The gap phenomenon in parabolic geometries

Dennis The

Mathematical Sciences Institute
Australian National University

August 2013

(Joint work with Boris Kruglikov)

Differential Geometry and its Applications

The gap problem

Q: **Among (reg./nor.) parabolic geometries of type (G, P), what is the gap between maximal and submaximal (infinitesimal) symmetry dimensions?**

Note: Maximum $= \dim(G)$ (flat model G/P).

Motivation:

Example (Riemannian geometry)

- Fubini (1903): $\binom{n+1}{2}$-1 is not possible.

n	max	submax	Citation
2	3	1	Darboux / Koenigs ($\sim$1890)
3	6	4	Wang (1947)
4	10	8	Egorov (1955)
≥ 5	$\binom{n+1}{2}$	$\binom{n}{2}+1$	Wang (1947), Egorov (1949)

Sharp gap results for parabolic geometries

1. ≤ 2012:
   - (i) 2-d projective & scalar 2nd order ODE (Tresse, 1896)
   - (ii) $(2, 3, 5)$-distributions (Cartan, 1910)
   - (iii) n-dim projective (Egorov, 1951)
   - (iv) scalar 3rd order ODE (Wafo Soh et al., 2002)
   - (v) pairs of 2nd order ODE (Casey et al., 2012)
2. 2013:
   - (i) any parabolic geometry modelled on complex or split-real G/P + non-Riem./Lor. conformal (Kruglikov & T.)
     Note: We make no additional assumptions such as transitivity, or curvature type being locally constant, etc.
   - (ii) Riem./Lor. conformal (Doubrov & T.)
   - (iii) Metric projective & metric affine (Kruglikov & Matveev)

Outline of the talk

1. Background
2. Formulation of results
3. Outlines of some proofs

Background

Example: Conformal geometry

Flat model:

- conf. sphere $\mathbb{S}^{p,q} = SO_{p+1,q+1}/P_1$ = null lines in $\mathbb{R}^{p+1,q+1}$
- sym. dim $= \binom{n+2}{2}$, where $n = p + q$.

Submax models:

Case	Submax	Model
$p, q \geq 2$	$\binom{n-1}{2} + 6$	$y^2 dw^2 + dwdx + dydz + g_{flat}^{p-2,q-2}$
Lorentzian (†)	$\binom{n-1}{2} + 4$	$y^2 dw^2 + dwdx + dy^2 + dz^2 + g_{flat}^{0,q}$
Riemannian (†)	$\binom{n-1}{2} + 3$	$\mathbb{S}^{n-2} \times \mathbb{S}^2$ $(5 \leq n \neq 6)$
	$\frac{n^2}{4} + n$	$\mathbb{CP}^{n/2}$ $(n = 4, 6, 8)$

(†) : Doubrov–T.

Example: $(2,3,5)$-distributions

$(2,3,5)$-distributions $\rightsquigarrow$ G_2/P_1 geometries (Cartan, 1910).

Goursat: Locally, D is spanned by

$$\partial_x + p\partial_y + q\partial_p + f\partial_z, \qquad \partial_q.$$

This is $(2,3,5)$ iff $f = f(x,y,p,q,z)$ satisfies $f_{qq} \neq 0$.

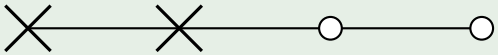
- Flat model: $f = q^2$ has 14-dim sym (Hilbert–Cartan eqn);
- Submax sym model: $f = q^m$ has 7-dim sym when $m \notin \{-1, 0, \frac{1}{3}, \frac{2}{3}, 1, 2\}$.

