On exceptional contact geometries

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Plan

1) Motivation

2) G_2 contact geometry

3) F_4, E_6, E_7, E_8 contact geometries

4) G_2 contact geometry as a reduction of $SO(4,3)$ contact geometry

5) Relations with 4^{th} order ODEs considered modulo contact transformations and exotic holonomy in dimension 4.

Example 1

$[X_4, X_5] = X_3, \quad [X_4, X_3] = X_2, \quad [X_5, X_3] = X_1$

$\mathfrak{g}_{-1} = \langle X_4, X_5 \rangle$

$\mathfrak{g}_{-2} = \langle X_3 \rangle$

$\mathfrak{g}_{-3} = \langle X_1, X_2 \rangle$

5	2	1	1	dim

$\mathfrak{g}_- = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$

5-dimensional nilpotent Lie algebra.

$\mathfrak{g}_-$ nilpotent $\xRightarrow{\text{TANAKA PROLONGATION}}$ (iterative procedure) Algebraic

1st step $\quad \mathfrak{g}_0 = \mathfrak{gl}(2, \mathbb{R}) = \langle X_6, X_7, X_8, X_9 \rangle$

2nd step $\quad \mathfrak{g}_1 = \langle X_{10}, X_{11} \rangle$

3rd step $\quad \mathfrak{g}_2 = \langle X_{12} \rangle$

4th step $\quad \mathfrak{g}_3 = \langle X_{13}, X_{14} \rangle$

5th step $\quad \underline{0} \quad$ STOP

dim	14	2	1	2	4	2	1	2

$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$

$\uparrow$

isomorphic to the split real form of the exceptional simple Lie algebra $\underline{\mathfrak{g}_2}$

Example 2

$[X_4, X_1] = X_0, \quad [X_2, X_3] = 3X_0$

$\mathfrak{g}_{-1} = \langle X_1, X_2, X_3, X_4 \rangle$

$\mathfrak{g}_{-2} = \langle X_0 \rangle$

5	1	4	dim

$\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$

$\mathfrak{g}_-$ nilpotent $\xRightarrow{\text{Tanaka prolongation}}$

1st step $\quad \mathfrak{g}_0 = \langle X_5, X_6, \dots X_{15} \rangle \quad$ <u>much to much!!!</u>

Tanaka prolongation goes to ∞ order!!!

<u>What is wrong?</u>

Roughly: the 1st step prolongation is too large!

More formally:

$\mathfrak{g}_0$ acts on $\mathfrak{g}_-$ linearly via Adjoint transformation

i.e. $\mathfrak{g}_0 \ni X \longrightarrow Ad_X = [X, \cdot] \in End(\mathfrak{g}_-)$

<u>Criterion</u> (I know it from Ben Warhurst)

If $\mathfrak{g}_0$ contains X s.t. $Ad_X = [X, \cdot]$ has rank 1 then the Tanaka prolongation of $\mathfrak{g}_-$ is INFINITE.

<u>In particular</u> in our example

$[X_9, \cdot]$ in basis X_0, X_1, X_2, X_3, X_4 looks like

$$\| \\ \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$$ – evidently element of rank 1 .!

/home/pawel/worksheets/tanaka_physics/TanakaForG2contact.mv.

How to remedy the situation?

Ask the Tanaka prolongation at the 1st step not only to preserve stratification of $\mathfrak{g}_-$ <u>but also</u> some additional structure on $\mathfrak{g}_{-1}$.

<u>What is this structure?</u>

Consider $X_0, X_1 \dots X_{15}$ and the dual $\theta^0, \theta^1, \dots \theta^{15}$.

Consider $\Upsilon = \theta^{12}\theta^{42} - 6\theta^1\theta^2\theta^3\theta^4 + 4\theta^1\theta^{32} + 4\theta^{23}\theta^4 - 3\theta^{22}\theta^{32}$

and look for $X = a_5 X_5 + \dots + a_{15} X_{15}$ s.t.

$$\underset{X}{\mathcal{L}}\Upsilon \sim \Upsilon. \implies X \in \overbrace{\langle X_5 + \tfrac{1}{3}X_{11}, X_6 + \tfrac{1}{3}X_{4}, X_7 + \tfrac{2}{3}X_{12}, X_{10} + 2X_{14} \rangle}^{\mathfrak{g}_0'}$$

Take $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_-$ and replace $\mathfrak{g}_0$ by $\mathfrak{g}_0^* = \mathfrak{g}_0'$

Tanaka prolong $\mathfrak{g}_- \oplus \mathfrak{g}_0'$.

<u>Fact</u> the prolongation is $\mathfrak{g} = \overset{1}{\mathfrak{g}_{-2}} \oplus \overset{4}{\mathfrak{g}_{-1}} \oplus \overset{4}{\mathfrak{g}_0'} \oplus \overset{4}{\mathfrak{g}_1} \oplus \overset{1}{\mathfrak{g}_2}$ Isomorphic to $\mathfrak{g}_2$.

Example 3

$\mathbb{R}^5 \supset \mathcal{U} \ni (x_0, x_1, x_2, x_3, x_4)$

$$\lambda = dx_0 + x_1 dx_4 - 3x_2 dx_3$$

$$\Upsilon = -3\,dx_2^2\,dx_3^2 + 4\,dx_1 dx_3^3 + 4\,dx_2^3 dx_4 - 6\,dx_1 dx_2 dx_3 dx_4 + dx_1^2 dx_4^2$$

Consider vector field X on $\mathcal{U}$ s.t.

$$(*)\quad \begin{cases} \mathcal{L}_X \lambda = f\lambda \\ \mathcal{L}_X \Upsilon = h\Upsilon + \lambda \odot T \end{cases}$$

Fact 1

- Vector fields X as in $(*)$ form a Lie algebra $\mathfrak{g}$.
- $\mathfrak{g} \simeq \tilde{\mathfrak{g}}_2$ – split real form of the exceptional simple Lie group $\mathfrak{g}_2$.

