Rigidity of Schubert varieties in compact Hermitian symmetric spaces

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(joint work with Colleen Robles)

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This talk is about a problem from topology / algebraic geometry originating in the 1960's.

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Since 1997, substantial progress has been made using differential geometry, representation theory, and Lie algebra cohomology.

(Irreducible) CHSS

$X = G/P_{\mathtt{i}}$: $\mathfrak{g}$ is $\mathbb{C}$-simple and 1-graded ($\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$).

Classical	Grassmannians	$A_n/P_{\mathtt{i}} = \mathrm{SL}_{n+1}\mathbb{C}/P_{\mathtt{i}}$
	Odd-dim. quadrics	$B_n/P_1 = \mathrm{SO}_{2n+1}\mathbb{C}/P_1$
	Even-dim. quadrics	$D_n/P_1 = \mathrm{SO}_{2n}\mathbb{C}/P_1$
	Lagrangian Grassmannians	$C_n/P_n = \mathrm{Sp}_{2n}\mathbb{C}/P_n$
	Spinor varieties	$D_n/P_n = \mathrm{SO}_{2n}\mathbb{C}/P_n$
Exceptional	Cayley plane	E_6/P_6
	Freudenthal variety	E_7/P_7

Take minimal embedding $X \hookrightarrow \mathbb{P}V_{\omega_{\mathtt{i}}}$ to realize X as a projective algebraic variety.

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Smoothability and Schur rigidity

Theorem (Kostant)

$H_*(G/P, \mathbb{Z})$ is generated by Schubert varieties X_w, $w \in W^{\mathfrak{p}}$.

FACT: X_w are irreducible and most X_w are singular!

Question (Borel–Haefliger 1961):

Can $[X_w]$ be represented by a smooth complex variety?

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Definition

X_w is *Schur rigid* if for any irred. subvar. $Y \subset X$ with $[Y] = r[X_w]$ $(\exists r \in \mathbb{Z})$, have $Y = g \cdot X_w$ $(\exists g \in G)$. Otw, it's *Schur flexible*.

(X_w is singular and Schur rigid $\Rightarrow$ X_w is *not* smoothable.)

CHSS case: $\exists$ diff. geo. approach to studying Schur rigidity!

Examples of Schubert varieties

Example (Smooth and singular X_w in $X = G/P_{\mathtt{i}}$ CHSS)

- Smooth:
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  - sub-Grassmannian, sub-quadric, sub-LG, sub-spinor variety.

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  - sub-Grassmannian, sub-quadric, sub-LG, sub-spinor variety.
- Singular:
  - In $\mathrm{Gr}(2,4)$, $X_w = \{E \in \mathrm{Gr}(2,4) : E \cap \mathbb{C}^2 \neq 0\}$ has dim 3 and is smoothable.
  - In $\mathrm{Gr}(3,6)$, $X_w = \{E \in \mathrm{Gr}(3,6) : E \cap \mathbb{C}^2 \neq 0\}$ has dim 7 and is *not* smoothable. (Hartshorne, Rees, Thomas - 1974; used topological techniques of Thom)

Walters (1997) and Bryant (2001)

Theorem

Let X be CHSS. Then $[Y] = r[X_w]$ iff Y is an integral variety of a differential system $\mathcal{R}_w \subset \mathrm{Gr}(|w|, TX)$, where $|w| = \dim_{\mathbb{C}} X_w$.

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Proof sketch.

Use dual basis $\{\phi_v\}$ in cohom: $[Y] = r[X_w]$ iff $\forall v \neq w$, $|v| = |w|$,

$$\int_Y \phi_v = 0 \quad \overset{(*)}{\Longleftrightarrow} \quad \phi_v|_Y = 0. \qquad (*) : \left\{ \begin{array}{l} \text{Kostant (1963):} \\ \phi_v \text{ are "positive".} \\ \text{(CHSS used here.)} \end{array} \right.$$

$$\mathcal{R}_w|_x := \{E \in \mathrm{Gr}(|w|, T_x X) : \phi_v|_E = 0,\ \forall v \neq w,\ |v| = |w|\}. \qquad \square$$

Walters and Bryant:

1. smooth X_w in $\mathrm{Gr}(m, n)$ and $\mathrm{LG}(n, 2n)$
2. maximal linear spaces in classical CHSS
3. singular X_w of low codim. in $\mathrm{Gr}(m, n)$

Hong (2005, 2007)

Theorem (Smooth case, 2007)

Let X be CHSS, other than B_n/P_1. Any smooth $X_w \subset X$ is Schur rigid except when $X_w \subset X$ is a non-maximal $\mathbb{P}^k$ or $\mathbb{P}^1 \subset C_n/P_n$.

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Key obstruction:

Kostant's theorem does not apply in the singular cases!

Summary of new results (2011)

In all CHSS:

1. Key technical lemma: New encoding of Schubert varieties.
2. Uniform approach to smooth and singular cases: *If X_w satisfies* H_+ *(a "first-order obstruction"), then it is Schur rigid.*
3. Cohomology: Analogous to Kostant, defined an *algebraic Laplacian*; characterized vanishing of cohomology in terms of representation theory.

