

HESSIAN MEASURES I

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Dedicated to Olga Ladyzhenskaya

ABSTRACT: The Hessian partial differential operators, determined by the elementary symmetric functions, are extended as weakly continuous mappings from corresponding cones of continuous functions to Borel measures. The corresponding Dirichlet problem is shown to be uniquely solvable under appropriate conditions, thereby extending previous results for integrable inhomogeneous terms as well as the theory of generalized solutions of Monge-Ampère equations. We also consider extensions as signed measures on larger classes of functions and the application to Hessian integrals.

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§1. Introduction

Let Ω be a domain in Euclidean n -space, \mathbf{R}^n . For $k = 1, 2, \dots, n$ and $u \in C^2(\Omega)$ the k -Hessian operator, F_k , is defined by

$$F_k[u] = S_k(\lambda(D^2u)), \quad (1.1)$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ denotes the eigenvalues of the Hessian matrix of second derivatives, D^2u , and S_k is the k^{th} elementary symmetric function on \mathbf{R}^n , given by

$$S_k(\lambda) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \quad (1.2)$$

Alternatively we may write

$$F_k[u] = [D^2u]_k, \quad (1.3)$$

where $[\mathcal{A}]_k$ denotes the sum of the $k \times k$ principal minors of an $n \times n$ matrix \mathcal{A} . Our purpose in this paper is to extend the definition of the F_k to corresponding classes of continuous functions so that $F_k[u]$ is a Borel measure and to consider the Dirichlet problem in this setting. A function $u \in C^2(\Omega)$ is called *k-convex* (*uniformly k-convex*) in Ω if $F_j[u] \geq 0$ (> 0) for $j = 1, \dots, k$. The operator F_k is degenerate elliptic (elliptic) with respect to k -convex (uniformly k -convex) functions. When $k = 1$, we have $F_1[u] = \Delta u$ and 1-convex functions are subharmonic. When $k = n$, $F_k[u] = \det D^2u$, the Monge-Ampère operator, and n -convex functions are convex. To extend these notions to continuous functions, we call a function $u \in C^0(\Omega)$, *k-convex*, if there exists a sequence $\{u_m\} \subset C^2(\Omega)$ such that in any subdomain $\Omega' \subset\subset \Omega$, u_m is k -convex for sufficiently large m and converges uniformly to u . It is easily seen that $u \in C^0(\Omega)$ is k -convex if and only if $F_k[u] \geq 0$ in the viscosity sense [11],[16], that is, whenever there exists a point $y \in \Omega$ and function $v \in C^2(\Omega)$ satisfying $u(y) = v(y)$, $u \leq v$ in Ω , we must have $F_k[v](y) \geq 0$. As above a function $u \in C^0(\Omega)$ is 1-convex if and only if it is subharmonic and n -convex if and only if it is convex. In each of these cases, it is well known that the operator F_k can be defined as a Borel measure μ_k . For $k = 1$, μ_1 is the positive distribution given by

$$\mu_1(\varphi) = \int_{\Omega} u \Delta \varphi \quad (1.4)$$

for $\varphi \in C_0^\infty(\Omega)$, while for $k = n$,

$$\mu_n(e) = |\chi_u(e)| \quad (1.5)$$

for any Borel set $e \subset \Omega$, where χ_u is the normal (subgradient) mapping of the convex function u [1], [4]. Let $\Phi^k(\Omega)$ denote the class of k -convex functions. In this paper we shall prove that $F_k[u]$ may be extended to $\Phi^k(\Omega)$ as a Borel measure μ_k , for all $k = 1, \dots, n$, and that the corresponding mapping $u \rightarrow \mu_k[u]$ is weakly continuous on $C^0(\Omega)$. The resultant measure $\mu_k[u]$ will be called the k -Hessian measure generated by u .

Theorem 1.1. *For any $u \in \Phi^k(\Omega)$, there exists a Borel measure $\mu_k[u]$ such that*

(i)

$$\mu_k[u](e) = \int_e F_k[u]$$

for any Borel set $e \subset \Omega$, if $u \in C^2(\Omega)$, and

(ii) if $u_m \rightarrow u$ locally uniformly in Ω , then the corresponding measures $\mu_k[u_m] \rightarrow \mu_k[u]$ weakly, that is,

$$\int_\Omega g d\mu_k[u_m] \rightarrow \int_\Omega g d\mu_k[u], \quad (1.6)$$

for all $g \in C^0(\Omega)$ with compact support.

Theorem 1.1 is proved in Section 2 of this paper as a consequence of various integral inequalities for the operators F_k .

In Section 3 we consider the corresponding Dirichlet problem,

$$\begin{aligned} \mu_k[u] &= \nu && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega, \end{aligned} \quad (1.7)$$

in the class of k -convex functions. Under the hypotheses that the domain Ω is uniformly $(k-1)$ -convex, that is, $\partial\Omega \in C^2$ and $H_j[\partial\Omega] > 0$, $j = 1, \dots, k-1$, where $H_j[\partial\Omega]$ denotes the j -mean curvature of the boundary $\partial\Omega$ (see [17] [18]), and that the Borel measure ν can be decomposed as a sum,

$$\nu = \nu_1 + \nu_2, \quad (1.8)$$

where $\nu_1 \in L^1(\Omega)$ and ν_2 has compact support in Ω , we prove the following existence and uniqueness theorem.