Example 4

Take λ as above on $\mathcal{U}$ but change Υ by replacing

$$dx_4 \longmapsto dx_4 + f(x_3)dx_1$$

Now: $\underset{X^1}{\partial_0}, \underset{X^2}{\partial_4}, \underset{X^3}{\partial_2 + 3x_3\partial_0}, \underset{X^4}{\partial_1 - x_4\partial_0}$ are symmetries

But to find more is rather impossible.

For example if $f(x_3) = x_3^k$ then

$$X^5 = x_1\partial_1 + \frac{k-3}{2k-3}x_2\partial_2 - \frac{3}{2k-3}x_3\partial_3 - \frac{k+3}{2k-3}x_4\partial_4 + \frac{k-6}{2k-3}x_0\partial_0$$

is also a symmetry, and the claim is that if $k \neq \frac{3}{2}$ $(X^1, X^2, \dots X^5)$ is the full symmetry algebra.

Example 5

$\mathbb{R}^4$ and the irreducible representation of $\mathfrak{gl}(2,\mathbb{R})$

$$\mathfrak{gl}(2,\mathbb{R}) = \mathrm{Span}\langle E_1, E_2, E_3, E_4 \rangle$$

$$E_1 = \begin{pmatrix} 0 & 3 & & \\ & 0 & 2 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 2 & 0 & \\ & & 3 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -3 & & & \\ & -1 & & \\ & & 1 & \\ & & & 3 \end{pmatrix} \quad E_4 = \mathrm{id}.$$

Look for $\boxed{\overset{\neq 0}{\Upsilon_{i_1 i_2 \dots i_p}}}$ - symmetric invariant w.r.t. $\mathfrak{gl}(2,\mathbb{R})$ action, I.e. look for $\Upsilon_{i_1 \dots i_p}$ satisfying

$$\boxed{E^i_{I i_1}\Upsilon_{i i_2 \dots i_p} + E^i_{I i_2}\Upsilon_{i_1 i \dots i_p} + \dots + E^i_{I i_p}\Upsilon_{i_1 i_2 \dots i} = c_I \Upsilon_{i_1 \dots i_p} \quad \forall\, I = 1,2,3,4}$$

Fact 2

1) The lowest p for which such Υ exists is $p = 4$.
2) If $p = 4$ there is a UNIQUE up to scale Υ with this property
3) Writing $\Upsilon = \frac{1}{24}\Upsilon_{ijkl}\, x_i x_j x_k x_l$ one gets:

$$\Upsilon = -3x_2^2 x_3^2 + 4x_1 x_3^3 + 4x_2^3 x_4 - 6x_1 x_2 x_3 x_4 + x_1^2 x_4^2.$$

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Now we pass to section 2) i.e. to

G_2 contact geometries

Definition

A G_2 contact structure is a 5-dimensional manifold M^5 equipped with an equivalence class of pairs $[(\lambda,\Upsilon)]$ s.t.

1) λ is a 1-form, Υ is a 4^{th} rank symmetric tensor on M^5.
2) (λ,Υ) and $(\lambda',\Upsilon') \in [(\lambda,\Upsilon)]$ iff there exist $f\neq 0$, $h\neq 0$, T – 3^{rd} rank tensor on M^5 s.t.

$$\lambda' = f\lambda$$
$$\Upsilon' = h\Upsilon + \text{Symetrization}(T\otimes\lambda)$$

3) $d\lambda\wedge d\lambda\wedge\lambda \neq 0$ at each point of M^5

$\Rightarrow$ it defines a contact distribution $\mathcal{D} = \lambda^{\perp}$

4) $\Upsilon|_{\mathcal{D}}$ reduces $GL(\mathcal{D})$ at each point to the irreducible $GL(2,\mathbb{R})$
5) $\Omega = d\lambda|_{\mathcal{D}}$ is invariant w.r.t. $GL(2,\mathbb{R})$ defined by Υ.

Alternatively

A G_2 contact structure in dimension 5 is M^5 equipped with a contact distribution $\mathcal{D}$ and a pair (Υ, Ω) on $\mathcal{D}$ which reduce $GL(\mathcal{D})$ to the irreducible $GL(2,\mathbb{R})$.

4^{th} rank sym tensor

2-form

Moreover, if $\lambda = \mathcal{D}^{\perp}$ we require that

$$d\lambda|_{\mathcal{D}} = a\,\Omega$$ for some nonvanishing a.

Definition

A coframe $(\omega^0,\omega^1,\omega^2,\omega^3,\omega^4)=(\omega^0,\omega^i)$ is adapted to $(M^5,[(\lambda,\Upsilon)])$ if

1) $\omega^0 = f\lambda$ for some nonvanishing f

2) $\Upsilon = -3(\omega^2)^2(\omega^3)^2 + 4\omega^1(\omega^3)^3 + 4(\omega^2)^3\omega^4 - 6\omega^1\omega^2\omega^3\omega^4 + (\omega^1)^2(\omega^4)^2$

Then, in particular

$$\Omega = \omega^1\wedge\omega^4 - 3\omega^2\wedge\omega^3$$

Fact If (ω^0,ω^i) is a coframe adapted to $(M^5,[(\lambda,\Upsilon)])$ then the most general adapted coframe (θ^0,θ^i) is related to (ω^0,ω^i) via:

$$(T)\qquad \begin{pmatrix}\theta^0\\ \theta^1\\ \theta^2\\ \theta^3\\ \theta^4\end{pmatrix} = \underbrace{\begin{bmatrix} s_9 & 0 & 0 & 0 & 0\\ s_{10} & s_5^3 & 3s_5^2 s_6 & 3s_5 s_6^2 & s_6^3\\ s_{11} & s_5^2 s_7 & s_5(s_5s_8+2s_6s_7) & s_6(2s_5s_8+s_6s_7) & s_6^2 s_8\\ s_{12} & s_5 s_7^2 & s_7(2s_5s_8+s_6s_7) & s_8(s_5s_8+2s_6s_7) & s_6 s_8^2\\ s_{13} & s_7^3 & 3s_8s_7^2 & 3s_8^2 s_7 & s_8^3\end{bmatrix}}_{=(S^\mu{}_\nu)} \begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{pmatrix}$$