New description of Schubert varieties in CHSS

$w \in W^{\mathfrak{p}} \longleftrightarrow \mathfrak{n}_w \subset \mathfrak{g}_{-1}$. Stabilizer in $\mathfrak{g}_0$ of $\mathfrak{n}_w$ is parabolic $\rightsquigarrow$ submarking $\mathtt{J} \subset \delta_{\mathfrak{g}} \backslash \{\mathtt{i}\}$. Let $Z_{\mathtt{J}} = \sum_{\mathtt{j} \in \mathtt{J}} Z_{\mathtt{j}}$.

Lemma (Main technical lemma for $X = G/P_{\mathtt{i}}$ CHSS)

X_w is encoded by $(\mathsf{a}, \mathtt{J})$, where $\mathsf{a} \in \mathbb{Z}_{\geq 0}$, $\mathtt{J} \subset \delta_{\mathfrak{g}} \backslash \{\mathtt{i}\}$, and

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Moreover, X_w is smooth iff $\mathsf{a} = 0$.

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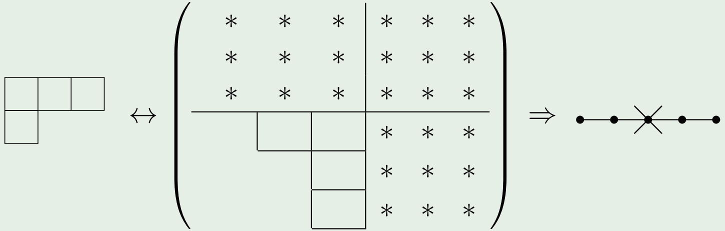
Example (X_w in $\mathrm{Gr}(3,6)$)



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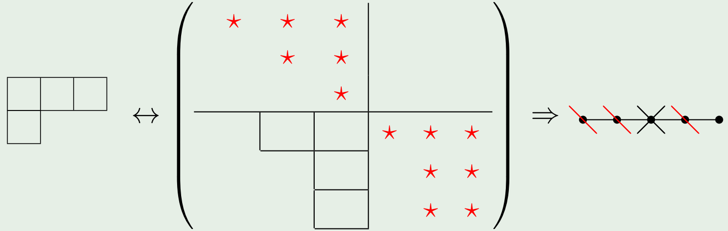

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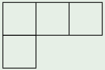

$$\leftrightarrow \left(\begin{array}{ccc|ccc} \star & \star & \star & & & \\ & \star & \star & & & \\ & & \star & & & \\ \hline -2 & -1 & 0 & \star & \star & \star \\ -3 & -2 & -1 & & \star & \star \\ -3 & -2 & -1 & & \star & \star \end{array}\right) \leftrightarrow$$

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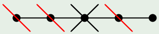

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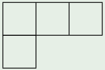

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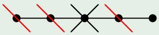


(singular)

Rigidity result for $\mathrm{LG}(n, 2n)$

Theorem

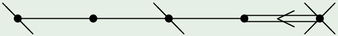
Let $X_w \subset \mathrm{LG}(n, 2n)$. *Let* $\mathtt{S} = \mathtt{J} \cup \{n\} \subset \delta_{C_n}$. *If*

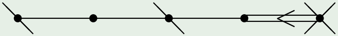
$$|\mathtt{J}| = \mathsf{a} : \quad \mathtt{S}^* \perp \qquad \textit{or} \qquad |\mathtt{J}| = \mathsf{a}+1 : \quad \mathtt{S} \perp$$

then X_w *is Schur rigid.*

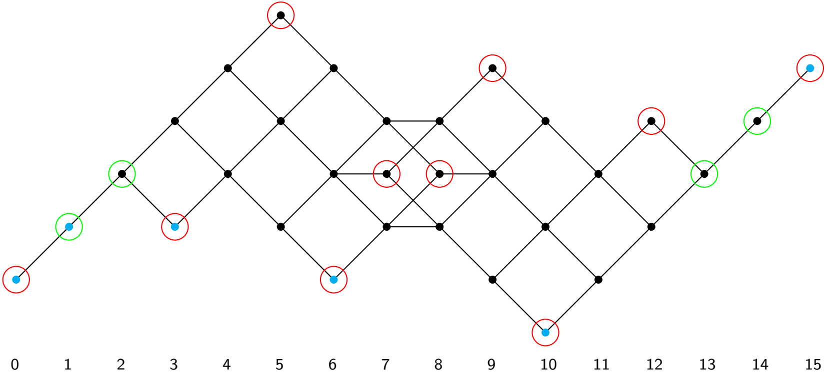
Note: $\mathtt{S}^* =$ do a Dynkin diagram automorphism of $\mathtt{S} \backslash \{n\}$.