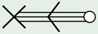
Parabolic subalgebras and gradings

$\mathfrak{g}$: semisimple Lie algebra

$(\mathfrak{g}, \mathfrak{p}) \rightsquigarrow \mathbb{Z}$-grading: $\quad \mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}.$

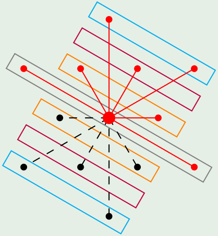
Example ($A_4/P_{1,2}$ and G_2/P_1)





$$A_4 = \mathfrak{sl}_5 = \begin{pmatrix} 0 & 1 & 2 & 2 & 2 \\ -1 & 0 & 1 & 1 & 1 \\ -2 & -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$G_2 =$



Curved versions:

System of three 2nd order ODE $\qquad (2,3,5)$-distribution

For (reg./nor.) G/P geometries $(\mathcal{G} \to M, \omega)$, have harmonic curvature κ_H, valued in $H^2_+(\mathfrak{g}_-, \mathfrak{g})$.

$$\boxed{(\mathcal{G} \to M, \omega) \text{ is locally flat iff } \kappa_H = 0}.$$

Examples (Harmonic curvature)

- conformal geometry: Weyl ($n \geq 4$) or Cotton ($n = 3$);
- $(2,3,5)$-distributions: binary quartic.

The (locally) flat model is the *unique* max. sym. model. $\therefore$ Want:

$$\boxed{\mathfrak{S} := \max\{\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \mid \kappa_H \not\equiv 0\}}.$$

Kostant's Bott–Borel–Weil theorem

Kostant (1961), Baston–Eastwood (1989): Dynkin diagram algorithm to calculate $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ as a $\mathfrak{g}_0$-module.

Example ((2, 3, 5)-distributions: G_2/P_1 geometry)

As a $\mathfrak{g}_0 = \mathfrak{gl}_2(\mathbb{R})$ module,

$$H^2_+(\mathfrak{g}_-, \mathfrak{g}) = \overset{-8}{\times}\!\Lleftarrow\!\overset{4}{\circ} = \bigodot^4 (\mathbb{R}^2)^*,$$

i.e. binary quartic, c.f. Cartan (1910) via method of equivalence.

Tanaka prolongation

Given $(\mathfrak{g}, \mathfrak{p})$, we have $\mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}$. Let $\mathfrak{a}_0 \subset \mathfrak{g}_0$.

Tanaka prolongation of $\mathfrak{a}_0$ in $\mathfrak{g}$:

$$\mathrm{pr}_{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{a}_0) = \mathfrak{g}_- \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \ldots$$

by

$$\begin{aligned} \mathfrak{a}_1 &= \{X \in \mathfrak{g}_1 \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_0\}, \\ \mathfrak{a}_2 &= \{X \in \mathfrak{g}_2 \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_1\}, \\ &\vdots \end{aligned}$$

Given $0 \neq \phi \in H^2_+$, interested in $\mathfrak{a}_0 = \mathfrak{ann}(\phi)$. Let

$$\boxed{\mathfrak{a}^\phi := \mathrm{pr}_{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{ann}(\phi)).}$$

Formulation of results

New results (2013)

Fix (G, P). Among regular, normal G/P geometries $(\mathcal{G} \to M, \omega)$,

$$\mathfrak{S} := \max\{\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \mid \kappa_H \not\equiv 0\}$$
$$\mathfrak{U} := \max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})\}$$

Theorem (Universal upper bound)

$\mathfrak{S} \leq \mathfrak{U} < \dim(\mathfrak{g})$.

Theorem (Local realizability)

If G/P is complex or split-real, then $\mathfrak{S} = \mathfrak{U}$ almost always. Complete exception list when G is simple: A_2/P_1, $A_2/P_{1,2}$, B_2/P_1.

Theorem

If G/P is complex or split-real, can read $\mathfrak{U}$ from a Dynkin diagram!

