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Regularity of potential functions of the optimal transportation problem

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Abstract

The potential function to the optimal transportation problem satisfies a partial differential equation of Monge-Ampère type. In this paper we introduce the notion of generalized solution, and prove the existence and uniqueness of generalized solutions to the problem. We also prove the solution is smooth under certain structural conditions on the cost function.

1. Introduction

The optimal transportation problem, as proposed by Monge in 1781 [22], is to find an optimal mapping from one mass distribution to another such that a cost functional is minimized among all measure preserving mappings. The original cost function of Monge was proportional to the distance moved and surprisingly this problem was only recently completely solved [9, 29], see also [2]. When the cost function is strictly convex, it was proved [6, 15] that a unique optimal mapping can be determined by the potential functions, which are maximizers of Kantorovich's dual functional.

The potential function satisfies a fully nonlinear equation of Monge-Ampère type, subject to a natural boundary condition. When the cost function c is given by $c(x,y) = |x-y|^2$, or equivalently $c(x,y) = x \cdot y$, the equation can be reduced to the standard Monge-Ampère equation

$$\det D^2 u = h(\cdot, Du), \tag{1.1}$$

and the regularity of solutions was obtained in [7, 8, 30], and earlier in [12] in dimension 2. See also §7.1 below. In this paper we study the regularity of potential functions for general cost functions. We establish an a priori interior second order derivative estimate for solutions to the corresponding Monge-Ampère equation when the cost function satisfies an additional structural condition, namely the assumption (A3) in Section 2. Although condition (A3) depends upon derivatives up to order four of the cost function, it is the natural condition for interior regularity of the associated Monge-Ampère type equation. Moreover we do not expect the interior regularity without this or a similar condition. To apply the a priori estimate to the potential functions we introduce and develop the notion of generalized solution and prove that a potential function is indeed a generalized solution. The regularity of potential functions then follows by showing that a generalized solution can be approximated locally by smooth ones.

In Section 2 we introduce Kantorovich's dual functional and deduce the Monge-Ampère type equation satisfied by the potential function. We then introduce the appropriate convexity notions relative to the cost function and state the main regularity result of the paper (Theorem 2.1). In Section 3 we introduce the generalized solution and show that a generalized solution is a potential function to the optimal transportation problem, from which follows the existence and uniqueness of generalized solutions (Theorem 3.1). In Section 4 we establish the a priori second order derivative estimates (Theorem 4.1) under assumption (A3). In Section 5 we prove that a generalized solution can be approximated locally by smooth solutions and so complete the proof of Theorem 2.1.

In Section 6 we verify the assumption (A3). We show that (A3) is satisfied by the two important cost functions [5], $c(x,y)=\sqrt{1-|x-y|^2}$ and $c(x,y)=\sqrt{1+|x-y|^2}$; and more generally by the cost function

$$c(x,y)=(\varepsilon+|x-y|^2)^{p/2} \tag{1.2}$$

for $1\le p<2$, where $\varepsilon>0$ is a constant. We also consider the cost function

$$c(x,y)=|X-Y|^2, \tag{1.3}$$

which is a function of the distance between points on graphs, where $X=(x,f(x))$ and $Y=(y,g(y))$ are points of the graphs of f and g. We show that condition (A3) is satisfied if f and g are uniformly convex or concave and their gradients are strictly less than one. The final Section 7 contains various remarks. First in §7.1 we indicate a simpler proof for the existence and interior regularity of solutions to equation (1.1) subject to the second boundary condition. We then recall in §7.2 the reflector antenna design problem treated in [32,33], which is an optimal transportation problem with cost function satisfying assumption (A3). In §7.3 we show that a potential function may not be smooth if the target domain is not convex relative to the cost function, while in §7.4 we provide conditions ensuring the convexity property for our examples in §6. We conclude in §7.5 with further remarks pertaining to the scope of our conditions.

Our treatment of c-concavity builds upon that in [16] for strictly convex cost functions and in [32] for the reflector antenna problem. We are very grateful to Robert McCann for useful discussions about this and other aspects of this paper.

2. Potential functions

Let Ω, Ω^* be two bounded domains in the Euclidean space $\mathbf{R}^n$, and let f,g be two nonnegative functions defined on Ω and Ω^*, and satisfying

$$\int_\Omega f=\int_{\Omega^*} g. \tag{2.1}$$

Let c be a C^4 smooth function defined on $\mathbf{R}^n \times \mathbf{R}^n$. The optimal transportation problem concerns the existence of a measure preserving mapping $s_0 : \Omega \to \Omega^*$ which minimizes the cost functional

$$\mathcal{C}(s) = \int_\Omega f(x) c(x, s(x))\, dx \tag{2.2}$$

among all measure preserving mappings from Ω to Ω^*. A map s is called measure preserving if it is Borel measurable and for any Borel set $E \in \Omega^*$,

$$\int_{s^{-1}(E)} f = \int_E g. \tag{2.3}$$

The above condition is equivalent to that for any continuous function $h \in C(\Omega^*)$,

$$\int_\Omega h(s(x)) f(x)\, dx = \int_{\Omega^*} h(y) g(y)\, dy.$$

To study the existence of optimal mappings to the minimization problem, Kantorovich introduced the dual functional

$$I(\varphi, \psi) = \int_\Omega f(x)\varphi(x) dx + \int_{\Omega^*} g(y)\psi(y) dy, \quad (\varphi, \psi) \in K, \tag{2.4}$$

where

$$K = \{(\varphi, \psi) \mid \ \varphi(x) + \psi(y) \le c(x, y), \quad x \in \Omega, y \in \Omega^*\}. \tag{2.5}$$

A fundamental relation between the cost functional $\mathcal{C}$ and its dual functional I is the following

$$\inf_{s \in S} \mathcal{C}(s) = \sup_{(\varphi,\psi)\in K} I(\varphi, \psi). \tag{2.6}$$

It is readily shown (under appropriate conditions) that the maximizer to the right hand side always exists, and is unique up to a constant [6,15]. Let $(u, v) \in K$ be a maximizer. The component functions u and v are called the *potential functions* of the optimal transportation problem. In this paper we study their regularity.

We assume the following conditions:
(A1) For any $x, z \in \mathbf{R}^n$, there is a unique $y \in \mathbf{R}^n$, depending continuously on x, z, such that $D_x c(x, y) = z$, and for any $y, z \in \mathbf{R}^n$, there is a unique $x \in \mathbf{R}^n$, depending continuously on y, z, such that $D_y c(x, y) = z$.
(A2) For any $x, y \in \mathbf{R}^n$,

$$\det D^2_{xy} c \ne 0,$$

where $D^2_{xy}c$ is the matrix whose element at the i^{th} row and j^{th} column is $\frac{\partial^2 c(x,y)}{\partial x_i \partial y_j}$.
(A3) There exists a constant $c_0 > 0$ such that for any $x \in \Omega, y \in \Omega^*$, and $\xi, \eta \in \mathbf{R}^n$, $\xi \perp \eta$,

$$\sum_{i,j,k,l,p,q,r,s} (c^{p,q} c_{ij,p} c_{q,rs} - c_{ij,rs}) c^{r,k} c^{s,l} \xi_i \xi_j \eta_k \eta_l \ge c_0 |\xi|^2 |\eta|^2,$$

where $c_{i,j}(x, y) = \frac{\partial^2 c(x,y)}{\partial x_i \partial y_j}$, and $(c^{i,j})$ is the inverse matrix of $(c_{i,j})$.