In particular $\theta^0 = s_9\lambda$

$$\tilde\Omega = \theta^1\wedge\theta^4 - 3\theta^2\wedge\theta^3 = (s_5s_8-s_6s_7)^3\overbrace{(\omega^1\wedge\omega^4-3\omega^2\wedge\omega^3)}^{\Omega} + (\dots)\wedge\omega^0$$

$$\tilde\Upsilon = -3(\theta^2)^2(\theta^3)^2 + 4\theta^1(\theta^3)^3 + \dots = (s_5s_8-s_6s_7)^6\overbrace{\left(-3(\omega^2)^2(\omega^3)^2+\dots\right)}^{\Upsilon} + (\dots)\cdot\omega^0$$

(Note: $\det(S) = (s_6s_7-s_5s_8)^6 s_9$)

This enables us to define a G_2 contact structure in terms of an adapted coframe $\omega=(\omega^0,\omega^i)$ given modulo transformations (T).

A simple invariant can be constructed via

Cartan equivalence method.

This consists in evaluating $d\theta^\mu = \frac{1}{2}C^\mu{}_{\nu\rho}\,\theta^\nu\wedge\theta^\rho$ and an apropriate normalization of $C^\mu{}_{\nu\rho} \equiv C^\mu{}_{\nu\rho}(S)$ and definition of 'vertical' forms ΩZ

Example 3 (continued).

$$\omega^0 = dx_0 + x_1 dx_4 - 3x_2 dx_3$$
$$\omega^1 = dx^1$$
$$\omega^2 = dx^2$$
$$\omega^3 = dx^3$$
$$\omega^4 = dx^4$$

$$\begin{cases} d\omega^0 = \omega^1\wedge\omega^4 - 3\omega^2\wedge\omega^3 \\ d\omega^1 = d\omega^2 = d\omega^3 = d\omega^4 = 0 \end{cases}$$

Thm

There is a unique way of normalizing $c^{\mu}_{\nu\rho}$'s and defining a 14-dimensional manifold $\mathcal{G}^{14}$ with an additional 9 one forms $\theta^5, \theta^6, \theta^7, \theta^8, \theta^9, \theta^{10}, \theta^{11}, \theta^{12}, \theta^{13}$ so that the 1-forms $\theta^0, \theta^1, \dots \theta^{13}$ are lineally independent at each point of $\mathcal{G}^{14}$ and satisfy the system

$\mathfrak{g}_{-2}$
$$\Big[\; d\theta^0 = -6\theta^0\wedge\theta^5 + \theta^1\wedge\theta^4 - 3\theta^2\wedge\theta^3$$

$\mathfrak{g}_{-1}$
$$\left[\begin{aligned} d\theta^1 &= 6\theta^0\wedge\theta^9 - 3\theta^1\wedge\theta^5 - 3\theta^1\wedge\theta^8 + 3\theta^2\wedge\theta^7 \\ d\theta^2 &= 2\theta^0\wedge\theta^{10} + \theta^1\wedge\theta^6 - 3\theta^2\wedge\theta^5 - \theta^2\wedge\theta^8 + 2\theta^3\wedge\theta^7 \\ d\theta^3 &= 2\theta^0\wedge\theta^{11} + 2\theta^2\wedge\theta^6 - 3\theta^3\wedge\theta^5 + \theta^3\wedge\theta^8 + \theta^4\wedge\theta^7 \\ d\theta^4 &= 6\theta^0\wedge\theta^{12} + 3\theta^3\wedge\theta^6 - 3\theta^4\wedge\theta^5 + 3\theta^4\wedge\theta^8 \end{aligned}\right.$$

$\mathfrak{g}_0$
$$\left[\begin{aligned} d\theta^5 &= 2\theta^0\wedge\theta^{13} - \theta^1\wedge\theta^{12} + \theta^2\wedge\theta^{11} - \theta^3\wedge\theta^{10} + \theta^4\wedge\theta^9 \\ d\theta^6 &= 6\theta^2\wedge\theta^{12} - 4\theta^3\wedge\theta^{11} + 2\theta^4\wedge\theta^{10} + 2\theta^6\wedge\theta^8 \\ d\theta^7 &= -2\theta^1\wedge\theta^{11} + 4\theta^2\wedge\theta^{10} - 6\theta^3\wedge\theta^9 - 2\theta^7\wedge\theta^8 \\ d\theta^8 &= -3\theta^1\wedge\theta^{12} + \theta^2\wedge\theta^{11} + \theta^3\wedge\theta^{10} - 3\theta^4\wedge\theta^9 - \theta^6\wedge\theta^7 \end{aligned}\right.$$

$\mathfrak{g}_1$
$$\left[\begin{aligned} d\theta^9 &= -\theta^1\wedge\theta^{13} - 3\theta^5\wedge\theta^9 - \theta^7\wedge\theta^{10} + 3\theta^8\wedge\theta^9 \\ d\theta^{10} &= -3\theta^2\wedge\theta^{13} - 3\theta^5\wedge\theta^{10} - 3\theta^6\wedge\theta^9 - 2\theta^7\wedge\theta^{11} + \theta^8\wedge\theta^{10} \\ d\theta^{11} &= -3\theta^3\wedge\theta^{13} - 3\theta^5\wedge\theta^{11} - 2\theta^6\wedge\theta^{10} - 3\theta^7\wedge\theta^{12} - \theta^8\wedge\theta^{11} \\ d\theta^{12} &= -\theta^4\wedge\theta^{13} - 3\theta^5\wedge\theta^{12} - \theta^6\wedge\theta^{11} - 3\theta^8\wedge\theta^{12} \end{aligned}\right.$$