Example (Singular X_w in $\mathrm{LG}(5, 10)$)

$\mathsf{a} = 1$,  : Schur rigid, so non-smoothable

$\mathsf{a} = 2$,  : inconclusive

Schubert varieties in $X^{15} = \mathrm{LG}(5,10)$



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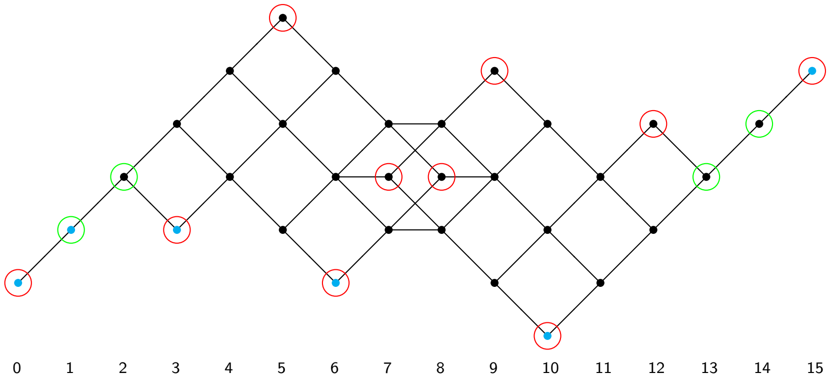

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A posteriori: Our Schur rigid list is invariant under Poincare duality!

Ideas in the analysis - 1: Schubert rigidity

Let B_w be the P-orbit of $\mathfrak{n}_w$. Define $\mathcal{B}_w|_{gP} = g_* B_w$. Always have

$$\mathcal{B}_w \subset \mathcal{R}_w.$$

Definition

X_w is Schubert rigid *if every irreducible integral variety Y of $\mathcal{B}_w$ is of the form $Y = g \cdot X_w$ for some $g \in G$.*

Theorem (Bryant, Hong)

X_w is Schur rigid iff X_w is Schubert rigid and $B_w = R_w$.

$B_w = R_w$ is assessed by comparing tangent spaces.

How to study Schubert rigidity?

Ideas in the analysis - 2: Lift to a frame bundle

IDEA: Lift $\mathcal{B}_w$ to a frame bundle $\mathcal{G} \to X$.

- $o \in X \longleftrightarrow$ h.w. line $[\mathrm{v}_0] \in \mathbb{P}V$.
- Distinguished flag: $\mathbb{C}\mathrm{v}_0 \subset \mathbb{C}\mathrm{v}_0 \oplus \mathfrak{n}_w \cdot \mathrm{v}_0 \subset \mathbb{C}\mathrm{v}_0 \oplus \mathfrak{g}_{-1} \cdot \mathrm{v}_0$.
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ϑ : MC form on $\mathcal{G}$. Let $\mathfrak{n}_w^\perp \subset \mathfrak{g}_{-1}$ compl. root spaces to $\mathfrak{n}_w$.
Schubert system $\tilde{\mathcal{B}}_w$ on $\mathcal{G}$:

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Proposition

Int. mflds of $\tilde{\mathcal{B}}_w$ are adapted frame bdles $\mathcal{F}$ over int. mflds of $\mathcal{B}_w$.

$$\vartheta_{\mathfrak{n}_w^\perp} = 0 \quad \stackrel{\text{MC eqn}}{\Rightarrow} \quad \vartheta_{0,-} = \lambda(\vartheta_{\mathfrak{n}_w}), \quad \text{where} \quad \lambda : \mathcal{F} \to \mathfrak{g}_{0,-} \otimes \mathfrak{n}_w^*.$$

In fact, $\lambda : \mathcal{F} \to \frac{\ker \delta^1}{\mathrm{im}\delta^0} = \frac{\text{torsion constraints}}{\text{frame normalizations}}$. $\boxed{\text{Cohomology!}}$

Ideas in the analysis - 3: Lie algebra cohomology

1. G_w = sym. group of $wX_w = \overline{N_w \cdot o}$ and $\mathfrak{g}_w$ its Lie algebra.
2. $\mathfrak{g}_w^\perp$ = complementary root spaces to $\mathfrak{g}_w$ in $\mathfrak{g}$.

Cohomology group of interest:

$$H^1 = H^1(\mathfrak{n}_w, \mathfrak{g}_w^\perp).$$

FACT: $\mathfrak{g}_w^\perp$ is an $\mathfrak{n}_w$–module: $u \cdot z = [u, z]_{\mathfrak{g}_w^\perp}$. $\therefore$ $\boxed{\text{Lie alg cohom!}}$

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Grade H^1 using $Z_{\mathtt{i}}$ and $Z_{\mathtt{J}}$.

$\boxed{\mathrm{H}_+ \text{ condition: } H^1_{1,\mathsf{a}-1} = 0 \quad \& \quad H^1_{2,2\mathsf{a}-1} = 0.}$

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Key advance: Define Laplacian $\Box$ ($\mathfrak{g}_{0,0}$–module map), $\ker \Box = $ *cohomology*, do spectral analysis. Characterize H_+ in terms of rep.theory.

What's next?

- Invariance of Schur rigidity under duality is still mysterious.
- Does Schubert rigidity imply Schur rigidity? i.e. is checking $B_w = R_w$ necessary? (No known counterexamples!)
- H_+ is a first-order obstruction to *Schur flexibility*. If H_+ fails, our result is inconclusive. Analyze the EDS further to uncover higher order obstructions.
- If X_w is Schur flexible, classify the representatives of $[X_w]$ (modulo G).