{det(A + t \cdot Id)}g((A + t \cdot Id)^{-1}\ ,\ )$.

Remark. $\sqrt{det(A + t \cdot Id)}(A + t \cdot Id)^{-1} = \mathrm{Comatrix}_{\mathbb{C}}(A + t \cdot Id)$ and is well defined for all t and is polynomial in t.
Remark. The same construction exists in the projective geometry.

It is easy linear algebra to see that the number of linear (and functionally) independent Killing tensors is the degree of the minimal polynomial of g.
Corollary. The structure of the Jordan normal form of A is the same at a generic point (i.e., if the number of multiple roots of each of the polynomials $Min(A)$ and of $Char(A)$ is the same of the generic point though the roots may depend on the point.)
Proof of Corollary is based on linear algebra + the following
Observation: The Killing equation is an overdetermined linear system of finite type, so if a solution vanishes on an open subset it vanishes everywhere.

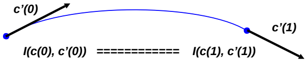
Corollary. Suppose the function $\rho(x)$ is an eigenvalue of A_x of geometric multiplicity $\geqslant 4$ at every point of $U \subseteq M$. Then, $\rho(x) = \text{const} := \rho$ and at every point of M this constant ρ is an eigenvalue of geometric multiplicity $\geqslant 4$.

Proof. We repeat: the tensor $K_t(\ ,\ ) = g(\mathbf{Comatrix}_{\mathbb{C}}(A + t \cdot Id)\ ,\ )$ is a Killing tensor.

Then, for every t the function

$$I_t : TM \to \mathbb{R}, I_t(\xi) := K_t(\xi, \xi)$$

is an integral: for any affine parameterized geodesic $C(s)$ and any s_1, s_2 we have $I(C'(s_1)) = I(C'(s_2))$.



Theorem 3. For generic h-projectively equivalent metrics g and $\bar{g}$ every Killing tensor of is a tensor combination of the Killing tensors and Killing 1-forms constructed above.

We already have seen the usefulness of the (quadratic in velocities) integrals I_t constructed by $\bar{g}$.

There also exists canonical linear (in velocities) integrals, namely the functions $L_t(\xi) := K(t)_i\xi^i$ where $K(t)_i$ are the canonical Killings form constructed above by $\bar{g}$.

Theorem (Topalov 2001; Topalov-Kiyohara 2011, Matveev-Rosemann 201?). The integrals L_t and I_t $\underbrace{\text{mutually commute}}_{\text{will be explained}}$.

We identify $TM \overset{g}{\equiv} T^*M$. Now, T^*M has a canonical Poisson structure (i.e., Poisson bracket: $\{\ ,\ \} : C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$. Theorem above means that for any t_1, t_2 we have

$$\{I_{t_1}, I_{t_2}\} = \{L_{t_1}, I_{t_2}\} = \{L_{t_1}, L_{t_2}\} = 0.$$

In the case A has n (=maximal number) of nonconstant eigenvalues we have $2n$ commuting functionally independent (=differentials are linearly independent a.e.) integrals; this situation is called Liouville integrability and there is a well-developed theory to work with Liouvulle-integrable systems (i.e. geodesic flows in our case).

Fakt (Liouville Theorem (Arnold 1954)). If a geodesic flows on a closed Riemannian manifold is Liouville integrable, then one can solve the geodesic (ordinary differential) equation in quadratures, i.e., find the formula for almost every geodesic in terms of integration of 1-forms.

This was actually the essential motivation of Kiyohara and later Topalov to study h-projectively equivalent metrics: there are (or were until Kiyohara 1996) no nonhomogeneoous examples of Liouville-integrable Kähler metrics of dimension $2n \geqslant 4$.

For our Killing vectors and Killing tensors we consider the linear operators

$$\mathcal{I}_t : C^\infty(M) \to C^\infty(M),\ \mathcal{I}_t(f) = \nabla_i K(t)^{ij} \nabla_j,\ \ \mathcal{L}_t(f) = K(t)^i \nabla_i f\ (= df(K(t)^i)$$

Conjecture. The operators mutially commute in the usual sense, as operators, e.g. (for any $f : M \to \mathbb{R}$):

$$[\mathcal{I}_{t_1}, \mathcal{I}_{t_2}](f) := \mathcal{I}_{t_1}(\mathcal{I}_{t_2}(f)) - \mathcal{I}_{t_2}(\mathcal{I}_{t_1}(f)) \equiv 0.$$

Proven parts of the conjecture.