$\mathfrak{g}_2$
$$\Big[\; d\theta^{13} = -6\theta^5\wedge\theta^{13} - 6\theta^9\wedge\theta^{12} + 2\theta^{10}\wedge\theta^{11}$$

Cartan structure equations

Cartan connection for this example

Cofrane $(\theta^0, \theta^1, \dots \theta^{13})$ defines its dual $(X_0, X_1, \dots X_{13})$

The distribution $\mathfrak{p} = (X_5, X_6, \dots X_{13})$ is integrable on $\mathcal{G}^{14}$ as can be seen looking at $d\theta^0, d\theta^1, d\theta^2, d\theta^3, d\theta^4$

Thus $\mathcal{G}^{14}$ is a fibre bundle

$$P \to \mathcal{G}^{14} \to \mathcal{G}^{14}/P = M^5$$

where M^5 is obtained as a quotient of $\mathcal{G}^{14}$ by an equivalence relation $\sim$ identifying points lying on integral manifolds of $\mathfrak{p}$.

$\mathcal{G}^{14}$ is a Lie group - isomorphic to the simple exceptional Lie group $\tilde{G}_2$

The matrix of 1-forms

$$\omega_{cart} = \begin{bmatrix} 3\theta^5+\theta^8 & -2\theta^6 & -4\theta^{10} & 4\theta^{11} & 6\theta^{12} & 6\theta^{13} & 0 \\ -\frac{1}{2}\theta^7 & 3\theta^5-\theta^8 & 6\theta^9 & -2\theta^{10} & -\theta^{11} & 0 & -6\theta^{13} \\ \frac{1}{2}\theta^3 & \theta^4 & 2\theta^8 & \theta^6 & 0 & \theta^{11} & -6\theta^{12} \\ \theta^2 & 2\theta^3 & 2\theta^7 & 0 & -\theta^6 & 2\theta^{10} & -4\theta^{11} \\ -\theta^1 & -2\theta^2 & 0 & -2\theta^7 & -2\theta^8 & -6\theta^9 & 4\theta^{10} \\ \theta^0 & 0 & 2\theta^2 & -2\theta^3 & -\theta^4 & -3\theta^5+\theta^8 & 2\theta^6 \\ 0 & -\theta^0 & \theta^1 & -\theta^2 & -\frac{1}{2}\theta^3 & \frac{1}{2}\theta^7 & -3\theta^5-\theta^8 \end{bmatrix}$$

is a $\tilde{\mathfrak{g}}_2$-valued Cartan connection on $P \to \mathcal{G}^{14} \to \mathcal{G}/P$.

IT is a Maurer-Cartan form on $\mathcal{G}^{14} \simeq \tilde{G}_2$ so that the equations 'Cartan structure equations' from the previous page are just

$$d\omega_{cart} + \omega_{cart} \wedge \omega_{cart} = 0$$

Producing a simple invariant for a general G_2 contact structure

Given a G_2 contact structure $(M^5, [(\alpha, r)])$ we start with an adopted coframe $\omega^\mu = (\omega^0, \omega^i)$ in the most convenient form.

We can always achieve:

$$\begin{cases} d\omega^0 = \omega^1\wedge\omega^4 - 3\omega^2\wedge\omega^3 + \text{terms not involving } \omega^1\wedge\omega^4,\ \omega^2\wedge\omega^3 \\ d\omega^i = -\frac{1}{2}\boxed{f^i{}_{\mu\nu}}\,\omega^\mu\wedge\omega^\nu \end{cases}$$

Now we define $\theta^\mu = S^\mu{}_\nu\omega^\nu$ with the most general $S^\mu{}_\nu$ making θ^μ still an adopted coframe.

We want that the 5 one forms θ^μ satisfy equations:

$$E_0 = d\theta^0 - (-6\theta^0\wedge\theta^5 + \theta^1\wedge\theta^4 - 3\theta^2\wedge\theta^3 + c^0{}_{0i}\,\theta^0\wedge\theta^i) \equiv 0$$

$$E_1 = d\theta^1 - (6\theta^0\theta^9 - 3\theta^1\theta^5 - 3\theta^1\theta^8 + 3\theta^2\theta^7 + c^1{}_{\mu i}\,\theta^\mu\wedge\theta^i) \equiv 0$$

with some additional forms $\theta^5, \theta^7, \theta^8, \theta^9$.

Now: $E_0\wedge\theta^0 \equiv 0$ implies $\underbrace{\boxed{S_9 = (S_5S_8 - S_6S_7)^3 = \Delta^3}}_{\text{normalization}}$

$\Longrightarrow$ $E_0 \equiv 0$ implies that

$$\underbrace{\theta^5 = \frac{d\Delta}{2\Delta} + (\ \ )\theta^1 + (\ \ )\theta^2 + (\ \ )\theta^3 + (\ \ )\theta^4 + b_1\theta^0}_{\text{definition of a new 'vertical' form}}$$

totally determined

and, eventually, the equation $E_1 \wedge \theta^0 \wedge \theta^1 \wedge \theta^2 \equiv 0$

gives

$$c'_{34} = \frac{s_5^7}{\Delta^5}\left[a_0 + 7a_1 x + 21 a_2 x^2 + 35 a_3 x^3 + 35 a_4 x^4 + 21 a_5 x^5 + 7 a_6 x^6 + a_7 x^7\right]$$

where $x = \frac{s_6}{s_5}$ and the coefficients $\underline{a_0, a_1, \dots a_7}$ are <u>linear combinations</u> of the <u>structure functions</u>