Theorem 1.2. *For any $\varphi \in C^0(\bar{\Omega})$, there exists a unique $u \in \Phi^k(\Omega) \cap C^0(\bar{\Omega})$ satisfying (1.7), provided $k > n/2$.*

Theorem 1.2 extends the case, $p = 1$, in [20], where an equivalent formulation of the Dirichlet problem (1.7) is treated for inhomogeneous terms in L^p spaces.

In Section 4, we consider the extension of the measures μ_k as signed measures on more general classes of functions including semi-convex functions (as in [10]) and admissible functions, for which the operators F_k are degenerate elliptic. Finally, in Section 5, we apply Theorem 1.1 to extend Hessian integrals, (as defined in [7], [19], [24]), to continuous k -convex functions. In particular we derive a convergence theorem, Theorem 5.1, monotonicity results, Lemma 5.2, Corollary 5.3, and a variational formula, Theorem 5.4.

In an ensuing paper [23], we consider the extension of Theorem 1.1 to convergence in measure, with applications to the cases $k \leq n/2$ in Theorem 1.2.

§2. Integral inequalities

In this section we develop some basic integral properties for the operators F_k which lead to Theorem 1.1. First we establish a monotonicity property.

Lemma 2.1. *Let $u, v \in \Phi^k(\Omega) \cap C^2(\bar{\Omega})$ satisfy $u = v$ on $\partial\Omega$, $u \geq v$ in Ω . Then*

$$\int_{\Omega} F_k[u] \leq \int_{\Omega} F_k[v]. \quad (2.1)$$

Proof. By approximation of the functions u and v and use of Sard's theorem, we may assume $\partial\Omega \in C^2$. Setting, for a symmetric matrix r with eigenvalues $\lambda = (\lambda_1, \dots, \lambda_n)$,

$$S_k^{ij}(r) = \frac{\partial}{\partial r_{ij}} S_k(\lambda(r)), \quad (2.2)$$

and, using the identity [15],

$$D_i S_k^{ij}(D^2 u) = 0, \quad (2.3)$$

we then obtain, by the divergence theorem,

$$\begin{aligned} \int_{\Omega} (F_k[v] - F_k[u]) &= \int_0^1 \int_{\Omega} S_k^{ij}(sD^2 u + (1-s)D^2 v) D_{ij}(v-u) \\ &= \int_0^1 \int_{\partial\Omega} S_k^{ij}(sD^2 u + (1-s)D^2 v) \gamma_i D_j(v-u) \end{aligned} \quad (2.4)$$

where γ denotes the unit outer normal to $\partial\Omega$. Letting ∂ denote the tangential gradient in $\partial\Omega$, given by

$$\partial = D - \gamma(\gamma \cdot D), \quad (2.5)$$

we can write the integrand in (2.4) as

$$\begin{aligned} S_k^{ij}(sD^2 u + (1-s)D^2 v) \gamma_i D_j(v-u) &= S_k^{ij}(sD^2 u + (1-s)D^2 v) \gamma \cdot D(v-u) \gamma_i \gamma_j \\ &\geq 0 \end{aligned} \quad (2.6)$$

since $\partial u = \partial v$ on $\partial\Omega$, $\gamma \cdot Dv \geq \gamma \cdot Du$ on $\partial\Omega$, and the function $su + (1-s)v$ will be k -convex for all $s \in [0, 1]$ (see Lemma 2.3 below). \square

Next we note that a global control on F_k is provided, for example, by Reilly's formula, [15] (see also [17]),

$$\int_{\Omega} F_k[u] = \frac{1}{k} \int_{\Omega} (\gamma \cdot Du)^k H_{k-1}[\partial\Omega], \quad (2.7)$$

when u vanishes on $\partial\Omega$. Our next estimate shows that we can control the integral of F_k *locally* in terms of the oscillation of u .

Lemma 2.2. *Let $u \in \Phi^k(\Omega) \cap C^2(\Omega)$. Then for any subdomain $\Omega' \subset\subset \Omega$, we have*

$$\int_{\Omega'} F_k[u] \leq C(\text{osc}_{\Omega} u)^k, \quad (2.8)$$

where C is a constant depending on Ω and Ω' .

To prove Lemma 2.2, we need a further property of k -convex functions.

Lemma 2.3. *Let $u_1, \dots, u_m \in \Phi^k(\Omega)$ and f be a convex, non-decreasing function in \mathbf{R}^m . Then the composite function*

$$w = f(u_1, \dots, u_m)$$

is also k -convex.