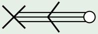
Dynkin diagram recipes - 1

1. From $\mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}$, we have $\mathfrak{g}_0 = \mathscr{Z}(\mathfrak{g}_0) \oplus (\mathfrak{g}_0)_{ss}$ with

$$\boxed{\begin{cases} \dim(\mathscr{Z}(\mathfrak{g}_0)) = \# \text{ crosses;} \\ (\mathfrak{g}_0)_{ss} \text{ D.D.} \to \text{remove crosses.} \end{cases}}$$

Since $\dim(\mathfrak{g}_-) = \dim(\mathfrak{g}_+)$, get $n = \dim(\mathfrak{g}/\mathfrak{p})$ and $\dim(\mathfrak{p})$.

Example (G_2/P_1)

 , $\dim(\mathfrak{g}_0) = 4, \quad n = 5.$

Dynkin diagram recipes - 2

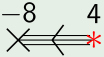
Let $\mathbb{V} \subset H^2_+$ be a $\mathfrak{g}_0$-irrep.

2. $\dim(\mathfrak{ann}(\phi))$ $(0 \neq \phi \in \mathbb{V})$ is max. on l.w. vector $\phi = \phi_0 \in \mathbb{V}$, $\mathfrak{q} := \{X \in (\mathfrak{g}_0)_{ss} \mid X \cdot \phi_0 = \lambda \phi_0\}$ is parabolic, and

$$\boxed{\dim(\mathfrak{ann}(\phi_0)) = (\#\mathit{crosses}) - 1 + \dim(\mathfrak{q})}.$$

D.D. Notation: If $\neq 0$ on uncrossed node, put $*$.

Example (G_2/P_1)

$H^2_+ =$ 

, $\dim(\mathfrak{ann}(\phi_0)) = 2$.

Dynkin diagram recipes - 3

Let $\mathbb{V} \subset H^2_+$ be a $\mathfrak{g}_0$-irrep.

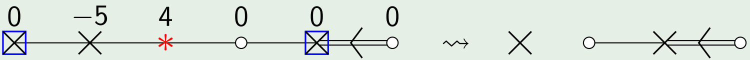
Lemma

$\dim(\mathfrak{a}^{\phi}_+)$ $(0 \neq \phi \in \mathbb{V})$ *is max. on l.w. vector* $\phi = \phi_0 \in \mathbb{V}$.

D.D. Notation: If 0 over $\times$ $\rightsquigarrow$ put $\square$.

3. Remove all $*$ and $\times$, except $\square$ (also remove adj. edges). Then remove connected components w/o $\square$. Obtain $(\bar{\mathfrak{g}}, \bar{\mathfrak{p}})$.

Example



Proposition

No $\square \Leftrightarrow \dim(\mathfrak{a}^{\phi_0}_+) = 0$. *Otw,* $\dim(\mathfrak{a}^{\phi_0}_+) = \dim(\bar{\mathfrak{g}}/\bar{\mathfrak{p}})$.

Example

G/P	H^2_+ components	n	$\dim(\mathfrak{a}_0^{\phi_0})$	$\dim(\mathfrak{a}_+^{\phi_0})$	$\dim(\mathfrak{a}^{\phi_0})$
G_2/P_1	$-8 \quad 4$	5	2	0	7
$A_4/P_{1,2}$	$0 \quad -4 \quad 3 \quad 1$	7	6	1	14
	$-4 \quad 1 \quad 1 \quad 1$	7	6	0	13
E_8/P_8	$0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad -4$; 0	57	90	0	147

Proposition (Maximal parabolics)

Single cross $\Rightarrow$ no $\square$, so $\mathfrak{a}_+^{\phi_0} = 0$.