Assumption (A3) will be used in the proof of the regularity of u. For the regularity of v we assume instead

$$\sum(c^{p,q}c_{rs,p}c_{q,ij} - c_{rs,ij})c^{k,r}c^{l,s}\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2.$$

In Section 4 we will show that assumption (A3) can be replaced by a slightly different one, see (4.21).

From the proofs in [6,15], the existence and uniqueness, up to constants, of maximizers to (2.4) is readily inferred under conditions (A1) (A2), (although one should note that the stated hypotheses in these papers is stronger). Let (u, v) be the potential functions. It is easy to verify the relation [15],

$$\begin{aligned} u(x) &= \inf_{y\in\Omega^*}\{c(x,y) - v(y)\}, \\ v(y) &= \inf_{x\in\Omega}\{c(x,y) - u(x)\}. \end{aligned} \tag{2.7}$$

It follows that u, v are Lipschitz continuous, and their Lipschitz constants are controlled by that of c. For any given $x_0 \in \Omega$, let $y_0 \in \Omega^*$ such that

$$u(x_0) = c(x_0, y_0) - v(y_0), \tag{2.8}$$

$$u(x) \leq c(x, y_0) - v(y_0) \quad \forall\ x \in \Omega. \tag{2.9}$$

Then we have, provided Du and D^2u exist at x_0,

$$Du(x_0) = D_x c(x_0, y_0), \tag{2.10}$$

$$D^2u(x_0) \leq D_x^2 c(x_0, y_0). \tag{2.11}$$

By assumption (A1) and (2.10), we obtain a mapping T_u, $T_u(x_0) = y_0$, such that for almost all $x \in \Omega$,

$$Du(x) = D_x c(x, T_u(x)). \tag{2.12}$$

Similarly there is a mapping T_v such that for almost all $y \in \Omega^*$,

$$Dv(y) = D_y c(T_v(y), y).$$

From the proof in [6,15], the mapping T_u is indeed measure preserving, and is optimal for the transportation problem (2.2). From (2.8) we have $T_u(x) = y$ if and only if $T_v(y) = x$. Hence T_v is the inverse of T_u. From (2.11) we see that if u is C^2 smooth, then

$$D^2u(x) \leq D_x^2 c(x, T_u(x)) \quad \forall\ x\ \in \Omega, \tag{2.13}$$

where $D_x^2 c$ is the Hessian matrix of c with respect to the x-variable. Similarly if v is C^2 smooth, we have

$$D^2v(y) \leq D_y^2 c(T_v(y), y) \quad \forall\ y\ \in \Omega^*.$$

Next we introduce some convexity notions relative to the cost function c. These notions coincide with the classical ones when

$$c(x, y) = x \cdot y. \tag{2.14}$$

First we introduce the c-concavity of functions; see also [16].

Definition 2.1. An upper semi-continuous function ϕ defined on Ω is c-concave if for any point $x_0 \in \Omega$, there exists a c-support function at x_0. Similarly, an upper semi-continuous function ψ defined on Ω^* is c^*-concave if for any point $y_0 \in \Omega^*$, there exists a c^*-support function at y_0.

A *c-support function* of φ at x_0 is a function of the form $a + c(x, y_0)$, where $y_0 \in \mathbf{R}^n$ and $a = a(x_0, y_0)$ is a constant, such that

$$\begin{aligned} \varphi(x_0) &= c(x_0, y_0) + a, \\ \varphi(x) &\le c(x, y_0) + a \quad \forall\, x \in \Omega. \end{aligned} \tag{2.15}$$

Similarly one can define c^*-support functions by switching x and y, Ω and Ω^*. One can also introduce the notion of c-convex functions by changing the direction of the inequality in (2.15).

As the cost function c is smooth, any c-concave function φ is semi-concave, namely there exists a constant C such that $\varphi(x) - C|x|^2$ is concave. It follows that a c-concave function is locally Lipschitz continuous in Ω and twice differentiable almost everywhere. If φ is twice differentiable at x_0, then T_φ is well defined at x_0 and one has $D^2\varphi(x) \le D_x^2 c(x, T_\varphi(x))$ at x_0. It is easy to show that if (φ_k) is a sequence of c-concave functions and $\varphi_k \to \varphi$, then φ is c-concave.

In the special case when $c(x, y) = x \cdot y$, the notion of c-concavity coincides with that of concavity, and the graph of a c-support function is a support hyperplane.

Obviously the potential function u is c-concave and v is c^*-concave. Next we derive the equation satisfied by (u, v). From (2.12) we have

$$D^2 u(x) = D_x^2 c(x, T(x)) + D_{xy}^2 c \cdot DT. \tag{2.16}$$

Hence

$$\begin{aligned} \det(D_x^2 c - D^2 u) &= \det(-D_{xy}^2 c) \det DT \\ &= |\det(D_{xy}^2 c)| \frac{f(x)}{g(T(x))} \quad x \in \Omega. \end{aligned} \tag{2.17}$$

Note that $\det DT$ could be negative and by (2.13), the matrix $(D_x^2 c - D^2 u)$ is non-negative. Hence (2.17) is *degenerate elliptic* if u is c-concave. For the optimal transportation problem we have the natural boundary condition

$$T_u(\Omega) = \Omega^*. \tag{2.18}$$

For the potential function v, we have,

$$D^2 v = D_y^2 c + D_{xy}^2 c \cdot DT_v. \tag{2.19}$$

Hence v satisfies the equation

$$\det(D_y^2 c - D^2 v) = |\det D_{xy}^2 c| \cdot \frac{g(y)}{f(T_v(y))} \quad y \in \Omega^*. \tag{2.20}$$

The corresponding boundary condition is

$$T_v(\Omega^*) = \Omega. \tag{2.21}$$

We remark that equations (2.17) and (2.20) can also be found in [31], and the notion of c-concavity is already in the literature; see for example, [16].

Equations (2.17) and (2.20) are of Monge-Ampère type. If $c(x, y) = x \cdot y$, then $T_u(x) = Du(x)$ and (2.17) is equivalent to the standard Monge-Ampère equation

$$\det D^2 u = h, \tag{2.22}$$

with the boundary condition

$$Du(\Omega) = \Omega^*, \tag{2.23}$$

where $h = g/f$. As remarked earlier, the regularity for the second boundary problem (2.22) (2.23), as originally proposed in [20], was established in [7,8,12,30].

To study the regularity of equation (2.17), we not only need the c-convexity of functions, but also the convexity of domains, as in the case for the standard Monge-Ampère equation (2.22). First we introduce a notion of c-segments, which plays a similar role as line segments for the Monge-Ampère equation.

Definition 2.2. A c-segment in Ω with respect to a point y is a solution set $\{x\}$ to $D_y c(x, y) = z$ for z on a line segment in $\mathbf{R}^n$. A c^*-segment in Ω^* with respect to a point x is a solution set $\{y\}$ to $D_x c(x, y) = z$ for z on a line segment in $\mathbf{R}^n$.

By assumptions (A1) and (A2), it is easy to check that a c-segment is a smooth curve, and for any two points $x_0, x_1 \in \mathbf{R}^n$ and any $y \in \mathbf{R}^n$, there is a unique c-segment connecting x_0 and x_1 relative to y.

Let $h(x) = c(x, y_0) + a$ be a c-support function. Then $T_h(x) = y_0$ for any $x \in \Omega$, namely T_h maps all points $x \in \Omega$ to the same point y_0. Hence we may regard y_0 as the "focus" of h. Geometrically a c^*-segment relative to x_0 is the set of foci of a family of c-support functions whose gradients at x_0 lie in a line segment. When $c(x, y) = x \cdot y$, a c-support function is a hyperplane and the "focus" of a hyperplane is its slope. In this case a c-segment connecting x_0 and x_1 is the Euclidean line segment, and the c-convexity of domains introduced below is equivalent to convexity.