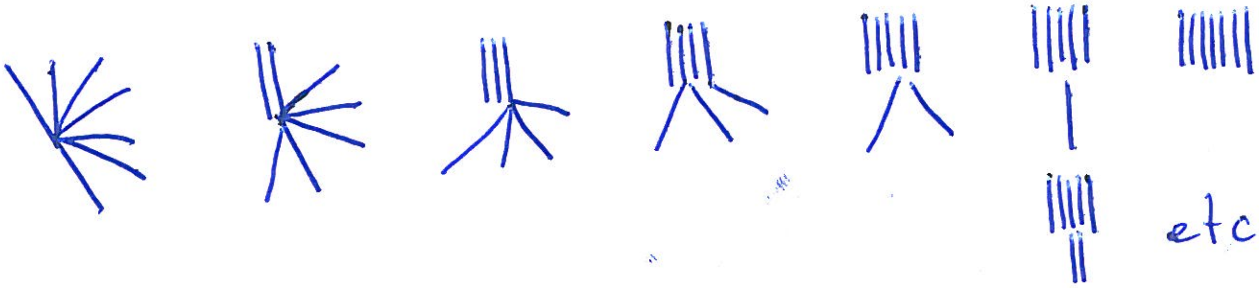
$$\boxed{f^i_{\mu\nu}}$$

Although the coefficient c'_{34} varies when we change $s_5, s_6, \dots s_{13}$, but the structure of its zero's ~~of~~ is unaffected by this change.

In other words the <u>number</u> of <u>roots</u> of the polynomial $P_7(x) = a_0 + 7a_1 x + \dots + a_7 x^7$ and <u>their multiplicities</u> are <u>invariant properties</u> of the structure $(M^5, [(\lambda, \gamma)])$.

<u>So</u> the G_2 contact structures split onto nonequivalent classes according to the structure of roots of $P_7(x) = 0$. The following cases can occur:

0) all $a_0 \equiv a_1 \equiv a_2 \equiv \dots \equiv a_7 \equiv 0$ and



etc.

Case 0) is special.

Thm

If $a_0 \equiv a_1 \equiv \dots \equiv a_7 \equiv 0$ then the G_2 contact structure is locally equivalent to the one defined by

$$\begin{cases} \omega^0 = dx^0 + x^1 dx^4 - 3x^2 dx^3 \\ \omega^i = dx^i \end{cases}$$

In such case the structure has $\tilde{G}_2$ symmetry. I will call it FLAT G_2 contact structure.

Coefficients $a_0, a_1, \dots a_7$ span the space of harmonic curvature for a G_2 contact structure.

Example Our Example 4 introduces G_2 contact structures with the root type IIIIIII.

Introducing Cartan normal connection for a contact G_2 structure

Given a G_2 contact structure and an adopted coframe $(\omega^0, \omega^i) = \omega^\mu$ we define $\theta^\mu = S^\mu{}_\nu \omega^\nu$ as before, and eventually

ω_{cart} as before with some additional 1-forms $\theta^5, \theta^6, \dots \theta^{13}$.

The art of defining these additional 9 forms consist in this that

1) $\theta^0, \dots, \theta^4, \theta^5, \dots \theta^{13}$ must be linearly independent on the 14 dimensional manifold which should be constructed

2) the matrix 2-form

$$K = d\omega_{cart} + \omega_{cart} \wedge \omega_{cart} =$$

$$= \frac{1}{2} K_{IJ}\, \theta^I \wedge \theta^J$$

must have all $K_{IJ} \equiv 0$ if at least one of I or J belongs to the set $\{5, 6, 7, \dots 13\}$.

Even such strong requirement for matrices K_{IJ} does <u>not</u> make θ^I unique.

One way of achieving <u>uniqueness</u> here is Tanaka normality condition.

We discuss this condition in a more general setting. passing smoothly to Section 3).

The Cartan structure equations for the flat G_2 contact structure, when writen on the bundle

$$P \to \mathfrak{g}^{14} = \tilde{G}_2 \to \tilde{G}_2/P$$

in terms of the canonical coframe $\theta^A = (\theta^0, \theta^1, \dots \theta^{13})$ look like:

$$(1)\quad \boxed{d\theta^A + \tfrac{1}{2} C^A{}_{BC}\, \theta^B \wedge \theta^C = 0,} \qquad A,B,C = 0,1,\dots 13$$

This defines <u>constants</u> $C^A{}_{BC}$. Because of the grading we have a split

$$\underbrace{\underbrace{\theta^0}_{\theta^0},\ \underbrace{\theta^1, \theta^2, \theta^3, \theta^4}_{\theta^i}}_{\theta^\mu}\quad \underbrace{\theta^5, \theta^6, \theta^7, \theta^8,}_{\theta^{4+I}}\quad \underbrace{\underbrace{\theta^9, \theta^{10}, \theta^{11}, \theta^{12},}_{\theta^{\bar{i}}}\ \underbrace{\theta^{13}}_{\theta^{\bar{0}}}}_{\theta^{\bar{\mu}}}$$

$$i = 1,2,3,4 = N$$
$$-8 + \bar{i} = 1,2,3,4 = N \qquad \mu = 0,1,2,3,4$$
$$I = 1,2,3,4 = d \qquad \bar{\mu} = 9,10,11,12,13$$

The Killing form for $\tilde{G}_2$, when written in the basis θ^A is $\boxed{B_{AB} = C^D{}_{AF}\, C^F{}_{BD}}$ Its inverse is B^{AB} s.t. $\boxed{B^{AC} B_{CB} = \delta^A{}_B}$

To define Tanaka normalization conditions for the <u>nonflat</u> G_2 contact structures we take the most general adapted coframe θ^μ and <u>force it</u> to satisfy

$$(CC)\quad \boxed{d\theta^A + \tfrac{1}{2} C^A{}_{BC}\, \theta^B \wedge \theta^C = \tfrac{1}{2} R^A{}_{\mu\nu}\, \theta^\mu \wedge \theta^\nu}$$

on a 14-dimensional manifold M^{14} with <u>constants</u> $C^A{}_{BC}$ as in (1) and forms θ^A s.t. the first five of them coincide with θ^μ, and the rest of them, together with θ^μ's form a <u>coframe</u> on M^{14}.