Proof. As a special case of Lemma 2.3, we see that linear combinations of k -convex functions with non-negative coefficients are also k -convex. This follows immediately from the convexity of the cones

$$\begin{aligned} \Gamma_k &= \{r \in \mathbb{S}^n \mid S_j(\lambda(r)) > 0, j = 1, \dots, k\}, \\ \bar{\Gamma}_k &= \{r \in \mathbb{S}^n \mid S_j(\lambda(r)) \geq 0, j = 1, \dots, k\} \end{aligned} \quad (2.9)$$

in \mathbb{S}^n , the space of real, $n \times n$, symmetric matrices. For the general case, it suffices to assume $u_1, \dots, u_m \in \Phi^k(\Omega) \cap C^2(\Omega)$ with $f \in C^2(\mathbf{R}^m)$. Then we have by calculation,

$$D_{ij}w = \frac{\partial f}{\partial u_p} D_{ij}u_p + \frac{\partial^2 f}{\partial u_p \partial u_q} D_i u_p D_j u_q$$

so that $D^2w \in \bar{\Gamma}_k$, since

$$\frac{\partial f}{\partial u_p} \geq 0, \quad p = 1, \dots, m, \quad \left[\frac{\partial^2 f}{\partial u_p \partial u_q} \right] \geq 0$$

and $\bar{\Gamma}_k$ is convex. \square

Proof of Lemma 2.2. Let $B = B_R(y)$ be a ball of radius R and centre y , lying in Ω and for $0 < \sigma < 1$, let $B_{\sigma R}$ denote the concentric ball of radius σR . Without loss of generality we may assume $y = 0$ and, by subtraction of a suitable constant, $u < -\varepsilon$ in B for some given positive constant ε . Setting

$$\begin{aligned}\psi(x) &= \frac{m_0}{(1-\sigma^2)} \left(1 - \frac{|x|^2}{R^2}\right), \quad m_0 = \inf_B u, \\ w(x) &= \max\{u, \psi\}\end{aligned}\tag{2.10}$$

it follows from Lemma 2.3, that w is k -convex in B and $w = u$ in $B_{\sigma R}$, $w = \psi$ on ∂B . Our desired result follows by applying Lemma 2.1 to the function w and ψ . To overcome the lack of smoothness of w , we replace it by

$$w_h = f_h(u, \psi),$$

where f_h , for $h > 0$, is the mollification,

$$f_h(x) = \int_{\mathbf{R}^2} \rho\left(\frac{x-y}{h}\right) \max(y_1, y_2) dy\tag{2.11}$$

and $\rho \geq 0$, $\in C_0^\infty(\mathbf{R}^2)$, with $\int \rho = 1$, is the usual mollifier. With h sufficiently small, we obtain from Lemma 2.1,

$$\begin{aligned}\int_{B_{\sigma R}} F_k[u] &\leq \int_B F_k[\psi] \\ &= \binom{n}{k} \omega_n \left(\frac{2|m_0|}{1-\sigma^2}\right)^k R^{n-2k} \\ &= \binom{n}{k} \omega_n \left(\frac{2}{1-\sigma^2}\right)^k R^{n-2k} (\text{osc}_B u)^k\end{aligned}\tag{2.12}$$

as $\varepsilon \rightarrow 0$. By covering Ω' with balls we conclude (2.8). \square

We are now ready to prove Theorem 1.1. Let $u \in \Phi^k(\Omega)$ and suppose $\{u_m\} \subset \Phi^k(\Omega) \cap C^2(\Omega)$ converges to u in $C^0(\Omega)$. By Lemma 2.2, the integrals

$$\int_{\Omega'} F_k[u_m]$$

are uniformly bounded, for any subdomain $\Omega' \subset\subset \Omega$ and hence a subsequence $\{F_k[u_{m_p}]\}$ converges weakly [2] (in the sense of measures) to a Borel measure $\mu_k[u]$ on Ω . It remains to show that the measure $\mu_k[u]$ is determined uniquely by the function u . Accordingly suppose that $\{u_m\}, \{v_m\} \subset \Phi^k(\Omega) \cap C^2(\Omega)$ are two sequences converging in $C^0(\Omega)$ to u and that the corresponding sequences of functions $\{F_k[u_m]\}, \{F_k[v_m]\}$ converge weakly

to Borel measures ν_1 and ν_2 respectively. Let $B = B_R(y) \subset\subset \Omega$ and fix some $\sigma \in (0, 1)$. Let $\eta \in C^2(\overline{B})$ be a convex function satisfying $\eta = 0$ in $B_{\sigma R}$, $\eta = 1$ on ∂B . For fixed $\varepsilon > 0$, it then follows, from the uniform convergence of $\{u_m\}$, $\{v_m\}$, that

$$u_m \leq v_m + \varepsilon\eta \quad (2.13)$$

on ∂B , for sufficiently large m . Let

$$G_m = \{x \in B \mid u_m > v_m + \varepsilon\eta\}. \quad (2.14)$$

Without loss of generality we may assume that ∂G_m is sufficiently smooth so that from Lemma 2.1 we have

$$\int_{G_m} F_k[u_m] \leq \int_{G_m} F_k[v_m + \varepsilon\eta]. \quad (2.15)$$

By adding $\varepsilon/2$ to u_m , we may also assume that $G_m \supset B_{\sigma R}$, so that from (2.15), we have