Definition 2.3. A set E is c-convex relative to a set E^* if for any two points $x_0, x_1 \in E$ and any $y \in E^*$, the c-segment relative to y connecting x_0 and x_1 lies in E. Similarly we say E^* is c^*-convex relative to E if for any two points $y_0, y_1 \in E^*$ and any $x \in E$, the c^*-segment relative to x connecting y_0 and y_1 lies in E^*.

The notion of c-convexity is not equivalent to convexity in the usual sense, rather it is stronger in general. For example, a ball may not be c-convex (relative to another given domain) at arbitrary location. On the other hand, a sufficiently small ball will be c-convex if c is C^3 smooth. Another way of expressing Definition 2.3 is that E is c-convex with respect to E^* if for each $y \in E^*$, the image $D_y c(\cdot, y)(E)$ is convex in $\mathbf{R}^n$ and E^* is c^*-convex with respect to E if for each $x \in E$, $D_x c(x, \cdot)(E^*)$ is convex in $\mathbf{R}^n$. We will examine this notion further in §7.4 in the light of our examples in Section 6.

We can now state the main regularity result of the paper.

Theorem 2.1. *Suppose $f \in C^2(\Omega)$, $g \in C^2(\Omega^*)$, f, g have positive upper and lower bounds, and (2.1) holds. Suppose the cost function c satisfies assumptions (A1)-(A3). Then if the domain Ω^* is c^*-convex relative to Ω, the potential function u is C^3 smooth in Ω. If Ω is c-convex relative to Ω^*, the potential function v is C^3 smooth in Ω^*.*

We have not studied the regularity near the boundary. In the special case $c(x, y) = x \cdot y$, it has been established in [8,12,30]. We also point out that if Ω^* is not c^*-convex (relative to Ω), the regularity assertion of Theorem 2.1 does not hold in general, see §7.3.

Remark 2.1. Instead of maximizers of Kantorovich's functional, one can also consider minimizers, namely functions (u, v) satisfying

$$I(u, v) = \inf_{(\varphi,\psi)\in K^*} I(\varphi, \psi), \tag{2.24}$$

where

$$K^* = \{(\varphi, \psi) \mid \ \varphi(x) + \psi(y) \geq c(x, y), x \in \Omega, y \in \Omega^*\}.$$

The existence of minimizers to (2.24) can also be proved along the line of [6,15]. In this paper we will study maximizers only, but the treatment in the paper also holds for minimizers. In particular Theorem 2.1 holds for minimizers if assumption (A3) is replaced by

(A3′) There exists a constant $c_0 > 0$ such that for any $x \in \overline{\Omega}, y \in \overline{\Omega}^*$, and $\xi, \eta \in \mathbf{R}^n$, $\xi \perp \eta$,

$$\sum_{i,j,k,l,p,q,r,s} (c^{p,q} c_{ij,p} c_{q,rs} - c_{ij,rs}) c^{r,k} c^{s,l} \xi_i \xi_j \eta_k \eta_l \leq -c_0 |\xi|^2 |\eta|^2,$$

Note that (A3) and (A3′) cannot hold simultaneously.

3. Generalized solutions

Let φ be a c-concave function in Ω. We define a set-valued mapping $T_\varphi : \ \Omega \to \mathbf{R}^n$, which is an extension of the normal mapping (super-gradient) for concave functions [24]. For any $x_0 \in \Omega$, let $T_\varphi(x_0)$ denote the set of points y_0 such that $c(x, y_0) + a$ is a c-support function of φ at x_0 for some constant $a = a(x_0, y_0)$. For any subset $E \subset \Omega$, we denote $T_\varphi(E) = \bigcup_{x\in E} T_\varphi(x)$.

If φ is C^1 smooth, then T_φ is single valued, and is exactly the mapping given by (2.12). In general, $T_\varphi(x)$ is single valued for almost all $x \in \Omega$ as φ is semi-concave and so twice differentiable almost everywhere. If $c(x, y) = x \cdot y$, T_φ is the normal mapping for concave functions. In this paper we call the mapping T_φ the *c-normal mapping* of φ. Similarly one can define the c^*-normal mapping for c^*-concave functions.

Remark 3.1. Since a c-concave function φ is semi-concave, its super-gradient

$$\partial^+\varphi(x_0) = \{p \in \mathbf{R}^n \mid \quad u(x) \leq u(x_0) + p \cdot x + o(|x - x_0|)\}$$

is defined everywhere. By (2.12), we see that if $y \in T_\varphi(x_0)$, then $D_x c(x_0, y) \in \partial^+\varphi(x_0)$. However if φ is not C^1 at some point x_0, the relation

$$T_\varphi(x_0) = \{y \in \mathbf{R}^n \mid \ D_x c(x_0, y) \in \partial^+\varphi(x_0)\}$$

does not hold in general, where $c(x,y)+a$ (for some constant a) is a c-support function of φ at x_0. Moreover, the set

$$E = \{x \in \mathbf{R}^n \mid \varphi(x) = c(x,y) + a\}$$

may be disconnected. See §7.5 for further discussion.

By the c-normal mapping we can introduce a measure $\mu = \mu_{\varphi,g}$ in Ω, where $g \in L^1(\mathbf{R}^n)$ is a nonnegative measurable function, such that for any Borel set $E \subset \Omega$,

$$\mu(E) = \int_{T_\varphi(E)} g(x)dx. \tag{3.1}$$

To prove that μ is a Radon measure, we need to show that μ is countably additive. We will use the following generalized Legendre transform.

Definition 3.1. Let φ be an upper semi-continuous function defined on Ω. The c-transform of φ is a function φ^* defined on $\mathbf{R}^n$, given by

$$\varphi^*(y) = \inf\{c(x,y) - \varphi(x) \mid x \in \Omega\}. \tag{3.2}$$

Similarly for an upper semi-continuous function ψ defined on Ω^*, the c^*-transform of ψ is the function

$$\psi^*(x) = \inf\{c(x,y) - \psi(y) \mid y \in \Omega^*\}.$$

From the definition, φ^* is obviously c^*-concave and ψ^* is c-concave. If φ is c-concave and $c(x,y_0)+a$ is a c-support function of φ at x_0, then $\varphi^*(y_0) = -a$ and $c(x_0,y) - \varphi(x_0)$ is a c^*-support function of φ^* at y_0. Hence the c^*-transform of φ^*, when restricted to Ω, is φ itself. In particular we see that $y \in T_\varphi(x)$ if and only if $x \in T_{\varphi^*}(y)$.

Lemma 3.1. *Let φ be a c-concave function. Let*

$$Y = Y_\varphi = \{y \in \mathbf{R}^n \mid \exists\ x_1 \neq x_2 \in \Omega, \text{such that y} \in \mathrm{T}_\varphi(\mathrm{x}_1) \cap \mathrm{T}_\varphi(\mathrm{x}_2)\}.$$

Then Y has Lebesgue measure zero.

Proof. Let ψ be the c-transform of φ. Then φ is the c^* transform of ψ. Observe that if $y \in T_\varphi(x_1) \cap T_\varphi(x_2)$, we have $x_1, x_2 \in T_\psi(y)$. Hence ψ is not twice differentiable at y. But since ψ is semi-concave, ψ is twice differentiable almost everywhere. □

It follows that μ is countably additive, see [3, 10, 24].