All complex $(\mathfrak{g}, \mathfrak{p})$ with $\mathfrak{a}_+^{\phi_0} \neq 0$ have been classified. ($\mathfrak{g}$ simple)

Outline of some proofs

Upper bound ($\mathfrak{S} \leq \mathfrak{U}$) - proof outline

Čap–Neusser (2009):

- Fix any $u \in \mathcal{G}$. Then $\omega_u : \mathfrak{inf}(\mathcal{G}, \omega) \hookrightarrow \mathfrak{g}$ (linearly).
- Bracket on $\mathfrak{f} = \mathrm{im}(\omega_u)$ is $[X, Y]_{\mathfrak{f}} := [X, Y]_{\mathfrak{g}} - \kappa_u(X, Y)$.
- Regularity: $\mathfrak{f}$ is filtered, so $\mathfrak{s} = gr(\mathfrak{f}) \subset \mathfrak{g}$ is a graded subalg.
- $\mathfrak{s}_0 \subset \mathfrak{ann}(\kappa_H(u))$.

$(*)$: $\boxed{[\mathfrak{s}_{i+1}, \mathfrak{g}_{-1}] \subset \mathfrak{s}_i}$ $(i \geq -1)$ $\quad \Rightarrow \quad$ $\mathfrak{s} \subset \mathrm{pr}_{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{s}_0) \subset \mathfrak{a}^{\kappa_H(u)}$,
so $\dim(\mathfrak{s}) \leq \mathfrak{U}$ when $\kappa_H(u) \neq 0$.

BUT: "Tanaka property" $(*)$ isn't always true!

Definition

$x \in M$ is a regular point iff $\forall i$, $\dim(\mathfrak{s}_i)$ is loc. constant near x.

Proof outline:

(1) Prop: At regular points, $(*)$ is true.
(2) Lemma: The set of regular points is open and dense in M.
(3) Any nbd of a non-flat point contains a non-flat regular pt. $\square$

Define $\mathfrak{f} = \mathfrak{a} := \mathfrak{a}^{\phi_0}$ as *vector spaces*, but with deformed bracket

$$[X, Y]_{\mathfrak{f}} := [X, Y]_{\mathfrak{a}} - \phi_0(X, Y).$$

(Kostant $\rightsquigarrow$ *explicit* l.w. $\phi_0 \in \mathbb{V} \subset H^2 \cong \ker(\Box) \subset \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$.)

Q: Is this even a Lie algebra? Want $\phi_0 \in \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{a}$. "Output" of ϕ_0 is in the $w(-\lambda)$ root space, $w \in W^{\mathfrak{p}}(2)$, $\lambda =$ h.w. of $\mathfrak{g}$.

Proposition

If $w(-\lambda) \in \Delta^-$ above, then $\mathfrak{f}$ is a filtered Lie algebra.

Lemma

If $\mathfrak{g}$ is simple, $w \in W^{\mathfrak{p}}(2)$, and $w(-\lambda) \in \Delta^+$, then $\mathrm{rank}(\mathfrak{g}) = 2$.

Non-exceptions: $\mathfrak{f}/\mathfrak{f}^0 \rightsquigarrow$ non-flat model, $\dim(\mathfrak{f}) = \mathfrak{U}$, so $\mathfrak{S} = \mathfrak{U}$.

Have algorithm for constructing an explicit submax. sym. model.

Conclusion

Summary: Gave a soln to the gap problem for complex or split-real G/P geometries.

Open questions:

- Non-split-real cases, e.g. CR geometry?
- Classification of *all* submaximally symmetric models?
- Non-parabolic geometries, e.g. Higher order ODE (systems)? Kähler geometry?
- Dim of submax space of solns of almost-Einstein scales, Killing tensors, etc. (more generally, of BGG operators)?

Appendix: At regular points, the Tanaka property holds

Let $u \in \pi^{-1}(x)$, $\widetilde{\mathcal{S}} := \mathfrak{inf}(\mathcal{G}, \omega)$, $\widetilde{\mathcal{S}}^j := \{\xi \in \widetilde{\mathcal{S}} \mid \omega_u(\xi) \in \mathfrak{g}^j\}$, $\mathfrak{f}^j := \omega_u(\widetilde{\mathcal{S}}^j)$.