The Tanaka normalization that specifies θ^I's <u>uniquely</u> (also uniquely defining local M^{14}) are given by:

$$(\mathrm{Tan})\begin{cases} R^0{}_{ij} \equiv 0 \quad \text{for all } i,j = 1,2,3,4 \\ 2\,R^A{}_{\mu\delta}\,B^{\mu\bar\beta}\,C^B{}_{\bar\beta A} + R^B{}_{\mu\nu}\,C^\mu{}_{\bar\beta\delta}\,B^{\nu\bar\beta} = 0 \end{cases}$$

These are linear equations for the $R^A{}_{BC}$, which when solved give constraints on θ^μ's. They are strong enough to ALGEBRAICALLY determine <u>all</u> the θ^A's satisfying (CC).

The equations (Tan) once solved should be inserted in (CC).
After insertion, <u>the Bianchi identities</u> imply that all $R^0{}_{\mu\nu} \equiv 0$.

Now Cartan equivalence method determines all θ^A's.

3) G_2, F_4, E_6, E_7, E_8 contact geometries

Consider the flat contact structure.

In the canonical coframe $\theta^0, \dots, \theta^{13}$ on G_2 we have:

- $$\Omega = \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3 = \frac{1}{2}\Omega_{ij}\,\theta^i \wedge \theta^j$$

  $i, j, k, \dots = N = 4 =$ dimension of the contact distribution

- $$\Upsilon = \frac{1}{6}\Upsilon_{ijkl}\,\theta^i\theta^j\theta^k\theta^l = \left(-3\theta^{2^2}\theta^{3^2} + 4\theta^1\theta^{3^3} + 4\theta^{2^3}\theta^4 - 6\theta^1\theta^2\theta^3\theta^4 + \theta^{1^2}\theta^{4^2}\right)$$

So we have $\boxed{\Omega_{ij} = \Omega_{[ij]}}$ and $\boxed{\Upsilon_{ijkl} = \Upsilon_{(ijkl)}}$

Ω_{ij} is invertible; Define Ω^{ij} by $\boxed{\Omega_{ik}\Omega^{kj} = \delta^j{}_i}$

We have the representation of $\mathfrak{gl}(2,\mathbb{R})$, which preserves (Ω, Υ):

$$e_{N+I}:\quad e_5 = \begin{pmatrix} -3 & & & \\ & -3 & & \\ & & -3 & \\ & & & -3 \end{pmatrix},\ e_6 = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 2 & 0 & \\ & & 3 & 0 \end{pmatrix},\ e_7 = \begin{pmatrix} 0 & 3 & & \\ & 0 & 2 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix},\ e_8 = \begin{pmatrix} -3 & & & \\ & -1 & & \\ & & 1 & \\ & & & 3 \end{pmatrix}$$

$I = 1, 2, 3, \dots, d = 4$ dimension of $\mathfrak{gl}(2,\mathbb{R})$

Define $$\Gamma^i{}_j = \sum_{I=1}^{d} \theta^{N+I}\,(e_{N+I})^i{}_j$$

$$\theta_i = \left(-6\theta^{12}, 6\theta^{11}, -6\theta^{10}, 6\theta^9\right) \quad\text{and}\quad \theta_0 = \theta^{13}$$

with this notation the flat Cartan structure equations read:

$$d\theta^0 + \frac{2}{N}\Gamma^i{}_j \wedge \theta^0 - \frac{1}{2}\Omega_{ij}\,\theta^i \wedge \theta^j = 0 \quad \Big]\ \mathfrak{g}_{-2}$$

$$d\theta^i + \Gamma^i{}_j \wedge \theta^j - \Omega^{ij}\,\theta_j \wedge \theta^0 = 0 \quad \Big]\ \mathfrak{g}_{-1}$$

(S)

$$d\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j - \frac{1}{2}\Omega_{jl}\,\theta^l \wedge \theta_k \Omega^{ki} + \frac{1}{2}\theta^i \wedge \theta_j + \frac{1}{2}\delta^i_j(\theta^k \wedge \theta_k + 2\theta^0 \wedge \theta_0) - \Omega^{ip}\Upsilon_{pjkq}\Omega^{ql}\,\theta^k \wedge \theta_l = 0 \quad \Big]\ \mathfrak{g}_0$$

$$d\theta_i + \theta_j \wedge \Gamma^j{}_i - \Omega_{ij}\,\theta^j \wedge \theta_0 = 0 \quad \Big]\ \mathfrak{g}_1$$

$$d\theta_0 + \frac{2}{N}\theta_0 \wedge \Gamma^i{}_i - \frac{1}{2}\Omega^{ij}\,\theta_i \wedge \theta_j = 0 \quad \Big]\ \mathfrak{g}_2$$

of course the system has also its nonflat version in which the zeroe's on the r.h.s. are replaced by $\frac{1}{2}R^A{}_{\mu\nu}\theta^\mu \wedge \theta^\nu$.

Interestingly, the system (S) can be considered for any <u>N</u> and <u>d</u>.

Here is the question.

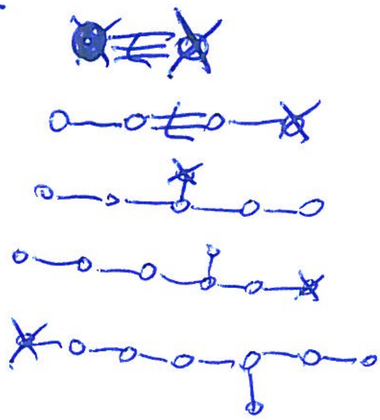
We know that there are $(\theta^0, \theta^i, \Gamma^i{}_j, \theta_i, \theta_0, \Omega_{ij}, \Upsilon_{ijkl})$ in dimension 14 corresponding to a flat G_2 contact structure that satisfy (S).