- $[\mathcal{I}_{t_1}, \mathcal{L}_{t_2}] = [\mathcal{L}_{t_1}, \mathcal{L}_{t_2}] = 0$ (more or less triviality).
- $[\Delta_g, \mathcal{I}_{t_1}] = [\Delta_g, \mathcal{L}_{t_1}] = 0$ follows from the combination of results of Kiosak and Schöbel.

Metaphysical argument that is actually a proof in the Riemannian situation: $[\mathcal{I}_{t_1}, \mathcal{I}_{t_2}]$ is an operator of the 2nd order. It commutes with Δ_g and therefore its symbol is a (quadratic in velocities) integral. But then for generic metrics it is (a linear combination) the integrals I_{t_i} which is "strange" since there is no difference between these integrals.
Interest from mathematical physics. For quantum integrable metrics the "Scrödinger" equation $\Delta_g(f) = C\ f$ de-couples (in the bests case in the system ODE).

Let $g \overset{h.p.}{\sim} \bar{g}$ and consider A as above. Let $\rho_1, ..., \rho_m$ be eigenvalues of A. Consider two distributions

$$D_{big} := \{\xi \in TM \mid d\rho_i(J(\xi)) = 0\} \supseteq D_{small} := span(grad(\rho_i))$$

(at a generic point the distributions are smooth) **Theorem 4.**

1. The distributions are integrable.
2. The integral manifolds are totally geodesics.
3. The integral manifolds can be prolonged over the nongeneric points.
4. THE RESTRICTION OF CONNECTIONS OF g AND $\bar{g}$ TO THESE DISTRIBUTIONS ARE GEODESICALLY EQUIVALENT.

In the Riemannian case, the restrictions on the metrics to these distributions are geodesically equivalent (in the pseudo-Riemannian case the restrictions could be degenerate)

$$g = \begin{bmatrix} \frac{x_1-x_2}{X(x_1)} & 0 & 0 & 0 \\ 0 & -\frac{x_1-x_2}{Y(x_2)} & 0 & 0 \\ 0 & 0 & -\frac{-X(x_1)+Y(x_2)}{x_1-x_2} & -\frac{-X(x_1)x_2+Y(x_2)x_1}{x_1-x_2} \\ 0 & 0 & -\frac{-X(x_1)x_2+Y(x_2)x_1}{x_1-x_2} & -\frac{-X(x_1){x_2}^2+Y(x_2){x_1}^2}{x_1-x_2} \end{bmatrix},$$

$$J^i{}_j = \begin{bmatrix} 0 & 0 & \frac{X(x_1)}{x_1-x_2} & \frac{X(x_1)x_2}{x_1-x_2} \\ 0 & 0 & \frac{Y(x_2)}{x_2-x_1} & \frac{Y(x_2)x_1}{x_2-x_1} \\ -\frac{x_1}{X(x_1)} & -\frac{x_2}{Y(x_2)} & 0 & 0 \\ (X(x_1))^{-1} & (Y(x_2))^{-1} & 0 & 0 \end{bmatrix},$$

$$a_{ij} = \begin{bmatrix} 2\,\frac{(x_1-x_2)x_1}{X(x_1)} & 0 & 0 & 0 \\ 0 & 2\,\frac{x_2(x_2-x_1)}{Y(x_2)} & 0 & 0 \\ 0 & 0 & 2\,\frac{x_1X(x_1)-x_2Y(x_2)}{x_1-x_2} & 2\,\frac{x_1x_2(X(x_1)-Y(x_2))}{x_1-x_2} \\ 0 & 0 & 2\,\frac{x_1x_2(X(x_1)-Y(x_2))}{x_1-x_2} & 2\,\frac{x_1x_2(X(x_1)x_2-Y(x_2)x_1)}{x_1-x_2} \end{bmatrix}$$

In this case both distributions coincide and the integral manifold is the plaque of the first two coordinates. The restriction of the metrics g and $\bar{g}$ to the integral manifold are essentially given by the Dini formulas.

The geodesically equivalent metrics on the integral manifold to the distribution D_{big} determines the h-projectively equivalent metrics.

- The statement of the frametitle is true if the degree of mobility is $D(g) = 2$.
- Also have sense if the degree of mobility is $\geqslant 3$.