Lemma 3.2. *Let φ_i be a sequence of c-concave functions which converges to φ locally uniformly in Ω. Then for any compact set $K \subset \Omega$ and open set $U \subset\subset \Omega$, we have*

$$\overline{\lim}_{i\to\infty}\mu_{\varphi_i,g}(K) \le \mu_{\varphi,g}(K), \tag{3.3}$$

$$\underline{\lim}_{i\to\infty}\mu_{\varphi_i,g}(U) \ge \mu_{\varphi,g}(U). \tag{3.4}$$

Proof. To prove (3.3) it suffices to prove

$$\bigcap_{i\to\infty}\bigcup_{j\geq i} T_{\varphi_j}(K)\subset T_\varphi(K). \tag{3.5}$$

To prove (3.5), we suppose $x_i\in K$ and $y_i\in T_{\varphi_i}(x_i)$ such that y_i converges to y_0. Let $h_i(x)=a_i+c(x,y_i)$ be a c-support function of φ_i at x_i. Obviously a_i is uniformly bounded. Letting $i\to\infty$ we obtain $h_i\to h_0$ and $h_0=a+c(x,y_0)$ is a c-support function of φ at x_0. Hence $y_0\in T_\varphi(x_0)$, and the inclusion (3.5) is proved.

To prove (3.4), we fix (as for example in [10]) a compact set $K\subset U$. Then for sufficiently large i, we have

$$T_\varphi(K)-Y_\varphi\subset T_{\varphi_i}(U),$$

so that by Lemma 3.1,

$$\begin{aligned}\mu_{\varphi,g}(U) &= \sup\nolimits_{K\subset U}\mu_{\varphi,g}(K)\\ &\leq \underline{\lim}_{i\to\infty}\mu_{\varphi_i,g}(U).\end{aligned}$$

□

Corollary 3.1. *Let $\{\varphi_i\}$ be a sequence of c-concave functions in Ω. If $\varphi_i\to\varphi$, then $\mu_{\varphi_i,g}\to\mu_{\varphi,g}$ weakly as measures in Ω.*

Let $P_k=\{p_1,\cdots,p_k\}$ be arbitrary k points in Ω and φ be a c-concave function. Let

$$\eta_k(x)=\inf\{c(x,y)+a\}, \tag{3.6}$$

where the infimum is taken over all $a\in\mathbf{R}$ and $y\in\mathbf{R}^n$ such that $c(p_i,y)+a\geq\varphi(p_i)$ for $i=1,\cdots,k$ and $c(x,y)+a\geq\varphi(x)$ for $x\in\partial\Omega$. It is easy to see that $\mu_{\eta_k,g}$ is a discrete measure, supported on the set P_k. Choosing a proper sequence of $\{P_k\}$ such that $\eta_k\to\varphi$, by Lemma 3.2 we see that $\mu_{\varphi,f}$ is a Radon measure.

For the standard Monge-Ampère equation (2.22), which corresponds to the cost function $c(x,y)=x\cdot y$, the above properties of the measure $\mu_{\varphi,g}$ can also be found in [10,24] for the case $g=1$, and in [3] for general positive function g.

Definition 3.2. A c-concave function φ is called a generalized solution of (2.17) if $\mu_{\varphi,g}=f\,dx$ in the sense of measures, that is for any Borel set $E\subset\Omega$,

$$\int_E f=\int_{T_\varphi(E)} g. \tag{3.7}$$

If furthermore φ satisfies

$$\Omega^*\subset T_\varphi(\overline{\Omega}),\quad |\{x\in\Omega\mid\ f(x)>0\text{ and }\mathrm{T}_\varphi(\mathrm{x})-\overline{\Omega}^*\neq\emptyset\}|=0, \tag{3.8}$$

then φ is a generalized solution of (2.17) (2.18).

For the standard Monge-Ampère equation (2.22), namely when $c(x, y) = x \cdot y$, the above generalized solution was introduced by Aleksandrov with $g = 1$ and Bakelman for general nonnegative locally integrable function g, see [3]. Note that for the boundary condition (3.8), we need to extend g to $\mathbf{R}^n - \Omega^*$ by letting $g = 0$ so that the mass balance condition (2.1) is satisfied.

Let u be a generalized solution of (2.17) and v its c-transform. Let E_u and E_v denote respectively the sets on which u and v are not twice differentiable. Then $|E_u| = |E_v| = 0$, and for any $x \in T_v(E_v)$, there is a point $y \in E_v$ such that $y \in T_u(x)$. Since $T_u(x)$ is a single point for all $x \in \Omega - E_u$, we have

$$\int_{T_v(E_v)} f = \int_{E_v} g = 0. \tag{3.9}$$

The above formula implies that T_u is one to one almost everywhere on $\{x \in \Omega \mid f(x) > 0\}$. It follows that T_u is a measure preserving mapping. Hence if u is a generalized solution of (2.17) (2.18), by condition (3.8) and the mass balance condition (2.1), we have $T_u(x) \in \Omega^*$ for almost all $x \in \{f > 0\}$. It follows that for any Borel set $E^* \subset \Omega^*$,

$$\int_{E^*} g = \int_{T_v(E^*)} f, \tag{3.10}$$

where v is the c-transform of u. Hence we have

Lemma 3.3. *Let u be a generalized solution of (2.17) (2.18). Let v be the c-transform of u. Then v is a generalized solution of (2.20) (2.21)*

The next lemma shows that a c-concave function is a generalized solution if and only if it is a potential function. In particular, if $c(x, y) = x \cdot y$, then a generalized solution of Aleksandrov-Bakelman (with $g = 0$ outside Ω^*) is equivalent to a generalized solution of Brenier [4].

Lemma 3.4. *Let (u, v) be a maximizer of the functional I over K. Then u is a generalized solution of (2.17) and (2.18).*

Conversely, if u is a generalized solution of (2.17) and (2.18), then (u, v) is a maximizer of the functional I over K, where v is the c-transform of u.

Proof. Let (u, v) be a maximizer. Then by (2.7), u is c-concave and v is c^*-concave, and v is the c-transform of u. By [6,15], the mapping T_u, as determined by (2.12), is a measure preserving optimal mapping. Hence T_u is a one to one mapping from $\{x \in \Omega \mid f(x) > 0\}$ to $\{y \in \Omega^* \mid g(y) > 0\}$ almost everywhere. Hence u is a generalized solution of (2.17). The assumption (2.1) implies that (3.8) holds.

Conversely, if u is a generalized solution of (2.17) and (2.18), then v is a generalized solution of (2.20) and (2.21). Hence T_u is a one to one mapping from $\{x \in \Omega \mid f(x) > 0\}$ to $\{y \in \Omega^* \mid g(y) > 0\}$ almost everywhere. By (3.7) we have

$$\int_{\Omega^*} h(y)g(y) = \int_{\Omega} h(T_u(x))f(x))$$

for any continuous function $h \in C(\Omega^*)$. It follows that

$$\begin{aligned}\int u(x)f(x)dx + \int v(y)g(y)dy &= \int (u(x)f(x) + v(T_u(x))f(x))dx \\ &= \int c(x - T_u(x))f(x)dx. \end{aligned} \tag{3.11}$$

Hence (u, v) is a maximizer by (2.6). □

The existence and uniqueness of maximizers to the functional I can be found in [6,15], where the cost function is assumed to be of the form $c(x, y) = \tilde{c}(x-y)$ for some strictly convex or concave function $\tilde{c}$. But the argument there applies to general smooth cost functions satisfying (A1). Note that the uniqueness of optimal mappings in [6,15] implies the uniqueness of potential functions (up to a constant). Namely if (u_1, v_1) and (u_2, v_2) are two maximizers, then $Du_1 = Du_2$ a.e. on $\{f > 0\}$. By Lemma 3.4 we have accordingly the following existence and uniqueness of generalized solutions.