Are there examples of $(\theta^0, \theta^i, \Gamma^i{}_j, \theta_i, \theta_0, \Omega_{ij}, \Upsilon_{ijkl})$ in other N's, and d's that also satisfy (S)?

The answer is yes!

At least for

N	d	H	$\dim G \overset{\parallel}{} 2(1+N)+d$	G
4	4	$GL(2,\mathbb{R})$	14	G_2
14	22	$CSp(3,\mathbb{R})$	$2\cdot 15+22=52$	F_4
20	36	$GL(6)$	$2\cdot 21+36=78$	E_6
32	67	$CSpin(6,6)$	$2\cdot 33+67=133$	E_7
56	134	CE_7	$2\cdot 57+134=248$	E_8.



First check: Use LiE:

dim([3], A1) = 4
sym_tensor(4, [3], A1) = 1X[0] + ...

dim([0,0,1], C3) = 14
sym_tensor(4, [0,0,1], C3) = 1X[0,0,0] + ...

dim([0,0,1,0,0], A5) = 20
sym_tensor(4, [0,0,1,0,0], A5) = 1X[0,0,0,0,0] + ...

dim([0,0,0,0,0,1], D6) = 32
sym_tensor(4, [0,0,0,0,0,1], D6) = 1X[0,0,0,0,0,0] + ...

dim([0,0,0,0,0,0,1], E7) = 56
sym_tensor(4, [0,0,0,0,0,0,1], E7) = 1X[0,0,0,0,0,0,0] + ...

The same for alt_tensor(2, [...], ...) = 1X[0...0] + ..

But also

dim([1,2], A1A1) = 6
sym_tensor(4, [1,2], A1A1) = 1X[0,0] + ..
alt_tensor(2, [1,2], A1A1) = 1X[0,0] + ..

N	d	H	G
6	7	$C(SL_2\times SL_2)$	$SO(4,3)$

Second check try to guess the representation of $\mathfrak{g}$ in dimension N;

Given such a representation

$$\mathfrak{g} = \langle E_I \rangle \quad \text{where } E_I = (E_I{}^i{}_j) \quad N \times N \text{ matrices}$$

look for $\Omega_{ij} = \Omega_{[ij]}$, $\Upsilon_{ijkl} = \Upsilon_{(ijkl)}$ s.t.

$$\begin{cases} E^i_{I i_1} \Omega_{i i_2} + E^i_{I i_2} \Omega_{i_1 i} = c_I \Omega_{i_1 i_2} \\ E^i_{I i_1} \Upsilon_{i i_2 i_3 i_4} + E^i_{I i_2} \Upsilon_{i_1 i i_3 i_4} + E^i_{I i_3} \Upsilon_{i_1 i_2 i i_4} + E^i_{I i_4} \Upsilon_{i_1 i_2 i_3 i} = q_I \Upsilon_{i_1 i_2 i_3 i_4}. \end{cases}$$

These are linear equations for Ω_{ij} and Υ_{ijkl}.

I solved them with Mathematica for all five exceptional cases, used the procedure described at page 15 to define $\theta^0, \theta^i, \Gamma^i{}_j, \theta_i, \theta_0$ and checked that all the 5 cases are solutions to the system (6)

Third check Ask Robert Bryant.

He essentially said something like this:

• Start with $\mathbb{R}^2$ equipped with $\epsilon_{AB} = -\epsilon_{BA}$.

$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Identify $\mathbb{R}^4$ with $\odot^3 \mathbb{R}^2$

$$\mathbb{R}^4 = \odot^3 \mathbb{R}^2 \ni \psi^{ABC} = (\psi^{000}, \psi^{001}, \psi^{011}, \psi^{111}) = (X^1, X^2, X^3, X^4)$$

Given $\psi^{ABC} \in \mathbb{R}^4$ define

$$L^A{}_H(\psi) = \psi^{ABC} \psi^{DEF} \epsilon_{CD} \epsilon_{BE} \epsilon_{FH}$$

This is an endomorphism of $\mathbb{R}^2$: $L^A{}_H(\psi) v^H = \tilde{v}^A$

Its trace $L^A{}_A(\Psi) \equiv 0$, but trace of its square is not!

$$L^A{}_B(\Psi)L^B{}_A(\Psi) := \Upsilon(\Psi) = \Upsilon(X)$$

quartic in X

precisely the same as the one I was working with so far.

Also, given two elements of $\mathbb{R}^4$, say, ψ^{ABC}, ϕ^{ABC}

define

$$\Omega(\psi,\phi) = \psi^{ABC}\phi^{DEF}\epsilon_{CD}\epsilon_{BE}\epsilon_{AF}$$

This is skew in ϕ,ψ: $\Omega(\psi,\phi) = -\Omega(\phi,\psi)$

and with the identification $(\psi^{000}, \psi^{001}, \psi^{011}, \psi^{111}) = (x^1, x^2, x^3, x^4)$

defines

$$\Omega(X,Y) = \Omega(\psi,\phi) = \tfrac{1}{2}\Omega_{ij}X^iY^j \quad \text{with}$$

$$\Omega_{ij}dx^i \wedge dx^j = dx^1 \wedge dx^4 - 3dx^2 \wedge dx^3 \quad \checkmark$$

- Now for $\mathbb{R}^{14}$ and (Υ, Ω) making reduction to $CSp(3,\mathbb{R})$.

Take standard representation of $Sp(3,\mathbb{R})$.