$$\begin{aligned} \int_{B_{\sigma R}} F_k[u_m] &\leq \int_B F_k[v_m + \varepsilon\eta] \\ &\leq \int_B [D^2 v_m + C\varepsilon I]_k \\ &\leq \int_B F_k[v_m] + C \sum_{j=0}^{k-1} \varepsilon^{k-j} \int_B F_j[v_m], \end{aligned} \quad (2.16)$$

where C is a constant depending on η . Using the estimate (2.11) and sending $m \rightarrow \infty$, $\varepsilon \rightarrow 0$, $\sigma \rightarrow 1$, we then obtain

$$\nu_1(B) \leq \nu_2(\overline{B}). \quad (2.17)$$

By replacing B by a sequence of balls $B_{\sigma_m R}$, with $\sigma_m \rightarrow 1$, satisfying

$$\nu_2(B_{\sigma_m R}) = \nu_2(\overline{B}_{\sigma_m R}),$$

we deduce $\nu_1(B) \leq \nu_2(B)$ and subsequently by interchanging $\{u_m\}$ and $\{v_m\}$, we have $\nu_1(B) = \nu_2(B)$, whence $\nu_1 = \nu_2$. This completes the proof of Theorem 1.1, as the above argument shows that $\mu_k[u]$ is well defined as the weak limit of $F_k[u_m]$ for any sequence $\{u_m\}$ converging to u in $C^0(\Omega)$ and the mapping, $\mu_k : C^0(\Omega) \rightarrow M(\Omega)$, the space of locally finite Borel measures in Ω is weakly continuous. \square

Using Theorem 1.1, our previous inequalities may be extended to functions in $\Phi^k(\Omega)$. In particular we have the following extensions of Lemmas 2.1, 2.2.

Corollary 2.4. *Let $u, v \in \Phi^k(\Omega) \cap C^0(\overline{\Omega})$ satisfying $u = v$ on $\partial\Omega$, $u \geq v$ in Ω . Then the corresponding measures μ_k satisfy*

$$\mu_k[u](\Omega) \leq \mu_k[v](\Omega). \quad (2.18)$$

Corollary 2.5. *Let $u \in \Phi^k(\Omega)$. Then for any subdomain $\Omega' \subset\subset \Omega$, we have*

$$\mu_k[u](\Omega') \leq C(\text{osc}_\Omega u)^k, \quad (2.19)$$

where C is a constant depending on Ω and Ω' .

§3. The Dirichlet problem

In the paper [20], existence and uniqueness results are obtained for the Dirichlet problem for weak solutions of the equation

$$F_k[u] = \psi \quad (3.1)$$

for inhomogeneous term $\psi \in L^p(\Omega)$ for $p \geq 1$. The classical case had been previously treated in [5] (see also [18]). A function u was called a *weak solution* of equation (3.1) in Ω if there existed a sequence $\{u_m\} \subset \Phi^k(\Omega) \cap C^2(\Omega)$ converging in $C^0(\Omega)$ to u with the corresponding sequence $\{F_k[u_m]\}$ converging in $L^1_{loc}(\Omega)$ to ψ . From Theorem 1.1, we have immediately, $\mu_k[u] = \psi$, so that the notion in (1.7) is more general. (Note that when a Borel measure μ is absolutely continuous and representable by a locally integrable function ψ we identify μ with ψ). A comparison principle for weak solutions is proved in [20] using estimates from [19]. From Corollary 2.4 we obtain a more general result as follows.

Theorem 3.1. *Let $u, v \in C^0(\bar{\Omega}) \cap \Phi^k(\Omega)$ satisfy*

$$\begin{aligned} \mu_k[u] &\geq \mu_k[v] \quad \text{in } \Omega, \\ u &\leq v \quad \text{on } \partial\Omega. \end{aligned} \quad (3.2)$$

Then $u \leq v$ in Ω .

Proof. Assume $\{o\} \in \Omega$ and set

$$\bar{u}(x) = u(x) + \varepsilon(|x|^2 - d^2)$$

for some $\varepsilon > 0$, where $d = \text{diam}\Omega$. Clearly we have

$$\mu_k[\bar{u}] \geq \mu_k[u] + \binom{n}{k} (2\varepsilon)^k,$$

and $\bar{u} \leq u \leq v$ on $\partial\Omega$. Accordingly setting

$$\Omega_\varepsilon = \{x \in \Omega \mid \bar{u}(x) > v(x)\},$$

and assuming Ω_ε is non-empty, we have, by Corollary 2.4,

$$\mu_k[u](\Omega_\varepsilon) < \mu_k[\bar{u}](\Omega_\varepsilon) \leq \mu_k[v](\Omega_\varepsilon),$$

which contradicts our hypothesis. Consequently, letting $\varepsilon \rightarrow 0$, we infer $u \leq v$ in Ω . \square