Theorem 3.1. *Let Ω and Ω^* be two bounded domains in $\mathbf{R}^n$. Suppose f and g are two positive, bounded, measurable functions defined respectively on Ω and Ω^*, satisfying condition (2.1). Suppose the cost function $c(x, y) \in C^2(\mathbf{R}^n \times \mathbf{R}^n)$ and satisfies assumption (A1). Then there is a unique generalized solution to (2.17) (2.18).*

Remark 3.2. The existence and uniqueness of generalized solutions can also be proved by the Perron method; see [32] for the treatment of the reflector antenna design problem, which is a special optimal transportation problem [33]. See also Section 7.1 below.

4. Second derivative estimates

Consider the equation

$$\det w_{ij} = \varphi \quad \text{in} \quad \Omega \tag{4.1}$$

where $w_{ij} = c_{ij}(x, T(x)) - u_{ij}$, and $\varphi = |\det(D^2_{xy}c)|\frac{f(x)}{g(T(x))}$. Suppose $\varphi > 0$ and the matrix $\{w_{ij}\}$ is positive definite. Write (4.1) in the form

$$\log \det w_{ij} = \psi, \tag{4.2}$$

where $\psi = \log \varphi$. Then we have, by differentiation,

$$\begin{aligned} &w^{ij} w_{ij,k} = \psi_k \\ &w^{ij} w_{ij,kk} = \psi_{kk} + w^{is} w^{jt} w_{ij,k} w_{st,k} \geq \psi_{kk}, \end{aligned} \tag{4.3}$$

where (w^{ij}) is the inverse of (w_{ij}). We use the notation $w_{ij,k} = \frac{\partial}{\partial x_k} w_{ij}$, $c_{ijk} = \frac{\partial}{\partial x_k} c_{ij}(x, y)$, $c_{ij,k} = \frac{\partial}{\partial y_k} c_{ij}(x, y)$, $T_{s,k} = \frac{\partial}{\partial x_k} T_s$ etc. That is

$$\begin{aligned} &-w^{ij}[u_{ijk} - c_{ijk} - c_{ij,s} T_{s,k}] = \psi_k, \\ &-w^{ij}[u_{ijkk} - c_{ijkk} - 2c_{ijk,s} T_{s,k} - c_{ij,s} T_{s,kk} - c_{ij,st} T_{s,k} T_{t,k}] \geq \psi_{kk}. \end{aligned} \tag{4.4}$$

Let

$$G(x,\xi)=\eta^2(x)w_{\xi\xi}, \tag{4.5}$$

where η is a cut-off function, and for a vector $\xi\in\mathbf{R}^n$ we denote $w_{\xi\xi}=\sum\xi_i\xi_j w_{ij}$. Suppose G attains a maximum at x_0 and $\xi_0\in S^{n-1}$. By a rotation of the coordinate system we assume $\xi_0=e_1$. At x_0 we have

$$\begin{aligned}(\log G)_i &= \frac{w_{11,i}}{w_{11}}+2\frac{\eta_i}{\eta}=0,\\ (\log G)_{ij} &= \frac{w_{11,ij}}{w_{11}}-\frac{w_{11,i}w_{11,j}}{w_{11}^2}+2\frac{\eta_{ij}}{\eta}-2\frac{\eta_i\eta_j}{\eta^2}\\ &= \frac{w_{11,ij}}{w_{11}}+2\frac{\eta_{ij}}{\eta}-6\frac{\eta_i\eta_j}{\eta^2}.\end{aligned} \tag{4.6}$$

Hence

$$0\ge w_{11}\sum w^{ij}(\log G)_{ij}=w^{ij}w_{11,ij}+2\frac{w_{11}}{\eta}w^{ij}\eta_{ij}-6w_{11}w^{ij}\frac{\eta_i\eta_j}{\eta^2}.$$

We have

$$w_{11,ij}=c_{11ij}+c_{11i,s}T_{s,j}+c_{11j,s}T_{s,i}+c_{11,s}T_{s,ij}+c_{11,st}T_{s,i}T_{t,j}-u_{11ij}.$$

It follows that

$$0\ge w^{ij}[c_{11,s}T_{s,ij}+c_{11,st}T_{s,i}T_{t,j}-u_{11ij}]-K,$$

where we use K to denote a positive constant satisfying

$$K\le C(1+w_{11}^2+\frac{w_{11}}{\eta^2}\sum w^{ii}).$$

By (4.4),

$$-w^{ij}u_{ij11}\ge -w^{ij}[c_{ij,s}T_{s,11}+c_{ij,st}T_{s,1}T_{t,1}]-K.$$

Hence we obtain

$$\begin{aligned}0\ge &-w^{ij}[c_{ij,s}T_{s,11}+c_{ij,st}T_{s,1}T_{t,1}]\\ &+w^{ij}[c_{11,s}T_{s,ij}+c_{11,st}T_{s,i}T_{t,j}]-K.\end{aligned} \tag{4.7}$$

Recall that

$$w_{ij}=c_{ij}-u_{ij}=-c_{i,k}T_{k,j}. \tag{4.8}$$

From the first relation,

$$\begin{aligned}w_{ki,j}&=c_{kij}+c_{ki,p}T_{p,j}-u_{kij}\\ &=w_{ij,k}+c_{ki,p}T_{p,j}-c_{ij,p}T_{p,k}.\end{aligned} \tag{4.9}$$

From the second one we have

$$-w_{ki,j}=c_{kj,p}T_{p,i}+c_{k,pq}T_{p,i}T_{q,j}+c_{k,p}T_{p,ij}. \tag{4.10}$$

Hence

$$\begin{aligned}T_{p,ij}&=-c^{p,k}w_{ki,j}-c^{p,k}[c_{kj,s}T_{s,i}+c_{k,st}T_{s,i}T_{t,j}]\\ &=-c^{p,k}w_{ij,k}-c^{p,k}[c_{ki,s}T_{s,j}-c_{ij,s}T_{s,k}+c_{kj,s}T_{s,i}+c_{k,st}T_{s,i}T_{t,j}].\end{aligned} \tag{4.11}$$

Note that by (4.8),

$$T_{k,j} = -c^{k,i} w_{ij}. \tag{4.12}$$

We have $|T| \le C w_{11}$. Hence

$$\begin{aligned} w^{ij} T_{p,ij} &= -c^{p,k} w^{ij} w_{ij,k} - c^{p,k} w^{ij} [2c_{ki,s} T_{s,j} - c_{ij,s} T_{s,k} + c_{k,st} T_{s,i} T_{t,j}] \\ &= -c^{p,k} w^{ij} c_{k,st} T_{s,i} T_{t,j} + O(K). \end{aligned} \tag{4.13}$$

By (4.12),

$$w^{ij} T_{p,i} T_{q,j} = c^{p,s} c^{q,t} w^{ij} w_{si} w_{tj} = O(w_{11}). \tag{4.14}$$

Inserting (4.13) (4.14) into (4.7) we obtain

$$0 \ge -w^{ij} [c_{ij,s} T_{s,11} + c_{ij,st} T_{s,1} T_{t,1}] - K. \tag{4.15}$$

We have from (4.11)

$$\begin{aligned} T_{s,11} &= -c^{s,k} w_{11,k} - c^{s,k} [2c_{k1,p} T_{p,1} - c_{11,p} T_{p,k} + c_{k,pq} T_{p,1} T_{q,1}] \\ &= -c^{s,k} c_{k,pq} T_{p,1} T_{q,1} + O(K). \end{aligned}$$

Hence we obtain, by (4.12)

$$\begin{aligned} 0 &\ge -w^{ij} c_{ij,st} T_{s,1} T_{t,1} + c^{l,k} c_{k,st} c_{ij,l} w^{ij} T_{s,1} T_{t,1} - K \\ &= w^{ij} [c^{k,l} c_{ij,k} c_{l,st} - c_{ij,st}] c^{s,p} c^{t,q} w_{p1} w_{q1} - K, \end{aligned} \tag{4.16}$$

where the summation runs over all parameters from 1 to n.