So we have $\mathbb{R}^6$, and $\epsilon_{AB} = -\epsilon_{BA}$ making the reduction from $GL(6,\mathbb{R})$ to $Sp(3,\mathbb{R}) \subset GL(6,\mathbb{R})$

Identify $\mathbb{R}^{14}$ with $(\Lambda^3\mathbb{R}^6)_0$

$$\mathbb{R}^{14} = \{\Lambda^3\mathbb{R}^6 \ni \psi^{ABC} \text{ s.t. } \psi^{ABC}\epsilon_{BC} = 0\}$$

Define $L^{A}{}_{H}(\psi) = \psi^{ABC}\psi^{DEF}\epsilon_{CD}\epsilon_{BE}\epsilon_{FH}$

and

$$\left.\begin{array}{l} \Upsilon(\psi) = L^{A}{}_{B}(\psi)L^{B}{}_{H}(\psi) \neq 0. \\ \text{Also} \quad \Omega(\psi,\phi) = \epsilon_{AF}\epsilon_{BE}\epsilon_{CD}\psi^{ABC}\phi^{DEF} \end{array}\right\} \quad (BF)$$

Both objects are now $CSp(3,\mathbb{R})$ invariant

Moreover, they are unique as such and,

in particular Υ_{ijkl} makes the reduction from $GL(14,\mathbb{R})$

to $CSp(3,\mathbb{R}) \subset GL(14,\mathbb{R})$

irred.

- For $\mathbb{R}^{20}$ and (Υ, Ω) making the reduction to $GL(6,\mathbb{R})$ also take $\mathbb{R}^6$ and ϵ_{AB} as in the previous case.

Identify $\mathbb{R}^{20}$ with $\Lambda^3\mathbb{R}^6$ and use (BF) to define Υ and Ω. They are now $GL(6,\mathbb{R})$ invariant and make reduction from $GL(20,\mathbb{R})$ to $GL(6,\mathbb{R})$.

- If a kind of formula (BF) exist for the other exceptional cases I do not know.

What I know instead is the following;

4) G_2 contact geometry as a reduction of $SO(4,3)$ contact geometry:

Start with $\mathbb{R}^2$ with $\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Identify $\mathbb{R}^6$ with $\mathbb{R}^2 \otimes \odot^2 \mathbb{R}^2 \ni \underbrace{\alpha^A \beta^{BC}}_{\psi^{ABC}}$ s.t. $\beta^{BC} = \beta^{CB}$.

$$\psi^{ABC} = \alpha^A \beta^{BC} = (\underset{X^1}{\psi^{000}}, \underset{X^2}{\psi^{001}}, \underset{X^2}{\psi^{010}}, \underset{X^3}{\psi^{011}}, \underset{X^2+2X^5}{\psi^{100}}, \underset{X^3+X^6}{\psi^{101}}, \underset{X^3+X^6}{\psi^{110}}, \underset{X^4}{\psi^{111}})$$

Use the same formula for $\Upsilon(\Psi)$ and $\Omega(\psi, \phi)$.

You get:

$$\Upsilon(X) = -3X_2^2X_3^2 + 4X_1X_3^3 + 4X_2^3X_4 - 6X_1X_2X_3X_4 + X_1^2X_4^2 + \ldots\ldots \text{ terms with } X_5, X_6$$

also

$$\Omega = \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3 - 2\theta^2 \wedge \theta^6 + 2\theta^3 \wedge \theta^5 - 2\theta^5 \wedge \theta^6$$

Υ and Ω make reduction of $GL(6,\mathbb{R})$ to $C(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$

and $\Upsilon_{ijk}, \Omega_{ij}, \theta^0, \theta^i, \Gamma^i{}_j, \theta_i, \theta_0$ satisfy system (S)

Moreover by setting $X_5 \to 0$, $X_6 \to 0$ we recover the G_2 case.

The Cartan connection matrix now looks like:

$$
\mathfrak{so}(4,1)\ \omega_{curv} = \begin{bmatrix}
3\Theta^7+\Theta^{10} & -\Theta^8 & 2(2\Theta^{19}-\Theta^{16}) & \Theta^{18} & -\Theta^{14} & \Theta^{20} & 0 \\
-\Theta^9 & 3\Theta^7-\Theta^{10} & \Theta^{17} & \Theta^{16}-\Theta^{19} & \frac{1}{2}(\Theta^{18}-\Theta^{15}) & 0 & -\Theta^{20} \\
\frac{1}{2}(\Theta^3+\Theta^6) & \Theta^4 & 2\Theta^{13} & \Theta^{11} & 0 & \frac{1}{2}(\Theta^{15}-\Theta^{18}) & \Theta^{14} \\
\Theta^2+\Theta^5 & 2\Theta^3+\Theta^6 & 2\Theta^{12} & 0 & -\Theta^{11} & \Theta^{15}-\Theta^{16} & -\Theta^{18} \\
-\Theta^1 & -2\Theta^2 & 0 & -2\Theta^{12} & \Theta^{13} & -\Theta^{17} & 2(\Theta^{16}-2\Theta^{19}) \\
\Theta^0 & 0 & 2\Theta^2 & -2\Theta^3-\Theta^6 & -\Theta^4 & 3\Theta^7+\Theta^{10} & \Theta^8 \\
0 & \Theta^0 & \Theta^1 & -\Theta^2-\Theta^5 & -\frac{1}{2}(\Theta^3+\Theta^6) & \Theta^9 & -3\Theta^7-\Theta^{10}
\end{bmatrix} \quad \dot{z}_2
$$

Passage to ω_{y_2}:

$\Theta^5 = 0$, $\Theta^6 = 0$, $\Theta^7 = \theta^5$, $\Theta^8 = 2\theta^6$, $\Theta^9 = \frac{1}{2}\theta^7$

$\Theta^{10} = \theta^8$, $\Theta^{11} = \theta^6$, $\Theta^{12} = \theta^7$, $\Theta^{13} = \theta^8$, $\Theta^{14} = -6\theta^2$, $\Theta^{15} = 6\theta^{11}$

$\Theta^{16} = -6\theta^{10}$, $\Theta^{17} = 6\theta^9$, $\Theta^{18} = 4\theta^{11}$, $\Theta^{19} = -4\theta^{10}$, $\Theta^{20} = 6\theta^{13}$.