Note that Corollary 2.4 and Theorem 3.1, were proved by completely different methods, (using the normal mapping), in the case $k = n$, [1] [4], [6]. The uniqueness assertion in Theorem 1.2 follows immediately from Theorem 3.1. We may obtain the existence part by approximation from the case $\nu_2 = 0$, ([20], Theorem 1.1), using the Hölder estimate there to guarantee the local equicontinuity of the approximating solutions. However, this estimate may be bypassed as k -convex functions are automatically Hölder continuous if $k > \frac{n}{2}$. To see this we fix a ball $B = B_R(y) \subset \Omega$ and observe that the function w given by

$$w(x) = C|x - y|^{2-n/k}, \quad (3.3)$$

where C is a positive constant, satisfies

$$F_k[w] = 0, \quad \text{for } x \neq y. \quad (3.4)$$

Consequently if $u \in \Phi^k(\Omega) \cap C^2(\Omega)$, we obtain, from the classical comparison principle in the punctured ball, $B_R(y) - \{y\}$,

$$u(x) - u(y) \leq \underset{B_R(y)}{\text{osc}} u \left(\frac{|x - y|}{R} \right)^{2-n/k}, \quad (3.5)$$

provided $k > \frac{n}{2}$. It follows then that $\Phi^k(\Omega) \subset C^{0,\alpha}(\Omega)$ for $\alpha = 2 - \frac{n}{k} > 0$ and moreover, for any $x, y \in \Omega$, $x \neq y$.

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq \frac{\text{osc } u}{d_{x,y}^\alpha}, \quad (3.6)$$

where $d_{x,y} = \min\{\text{dist}(x, \partial\Omega), \text{dist}(y, \partial\Omega)\}$. For $k > \frac{n}{2}$, the function w will be k -convex in any domain and from [19], (see, in particular, (3.15), (3.16) in [19]), we have

$$F_k[w] = [C(2 - \frac{n}{k})]^k \binom{n}{k} \omega_n \delta_y, \quad (3.7)$$

where δ_y denotes the Dirac delta measure at y .

To complete the proof of Theorem 1.2, we let $\{\psi_m\}$ be a sequence of non-negative functions in $C_0^\infty(\Omega)$, converging weakly as measures to ν_2 , with support lying in some

subdomain $\Omega' \subset\subset \Omega$. By virtue of the case $p = 1$, ([20], Theorem 1.1), there exists a sequence $\{u_m\} \subset C^0(\overline{\Omega}) \cap \Phi^k(\Omega)$ of weak solutions of the Dirichlet problems

$$\begin{aligned}\mu_k[u_m] &= \nu_1 + \psi_m \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega.\end{aligned}\tag{3.8}$$

From the L^∞ estimates in [19], [20], the sequence $\{u_m\}$ is uniformly bounded in $L^\infty(\Omega)$ and hence, from (3.6), (see also [20], Theorem 4.1), equicontinuous in Ω' , so that a subsequence converges uniformly in Ω' . Relabelling the subsequence as $\{u_m\}$, we fix $\varepsilon > 0$, so that for sufficiently large m, l , we have,

$$|u_m - u_l| \leq \varepsilon \quad \text{on } \Omega'.\tag{3.9}$$

Using the comparison principle, Theorem 3.1 (or [20], Theorem 2.2), in the domain $\Omega - \overline{\Omega'}$, we then obtain (3.9) on the whole domain Ω and Theorem 1.2 follows from Theorem 1.1.

□

We remark that the necessary L^∞ estimates for the above proof (and also that of Theorem 1.1 in [20]), also follow readily from the Sobolev inequality in [19, 24], and moreover, (in the case $k > n/2$), can be derived simply from comparison with the functions (3.3), [21].

As an example of Theorem 1.2, we see that for any uniformly $(k - 1)$ -convex domain Ω , and point $y \in \Omega$, there exists a bounded Greens function G_y , given by the solution of the Dirichlet problem,

$$\begin{aligned}\mu_k[G_y] &= \delta_y \quad \text{in } \Omega, \\ G_y &= 0 \quad \text{on } \partial\Omega.\end{aligned}\tag{3.10}$$

Furthermore it is readily shown that $G_y \in C^{0,\alpha}(\overline{\Omega}) \cap C^{0,1}(\overline{\Omega} - \{y\})$, where $\alpha = 2 - \frac{n}{k}$, and, in accordance with (3.7) (see also [19]), for $\Omega = B_R(y)$, we have

$$G_y(x) = \left[\frac{1}{\binom{n}{k} \omega_n} \right]^{1/k} \frac{1}{2 - \frac{n}{k}} \left(|x - y|^{2-n/k} - R^{2-n/k} \right).\tag{3.11}$$

The Greens function is used to sharpen maximum principles in [21].