Let us assume by a rotation of coordinates that the matrix $\{w_{ij}\}$ is diagonal at x_0. Then we obtain

$$w^{ii} [c^{k,l} c_{ii,k} c_{l,st} - c_{ii,st}] c^{s,1} c^{t,1} w_{11}^2 \le K.$$

By assumption (A3), we obtain

$$w_{11}^2 \sum w^{ii} \le K.$$

Observing that

$$\sum_{i=1}^n w^{ii} \ge \sum_{i=2}^n w^{ii} \ge [\prod_{i=2}^n w^{ii}]^{\frac{1}{n-1}} \ge C w_{11}^{\frac{1}{n-1}},$$

we obtain $\eta^2 w_{11} \le C$, namely $G \le C$. We have thus proved

Theorem 4.1. *Let $u \in C^4(\Omega)$ be a c-concave solution of (4.1). Suppose assumptions (A1)-(A3) are satisfied. Then we have the a priori second order derivative estimate*

$$|D^2 u(x)| \le C, \tag{4.17}$$

where C depends on n, $dist(x, \partial\Omega)$, $sup_\Omega |u|$, ψ up to its second derivatives, the cost function c up to its fourth order derivatives, the constant c_0 in (A3), and a positive lower bound of $|\det D^2_{xy} c|$.

With the estimate (4.17), equation (4.1) becomes a uniformly elliptic equation, and so higher order derivative estimates follows from the elliptic regularity theory [17].

Instead of the auxiliary function G given in (4.5), we can choose the auxiliary function $G = \eta^2 \sum w_{kk} = \Delta w$. Then by the same computation as above, we have

$$w^{ij}[c^{k,l}c_{ij,k}c_{l,st} - c_{ij,st}]c^{s,p}c^{t,q}w_{pm}w_{qm} \le K. \tag{4.18}$$

Hence we also need to assume condition (A3).

Remark 4.1. From our preceding argument or by direct calculation we have the formula

$$A_{ij,kl} =: \frac{\partial^2 A_{ij}}{\partial z_k \partial z_l} = (c_{ij,rs} - c^{p,q}c_{ij,p}c_{q,rs})c^{r,k}c^{s,l}, \tag{4.19}$$

where

$$A_{ij}(x,z) = c_{ij}(x, T(x,z)), \tag{4.20}$$

and $T(x,z)$ is the mapping determined by (2.12), namely $D_x c(x, T(x,z)) = z$. Consequently condition (A3) may be written in the form

$$A_{ij,kl}\xi_i\xi_j\eta_k\eta_l \le -c_0|\xi|^2|\eta|^2 \tag{4.21}$$

for all $\xi \perp \eta \in \mathbf{R}^n$.

Moreover our proof of Theorem 4.1 extends to Monge-Ampère equations of the form

$$\det[\mathcal{A}(x,u,Du) - D^2 u] = \mathcal{B}(x,u,Du), \tag{4.22}$$

where $\mathcal{A}$ and $\mathcal{B}$ are respectively C^2 matrix and scalar valued functions on $\Omega \times \mathbf{R} \times \mathbf{R}^n$ with

$$A_{ij,kl}(x,u,z) = \frac{\partial^2 A_{ij}}{\partial z_k \partial z_l}(x,u,z)$$

satisfying (4.21) with respect to the gradient variables and $\mathcal{B} > 0$. Any elliptic solution $u \in C^4(\Omega)$ of (4.22) will satisfy an interior estimate of the form (4.17), with constant C depending on n, $dist(x, \partial\Omega)$, $sup_\Omega(|u| + |Du|)$, $\mathcal{A}$ and $\mathcal{B}$. The Heinz-Lewy example [26] shows that there is no C^1 regularity for equation (4.22) without some restriction on $\mathcal{A}$.

If $D_x^2 c$ is positive definite, we can also choose the auxiliary function

$$G = \eta^2 \sum c^{ij} w_{ij}, \tag{4.23}$$

where $\{c^{ij}\}$ is the inverse matrix of $\{c_{ij}\}$. The advantage of this function is that it is invariant under linear transformation. Denote $H = \sum c^{ij} w_{ij}$. Suppose G attains a maximum at x_0. Then we have, at x_0,

$$\begin{aligned} 0 \ge H \sum w^{ij}(\log G)_{ij} &\ge \sum w^{ij} H_{ij} - K \\ &= w^{ij}\big[c^{kl}w_{kl,ij} - 2c^{ks}c^{lt}(\partial_{x_i}c_{st})w_{kl,j} \\ &\quad -c^{ks}c^{lt}(\partial^2_{x_i x_j}c_{st})w_{kl} + 2c^{kp}c^{sq}c^{lt}(\partial_{x_i}c_{st})(\partial_{x_j}c_{pq})w_{kl}\big] - K. \end{aligned}$$

To control the terms

$$A = w^{ij}\big[-2c^{ks}c^{lt}(\partial_{x_i}c_{st})w_{kl,j} + 2c^{kp}c^{sq}c^{lt}(\partial_{x_i}c_{st})(\partial_{x_j}c_{pq})w_{kl}\big],$$

one observes that G is invariant under linear transformations. Hence one may assume by a linear transformation that $\{c_{ij}\}$ is the unit matrix and w_{ij} is diagonal. Then

$$A = 2w^{ii}\big[-(\partial_{x_i}c_{kl})w_{kl,j} + (\partial_{x_i}c_{kl})^2 w_{kk}\big] \geq -\frac{1}{2}w^{ii}w^{kk}w^2_{kl,i}.$$

Hence A can be controlled by the term $w^{is}w^{jt}w_{ij,k}w_{st,k}$ in (4.3) (note that $w^{ii}(\partial_{x_i}c_{kl})^2 w_{kk} \leq K$ by (4.12)-(4.14)). It follows that

$$w^{ij}c^{kl}w_{kl,ij} - c^{ks}c^{lt}w_{kl}w^{ij}(\partial^2_{x_ix_j}c_{st}) \leq K. \tag{4.24}$$

The first term can be estimated in the same way as (4.16) or (4.18). For the second one, we have by (4.12)-(4.14) that

$$-c^{ks}c^{lt}w_{kl}w^{ij}(\partial^2_{x_ix_j}c_{st}) = c^{ks}c^{lt}c_{st,p}c_{ij,q}c^{p,a}c^{q,b}w_{kl}w_{ab}w^{ij} - K.$$

Hence by (4.18), we obtain

$$\begin{aligned} &c^{ab}w^{ij}[c^{k,l}c_{ij,k}c_{l,st} - c_{ij,st}]c^{s,p}c^{t,q}w_{pa}w_{qb} \\ &+c^{ks}c^{lt}c_{st,p}c_{ij,q}c^{p,a}c^{q,b}w_{kl}w_{ab}w^{ij} \leq K. \end{aligned} \tag{4.25}$$

If the matrix $\{c_{ij}\}$ is negative, we replace the auxiliary function G in (4.23) by $G = \eta^2\sum(-c^{ij})w_{ij}$ and obtain

$$-c^{ab}w^{ij}[c^{k,l}c_{ij,k}c_{l,st} - c_{ij,st}]c^{s,p}c^{t,q}w_{pa}w_{qb} - c^{ks}c^{lt}c_{st,p}c_{ij,q}c^{p,a}c^{q,b}w_{kl}w_{ab}w^{ij} \leq K.$$

If $\{c_{ij}\}$ is not definite, we can replace G by $G = \eta^2\sum c^{ki}c^{jk}w_{ij}$ and obtain similar sufficient condition.