WTS: $\boxed{[\mathfrak{f}^{i+1}, \mathfrak{g}^{-1}] \subset \mathfrak{f}^i + \mathfrak{g}^{i+1}, \quad \forall i \geq -1}$.

1. Have tower $\mathcal{G} = \mathcal{G}_\nu \to \ldots \to \mathcal{G}_0 \to M$ with $\mathcal{G}_i = \mathcal{G}/P_+^{i+1} \xrightarrow{\pi_i} M$. Then $\widetilde{\mathcal{S}}$ projects to $\mathcal{S}^{(i)} \subset \mathfrak{X}(\mathcal{G}_i)^{P/P_+^{i+1}}$.
   - x regular pt $\Rightarrow \mathcal{S}^{(i)}$ is constant rank (+ involutive). By Frobenius, $\exists$ fcns $\{F_j\}$ on $\mathcal{G}_i$; level sets foliate by int. submflds of $\mathcal{S}^{(i)}$. Thus, $\xi^{(i)} \cdot F_j = 0$, $\forall \xi \in \widetilde{\mathcal{S}}^{i+1}$.
   - If $\xi \in \widetilde{\mathcal{S}}^{i+1}$ and $\eta \in \Gamma(T\mathcal{G})^P$, then $\forall u_i \in \pi_i^{-1}(x)$, $\xi^{(i)}_{u_i} = 0$ and

$$[\xi^{(i)}, \eta^{(i)}]_{u_i} \cdot F_j = 0 \quad \Rightarrow \quad [\xi, \eta]_u = \xi'_u + \chi_u \qquad (*)$$

   where $\xi' \in \widetilde{\mathcal{S}}$ and $\chi_u \in T_u^{i+1}\mathcal{G}$.
2. Let $X = \omega_u(\xi) \in \mathfrak{f}^{i+1}$ and $Y = \omega_u(\eta) \in \mathfrak{g}^{-1}$. Since $[X, Y] = -\omega_u([\xi, \eta]) \in \mathfrak{g}^i$, then $[X, Y] \in \mathfrak{f}^i + \mathfrak{g}^{i+1}$ by $(*)$. □

Appendix 2: Tanaka prolongation

Let $\mathbb{V}$ be a $\mathfrak{g}_0$-irrep.

Lemma (Complex case)

$\dim(\mathfrak{a}_+^\phi)$ $(0 \neq \phi \in \mathbb{V})$ *is max. when* $\phi = \phi_0 \in \mathbb{V}$ *is a l.w. vector.*

Proof.

If $\mathfrak{g}_0^{ss} = 0$, then $\mathbb{V} = \mathbb{C}$ $\therefore$ trivial. So suppose $\mathfrak{g}_0^{ss} \neq 0$.

1. $\mathfrak{a}_k^\phi = pr_k(\mathfrak{g}_-, \mathfrak{ann}(\phi)) = \{X \in \mathfrak{g}_k : \mathrm{ad}_{\mathfrak{g}_{-1}}^k(X) \cdot \phi = 0\}$.
2. If $M(\phi)$ depends linearly on ϕ, then $\mathrm{rank}(M(\phi))$ is a lower semi-cts function.
3. $\phi \mapsto \dim(\mathfrak{a}_k^\phi)$ is upper semi-cts; it descends to $\mathbb{P}(\mathbb{V})$.
4. $\exists!$ closed G_0-orbit in $\mathbb{P}(\mathbb{V})$. $\therefore$ $[\phi_0] \in$ closure of *every* G_0-orbit.
5. Given $0 \neq \phi \in \mathbb{V}$, $\exists$ seq. $\{g_n\}$ in G_0 s.t. $g_n \cdot [\phi] \to [\phi_0]$.

By upper semi-continuity, $\dim(\mathfrak{a}_k^\phi) = \dim(\mathfrak{a}_k^{g_n \cdot \phi}) \leq \dim(\mathfrak{a}_k^{\phi_0})$. $\square$