When $k \leq n/2$, we cannot expect to obtain a *continuous* k -convex solution of the Dirichlet problem (1.7) without further restrictions on ν , for example, $\nu \in L^p(\Omega)$ for $p > n/2k$, as in [20]. In order to embrace this case, we extend our notion of k -convexity to upper semi-continuous functions analogously to the general notion of subharmonic functions in the case $k = 1$. Accordingly an upper semi-continuous function $u: \Omega \rightarrow$

$[-\infty, \infty)$, is called k -convex if $F_k[u] \geq 0$ in the viscosity sense, that is, whenever there exists a point $y \in \Omega$ and function $v \in C^2(\Omega)$ satisfying $u(y) = v(y)$, $u \leq v$ in Ω , we must have $F_k[v](y) \geq 0$. Because our comparison argument above automatically extends to upper semi-continuous k -convex functions, we infer again the estimate (3.5) and (3.6) when $k > n/2$ so that there is no gain in generality in this case. However the functions (3.3) will be k -convex for all $k = 1, \dots, n$ and corresponding Greens functions arise by solving (3.10) in an appropriate sense. The general case is treated in our ensuing paper [23], together with further local properties of k -convex functions.

Finally, we note that Theorem 1.2 extends to embrace more general boundary data in the presence of barriers and that the Perron process [12] is applicable also. In particular the condition $\nu_1 \in L^1(\Omega)$ may be replaced by

$$\nu_1 \leq \nu [\text{dist}(x, \partial\Omega)]^{\beta-k-1} \quad (3.12)$$

for positive constants ν and β , as in the case $k = n$, (see [4], [20]).

§4. Semi-convex and admissible functions

The theory in Section 2 extends to larger classes of functions. Analogously to the notion of semi-convexity, we may call a function $u \in C^0(\Omega)$, k -semi-convex if the function v given by

$$v(x) = u(x) + \frac{1}{2}Ax^2, \quad (4.1)$$

is k -convex for some fixed positive constant A . From the expansion,

$$F_k[u] = \sum_{j=0}^k c(j, k, n)(-A)^j F_{k-j}[v], \quad (4.2)$$

where $c(j, k, n) = \binom{n}{k} \binom{k}{j} / \binom{n}{k-j}$, we can then define μ_k as a signed Borel measure in Ω , by

$$\mu_k[u] = \sum_{j=0}^k c(j, k, n)(-A)^j \mu_{k-j}[v]. \quad (4.3)$$

If $\{u_m\}$ is a sequence of k -semi-convex functions, with the same constant A , converging in $C^0(\Omega)$ to a k -semi-convex function u , the corresponding sequence of measures $\mu_k[u_m]$ will converge weakly to $\mu_k[u]$. It follows that the definition (4.3) is independent of the expansion (4.2).

Following usual terminology ([16], [18]), we call a function $u \in C^2(\Omega)$, *admissible* with respect to the operator F_k (or simply k -admissible) if

$$S_k(D^2u + \eta) \geq S_k(D^2u) \quad (4.4)$$

for all non-negative matrices $\eta \in \mathbb{S}^n$. Condition (4.4) implies that the operator F_k is degenerate elliptic with respect to u , that is,

$$[S_k^{ij}(D^2u)] \geq 0, \quad (4.5)$$

and is weaker than k -convexity, although the two conditions coincide in the convex case, $k = n$. A function $u \in C^0(\Omega)$ is called *k-admissible* if there exists a sequence $\{u_m\} \subset C^2$ of k -admissible functions converging to u in $C^0(\Omega)$. If additionally the sequence $\{u_m\}$ satisfies

$$F_k[u_m] \geq -\binom{n}{k} A^k \quad (4.6)$$

for a positive constant A , then the function u will be k -semi-convex, (with the same constant A). To see this, we set

$$v_m = u_m + \frac{1}{2} A |x|^2$$

and expand

$$\begin{aligned} F_k[v_m] &= \sum_{j=0}^k c(j, k, n) A^j F_{k-j}[u_m] \\ &\geq F_k[u_m] + \binom{n}{k} A^k, \end{aligned}$$

since $F_j[u_m] \geq 0$, $j = 1, \dots, k-1$. Equivalently, if $u \in C^0(\Omega)$ satisfies the inequality

$$F_k[u] \geq -\binom{n}{k} A^k \quad (4.7)$$

in the viscosity sense ([11], [16]), then u is k -semi-convex with constant A . Consequently we can define signed Borel measures μ_k for such functions, which extend the smooth case and are weakly continuous with respect to convergence in $C^0(\Omega)$.

Alternatively, the existence of the signed measure μ_k can be approached directly since Lemma 2.1 holds, more generally, for k -admissible functions $u, v \in C^2(\overline{\Omega})$. In Lemma 2.2, we obtain, in place of (2.11), for k -admissible $u \in C^2(\Omega)$,

$$\int_{\Omega'} F_k[u] \leq C \left\{ \int_{\Omega} (F_k[u])^- + (\text{osc}_{\Omega} u)^k \right\}. \quad (4.8)$$