To verify (4.25), one may choose a proper coordinate system such that $\{c_{ij}\} = I$ is the unit matrix and $\{c^{i,j}\} = -I$ at x_0, and $\{w_{ij}\}$ is diagonal, as we will do in Section 6. Then (4.25) can be rewritten as

$$(-c_{ii,k}c_{k,jj} - c_{ii,jj})w^2_{jj}w^{ii} + c_{kk,l}c_{ii,l}w_{kk}w_{ll}w^{ii} < K. \tag{4.26}$$

5. Proof of Theorem 2.1

With the a priori estimates established in Section 4, we need only to show that a generalized solution of (2.17) (2.18) can locally be approximated by smooth ones.

We will use the notion of extreme points of convex sets. For a bounded convex set $E \subset \mathbf{R}^n$, we say x_0 is an extreme point of E if there is a hyperplane P such that $P \cap E = \{x_0\}$. It is well known that any point $x \in E$ can be represented as a linear combination of extreme points of E, namely there exists extreme points $x_1, \cdots, x_k$ and nonnegative constants $\alpha_1, \cdots, \alpha_k$ with $\sum \alpha_i = 1$ such that $x = \sum \alpha_i x_i$.

We also need an obvious property of concave functions. That is for any $x_0 \in \Omega$ and any extreme point $y \in \partial^+u(x_0)$, there is a sequence of points $\{x_i\} \subset \Omega - \{x_0\}$ such that u is twice differentiable at x_i and $Du(x_i) \to y$.

Lemma 5.1. *Let u be a generalized solution of (2.17). Suppose $f > 0$ in Ω, and Ω^* is c^*-convex relative to Ω. Then $T_u(\Omega) \subset \overline{\Omega}^*$.*

Proof. Let G denote the set of points where u is twice differentiable. Then for any $x \in G$, $T_u(x)$ is a single point. Moreover, for any $x_0 \in \Omega$ and any sequence $(x_i) \subset G$ such that $x_i \to x_0$ and $T_u(x_i) \to y_0$ for some $y_0 \in \mathbf{R}^n$, we have $y_0 \in T_u(x_0)$. It follows that if there is a point $x_0 \in G$ such that $T_u(x_0) \notin \overline{\Omega}^*$, then for almost all $x \in \Omega$, close to x_0, we have $T_u(x) \notin \overline{\Omega}^*$. The mass balance condition implies this is impossible.

Observe that if $x_i \to x_0$ and $T_u(x_i) \to y_0$, then $y_0 \in T_u(x_0)$. Hence for any point $x_0 \in \Omega - G$, if $T_u(x_0)$ is a single point, then it falls in $\overline{\Omega}^*$. If $T_u(x_0)$ contains more than one point, then the super-gradient $\partial^+ u(x_0)$ is a convex set. For every extreme point $y \in \partial^+ u(x_0)$, let z be the unique solution of $y = D_x c(x_0, z)$. Then $z \in \overline{\Omega}^*$ since there is a sequence $(x_i) \in G$ such that $T_u(x_i) \to z$. Hence $T_u(x_0) \subset \overline{\Omega}^*$ as Ω^* is c^*-convex and any point $z \in \partial^+ u(x_0)$ can be represented as a linear combination of extreme points of $\partial^+ u(x_0)$. $\square$

Lemma 5.2. *(Monotonicity) Let Ω be a bounded domain in $\mathbf{R}^n$ and $u, v \in C^0(\overline{\Omega})$, c-concave functions satisfying $u \le v$ in Ω and $u = v$ on $\partial\Omega$. Then $T_u(\Omega) \subset T_v(\Omega)$.*

Proof. This result follows, as in the Monge-Ampère case, by vertical translation upwards of c-support functions for u. $\square$

From Lemma 5.2 follows a simple comparison principle, namely if $u, v \in C^0(\overline{\Omega})$ are c-concave, $u \le v$ on $\partial\Omega$, $\mu_{u,g} < \mu_{v,g}$ in Ω, then $u \le v$ in Ω. For a more general result, based on the Aleksandrov argument [1], see [16].

Proof of Theorem 2.1. It suffices to show that u is smooth in any sufficiently small ball $B_r \subset\subset \Omega$. For this purpose we consider the approximating Dirichlet problems

$$\begin{aligned} \det(D_x^2 c - D^2 w) &= |\det(D_{xy}^2 c)| \frac{f}{g \circ T} \quad \text{in} \quad \mathrm{B_r}, \\ w &= u_m \qquad\qquad\qquad \text{on} \quad \partial\mathrm{B_r}, \end{aligned} \tag{5.1}$$

where $\{u_m\}$ is a sequence of smooth functions converging uniformly to u. We want to prove that (5.1) has a C^3 smooth solution $w = w_m$ such that the matrix $\{D_x^2 c - D^2 w\}$ is positive definite.

First we show that if w is a C^2 solution of (5.1), the cost function $c(x, y)$ satisfies assumption (A3) at any points $x \in B_r$ and $y \in T_w(B_r)$, so that the a priori estimate in §4 applies. Indeed we may assume that (A3) holds in $\tilde{\Omega} \times \tilde{\Omega}^*$, where $\tilde{\Omega}$ and $\tilde{\Omega}^*$ are respectively c-convex neighborhoods of $\overline{\Omega}$ and c^*-convex neighborhoods of $\overline{\Omega}^*$. By the semi-concavity of u, there exists a constant C such that the function $\tilde{u} = u - C|x|^2$ is concave and $|Du| < C$. For $h > 0$ sufficiently small, we let

$$\tilde{u}_h(x) = \int \rho(\frac{x - y}{h}) u(y), \tag{5.2}$$

denote the mollification of $\tilde{u}$, with respect to a symmetric mollifier $\rho \ge 0, \in C_0^\infty(\mathbf{R}^n)$, $\int \rho = 1$, $\mathrm{supp}\rho \subset B_1(0)$. Setting

$$u_m = \tilde{u}_{h_m} + C|x|^2, \tag{5.3}$$

where $h_m \to 0$, we then have $u_m \to u$ uniformly and

$$D^2 u_m \leq C, \qquad |Du_m| \leq C. \tag{5.4}$$

Moreover for sufficiently large k and small r, the functions

$$v = v_m = u_m + k(r^2 - |x|^2) \tag{5.5}$$

will be upper barriers for (5.1), that is, writing (5.1) in the form (4.22), we have $A(x, Dv) > D^2 v$ and

$$\det(A(x, Dv) - D^2 v) \geq B(x, Dv) \quad \text{in } \mathrm{B_r}, \tag{5.6}$$
$$v = u_m \quad \text{on } \partial\mathrm{B_r}.$$

Consequently by the classical comparison principle [17], we have

$$w_m \leq v_m \quad \text{in } \mathrm{B_r},$$

whence

$$T_{w_m}(\partial B_r) \subset T_{v_m}(\overline{B}_r).$$

It follows that $T_{w_m}(B_r) \subset \tilde{\Omega}$ if kr is small. Hence c is well defined on $\overline{B}_r \times \overline{T}_w(\overline{B}_r)$.