Consequently, by following the proof of Theorem 1.1, we see that Theorem 1.1 can be extended to the class $\Phi^k(\Omega; g)$ of k -admissible functions u which are limits in $C^0(\Omega)$ of sequences $\{u_m\} \subset C^2(\Omega)$ of k -admissible functions u_m satisfying

$$F_k[u_m] \geq -g, \quad (4.9)$$

where g is a fixed, non-negative, locally integrable function in Ω . Corollaries 2.4, 2.5 then extend also to $\Phi^k(\Omega; g)$ with (2.19) replaced by

$$\mu_k[u](\Omega') \leq C \left\{ \int_{\Omega} g + (\text{osc}_{\Omega} u)^k \right\}. \quad (4.10)$$

§5. Hessian integrals

For $u \in C^2(\overline{\Omega})$, we define the Hessian integral $I_k[u]$ by

$$I_k[u] = I_k[u; \Omega] = - \int_{\Omega} u F_k[u]. \quad (5.1)$$

If $u = 0$ on $\partial\Omega$, we have by integration by parts,

$$I_k[u] = \frac{1}{k} \int_{\Omega} S_k^{ij} D_i u D_j u, \quad (5.2)$$

so that $I_k[u] \geq 0$ if, also, u is k -admissible. Imbedding properties of Hessian integrals are treated in the papers [7], [19], [20], [24]. Using Theorem 1.1, we define an extension of I_k to $\Phi^k(\Omega) \cap C^0(\overline{\Omega})$ by

$$I_k[u] = - \int_{\Omega} u d\mu_k[u]. \quad (5.3)$$

Clearly $I_k[u]$ is finite if $\mu_k[u](\Omega) < \infty$. Letting $\Phi_0^k(\Omega)$ denote the subset of $\Phi^k(\Omega) \cap C^0(\overline{\Omega})$ of functions vanishing on $\partial\Omega$, we then obtain from the weak continuity of μ_k , the approximation result.

Theorem 5.1. *Let $\{u_m\} \subset \Phi_0^k(\Omega)$ converge uniformly to u and suppose $\{\mu_k[u_m](\Omega)\}$ is bounded. Then $I_k[u_m] \rightarrow I_k[u]$.*

Proof. For $\Omega' \subset\subset \Omega$, we have

$$\mu_k[u](\Omega') \leq \liminf_{m \rightarrow \infty} \mu_k[u_m](\Omega')$$

so that $\mu_k[u](\Omega) < \infty$. From (5.3) we have, for any $\eta \in C_0^0(\Omega)$, $0 \leq \eta \leq 1$,

$$\begin{aligned} I_k[u_m] - I_k[u] &= \int_{\Omega} (u - u_m) d\mu_k[u_m] + \int_{\Omega} u (d\mu_k[u] - d\mu_k[u_m]) \\ &\leq \sup_{\Omega} |u - u_m| \mu_k[u_m](\Omega) + \sup(1 - \eta) |u| (\mu_k[u_m](\Omega) + \mu_k[u](\Omega)) \\ &\quad + \int_{\Omega} \eta u (d\mu_k[u] - d\mu_k[u_m]) \\ &\rightarrow 0, \end{aligned}$$

as $\eta \rightarrow 1$, $m \rightarrow \infty$. Interchanging u and u_m we obtain $I_k[u_m] \rightarrow I_k[u]$ as required. \square

Remark. If we only assume $\{u_m\} \subset \Phi^k(\Omega)$ converges to u in $C^0(\Omega)$, we obtain $I_k[u_m; \Omega'] \rightarrow I_k[u; \Omega']$ for any subdomain $\Omega' \subset\subset \Omega$ satisfying $\mu_k[u_m](\partial\Omega') = 0$, $m = 1, 2, \dots$. If additionally, $\mu_k[u_m] \rightarrow \mu_k[u]$ strongly in $\Omega - \Omega'$ for some $\Omega' \subset\subset \Omega$, then we obtain $I_k[u_m; \Omega] \rightarrow I_k[u; \Omega]$ as above.

Monotonicity.

Hessian integrals enjoy corresponding monotonicity properties to the Hessian measures. Assuming $u, v \in C^2(\overline{\Omega})$, $u = v$ on $\partial\Omega$, $\partial\Omega \in C^2$, and writing

$$\begin{aligned} w_t &= (1 - t)u + tv, \quad 0 \leq t \leq 1, \\ f(t) &= I_k[w_t], \end{aligned} \tag{5.4}$$

we calculate

$$\begin{aligned} I_k[u] - I_k[v] &= f(0) - f(1) \\ &= \int_0^1 \int_{\Omega} (v - u) F_k[w_t] dt + \int_0^1 \int_{\Omega} w_t S_k^{ij} D_{ij}(v - u) dt \\ &= (k + 1) \int_0^1 \int_{\Omega} (v - u) F_k[w_t] dt + \int_0^1 \int_{\partial\Omega} u S_k^{ij} \gamma_i \gamma_j D_{\gamma}(v - u) dt, \end{aligned} \tag{5.5}$$

where, as in Section 2, γ denotes the unit outer normal to $\partial\Omega$. Accordingly, if $u \geq v$ in Ω , $u = v \leq 0$ on $\partial\Omega$, with u and v both k -convex in Ω , we infer $I_k[u] \leq I_k[v]$. More generally, if $M = \max_{\partial\Omega} u > 0$, we replace u, v by $u - M, v - M$ respectively, to obtain

$$\begin{aligned} I_k[u] &= I_k[u - M] + M\mu_k[u](\Omega) \\ &\leq I_k[v - M] + M\mu_k[v](\Omega) \\ &= I_k[v]. \end{aligned}$$

We therefore have the following analogue of Lemma 2.1.

Lemma 5.2. *Let $u, v \in \Phi^k(\Omega) \cap C^2(\overline{\Omega})$ satisfy $u = v$ on $\partial\Omega$, $u \geq v$ in Ω . Then*

$$I_k[u] \leq I_k[v]. \quad (5.6)$$

By approximation, using Theorem 5.1, we then infer the analogue of Corollary 2.4.

Corollary 5.3. *Let $u, v \in \Phi_0^k(\Omega)$ satisfy $u \geq v$ in Ω . Then*

$$I_k[u] \leq I_k[v]. \quad (5.7)$$

Remark. More generally, if $u, v \in \Phi^k(\Omega) \cap C^0(\overline{\Omega})$, $u \geq v$ in Ω , we obtain, using our previous remark after Theorem 5.1,

$$\liminf_{\delta \rightarrow 0} I_k[u; \Omega_\delta] \leq \limsup_{\delta \rightarrow 0} I_k[v; \Omega_\delta], \quad (5.8)$$

where $\Omega_\delta = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \delta\}$. Also if only $u, v \in \Phi^k(\Omega)$, then $I_k[u; \Omega'] \leq I_k[v; \Omega']$ for any subdomain $\Omega' \subset\subset \Omega$, where $u \geq v$ and $u = v$ on $\partial\Omega'$.

Variational derivatives.

From (5.5) we have

$$f'(0) = (k+1) \int_{\Omega} (u-v) F_k[u] - \int_{\partial\Omega} u S_k^{ij}(D^2u) \gamma_i \gamma_j D_\gamma(v-u). \quad (5.9)$$

Furthermore if $D_\gamma u = D_\gamma v$ (or $u = 0$) on $\partial\Omega$, we have

$$f'(0) = (k+1) \int_{\Omega} (u-v) F_k[u], \quad (5.10)$$

$$f''(t) = (k+1) \int_{\Omega} S_k^{ij}(D^2w_t) D_i(u-v) D_j(u-v) \geq 0,$$

if u, v are k -admissible in Ω . Moreover if $\varphi = u - v$ has compact support in Ω_δ for some $\delta > 0$, we have an upper bound,

$$\begin{aligned} f''(t) &\leq (k+1)(n-k+1) \int_{\Omega_\delta} F_{k-1}[w_t] \max |D\varphi|^2 \\ &\leq C(\text{osc}_\Omega u + \text{osc}_\Omega \varphi)^{k-1} \max |D\varphi|^2 \end{aligned} \quad (5.11)$$

by Lemma 2.2. By approximation we then obtain the following variational formula.

Theorem 5.4. *Let $u, v \in \Phi^k(\Omega) \cap C^0(\overline{\Omega})$ with $\varphi = u - v \in C_0^2(\Omega)$ and let $f(t) = I_k[w_t]$, $0 \leq t \leq 1$. Then*

$$f'(0) = (k+1) \int_{\Omega} (u-v) d\mu_k[u]. \quad (5.12)$$

Further Remarks.

Taking account of the preceding section, certain of the above results extend to semi-convex or admissible functions. In particular Theorem 5.1 extends to sequences $\{u_m\}$ of k -semi-convex functions (with same constant A) or sequences $\{u_m\} \subset \Phi^k(\Omega, g)$ for some $g \in L^1(\Omega)$, vanishing continuously on $\partial\Omega$. The variational formula (5.12) remains valid for u, v being k -semi-convex or k -admissible with $u \in \Phi^k(\Omega, g)$. Furthermore, if $u, v \in C^0(\bar{\Omega}) \cap \Phi^k(\Omega, g)$ with $u = v = 0$ on $\partial\Omega$, $u \geq v$ in Ω , we obtain from (5.10), the inequality

$$I_k[u] - I_k[v] \leq (k+1) \int_{\Omega} (v-u) d\mu_k[u], \quad (5.13)$$

which complements Lemma 5.2. In the case $k = n$, (5.7) and (5.13) were proved by Krylov [14]. Accordingly, if we define the functional $J_{k;\mu}$ on $\Phi_0^k(\Omega)$ by

$$J_{k;\mu}[u] = \frac{1}{k+1} I_k[u] + \int_{\Omega} u d\mu \quad (5.14)$$

for $u \in \Phi_0^k(\Omega)$, where μ is a finite Borel measure on Ω , we obtain

$$J_{k,\mu}[u] = \min_{u \geq v} J_{k;\mu}[v], \quad (5.15)$$

provided $\mu_k[u] = \mu$. Consequently the solution of the Dirichlet problem (1.7) for $\varphi \equiv 0$ solves the variational problem (5.15). Related variational problems are treated in [3], [8], [9], and [13].

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