The preceding arguments yield a priori bounds for solutions and their gradients of the Dirichlet problem (5.1). To conclude the existence of globally smooth solutions, we need global second derivative bounds. The argument of the previous section, with $\eta = 1$, clearly implies a priori bounds for second derivatives in terms of their boundary estimates. The latter can be established similarly to [32], (see also [17,19]), or by directly using the method introduced in [27,28] for the Monge-Ampère equation. The key observation again is that functions of the form $k(r^2 - |x|^2)$ provide appropriate barriers for large k and small r. From the interior estimates in Theorem 4.1, we finally infer the existence of locally smooth elliptic solutions w of the Dirichlet problem (5.1) with $w = u$ on ∂B_r. Finally it remains to show that $w = u$ in B_r. By perturbation of w, we may suppose that the set $\omega_\varepsilon = \{x \in B_r \mid w(x) > u(x) + \varepsilon\}$ has sufficiently small diameter, with w c-concave a strict supersolution of equation (5.1) in ω_ε, that is $\mu_{w,g} < \mu_{u,g}$ in ω_ε. From Lemma 5.2, we thus conclude $w \leq u$ in B_r and similarly $w \geq u$ follows. □

Our argument above may also be used to prove interior regularity for the generalized Dirichlet problem, the solvability of which depends upon the existence of barriers. Note that the condition of c-convexity is not directly relevant to the Dirichlet problem.

6. Verification of assumption (A3)

In this section we give some cost functions which satisfy assumption (A3). First we consider cost functions of the form

$$c(x, y) = \varphi(\frac{1}{2}|x - y|^2). \tag{6.1}$$

In general a function of the form (6.1) does not satisfies (A3), see for example §7.5.

For the cost function (6.1), we have $\partial_{y_i} c = -\partial_{x_i} c$. We can write $c(x,y)$ as $c(x-y)$. Then (A3) is equivalent to

$$\sum_{k,l,s,t} (c^{kl} c_{iik} c_{stl} - c_{iist}) c^{sj} c^{tj} \geq c_0 \tag{6.2}$$

for any fixed $1 \leq i, j \leq n$, $i \neq j$, where (c^{ij}) is the inverse matrix of (c_{ij}).

It is hard to verify (6.2) directly, as it involves fourth order derivatives. To verify (6.2), we recall the estimate (4.16), which can be written as

$$w^{ij}[c^{kl} c_{ijk} c_{lst} - c_{ijst}] c^{sp} c^{tq} w_{p\xi} w_{q\xi} \leq K, \tag{4.16$'$}$$

where ξ is the unit vector in which the auxiliary function $G = \eta(x) w_{\xi\xi}$ attains its maximum at some point x_0. Let $y_0 = T_u(x_0)$ and assume by a rotation of coordinates that $x_0 - y_0 = (r, 0, ..., 0)$. We first make a linear transformation (dilation) such that $\{c_{ij}\} = \varphi' I$ at $x_0 - y_0$, then make a rotation of coordinates such that $\{w_{ij}\}$ is diagonal. Then the unit vector $\xi = \alpha_k e_k$ for some constants α_k, and $(4.16)'$ becomes

$$w^{ii}[c_{iik} c_{kst} - c_{iist}] \alpha_s \alpha_t w_{ss} w_{tt} \leq K, \tag{4.16$''$}$$

where $\{e_k\}$ are the axial directions. After the coordinate transformations, it suffices to verify

$$\Sigma_k c_{iik} c_{kjj} - c_{iijj} \geq c_0 > 0 \tag{6.3}$$

for any $i \neq j$. Namely (6.3) implies (A3).

We compute

$$\begin{aligned} c_i &= \varphi' r_i, \\ c_{ij} &= \varphi' \delta_{ij} + \varphi'' r_i r_j \end{aligned}$$

where $r_i = x_i - y_i$. At the point $(r, 0, ..., 0)$, we have

$$\{c_{ij}\} = diag(\varphi' + r^2 \varphi'', \varphi', ..., \varphi').$$

Suppose that $\frac{\varphi'}{\varphi' + r^2 \varphi''} > 0$. Let

$$\begin{aligned} x_1 &= \beta \tilde{x}_1, \\ x_i &= \tilde{x}_i, \quad i > 1, \end{aligned}$$

where $\beta = \sqrt{\frac{\varphi'}{\varphi' + r^2 \varphi''}}$. Then

$$c(\tilde{x}) = \varphi(\frac{1}{2}(\beta^2 \tilde{x}_1^2 + \tilde{x}_2^2 + ... + \tilde{x}_n^2))$$

with

$$\{c_{ij}\} = \varphi' I \quad at \quad \tilde{x}_0 = (\frac{r}{\beta}, 0, ..., 0) \tag{6.4}$$

in the coordinates $\tilde{x}$, where I is the unit matrix.

We make a rotation $\tilde{x} = Ay$, where A is an orthogonal matrix. Then

$$c(y) = \varphi(\frac{1}{2}y'By),$$

where

$$B = A'\hat{B}A, \quad \hat{B} = diag(\beta^2, 1..., 1).$$

Denote $z = By$. We have, at the point $y_0 = A'\tilde{x}_0$,

$$\begin{aligned} |z|^2 &= z' \cdot z = y'A'\hat{B}^2Ay \\ &= \tilde{x}'\hat{B}^2\tilde{x} = \beta^2 r^2 = \frac{\varphi' r^2}{\varphi' + r^2\varphi''}. \end{aligned} \tag{6.5}$$

In the y-coordinates, we have

$$\begin{aligned} c_i &= \varphi' z_i, \\ c_{ij} &= \varphi' b_{ij} + \varphi'' z_i z_j, \\ c_{ijk} &= \varphi''(b_{ij}z_k + b_{ik}z_j + b_{jk}z_i) + \varphi''' z_i z_j z_k, \\ c_{ijkl} &= \varphi''(b_{ij}b_{kl} + b_{ik}b_{jl} + b_{il}b_{jk}) \\ &\quad +\varphi'''(b_{ij}z_k z_l + b_{ik}z_j z_l + b_{il}z_j z_k + b_{jk}z_i z_l + b_{jl}z_i z_k + b_{kl}z_i z_j) \\ &\quad +\varphi'''' z_i z_j z_k z_l. \end{aligned}$$

By (6.4) and since A is orthogonal, we have $\{c_{ij}\} = \varphi' I$ in the y-coordinates. From the above formula for c_{ij}, we obtain

$$b_{ij} = \delta_{ij} - \frac{\varphi''}{\varphi'} z_i z_j.$$

Hence

$$\begin{aligned} c_{ijk} &= \varphi''(\delta_{ij}z_k + \delta_{ik}z_j + \delta_{jk}z_i) + (\varphi''' - 3\frac{(\varphi'')^2}{\varphi'})z_i z_j z_k, \\ c_{ijkl} &= \varphi''(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &\quad +(\varphi''' - \frac{(\varphi'')^2}{\varphi'})(\delta_{ij}z_k z_l + \delta_{ik}z_j z_l + \delta_{il}z_j z_k + \delta_{jk}z_i z_l + \delta_{jl}z_i z_k + \delta_{kl}z_i z_j) \\ &\quad +[\varphi'''' - 6\frac{\varphi'''\varphi''}{\varphi'} + 3\frac{(\varphi'')^3}{(\varphi')^2}]z_i z_j z_k z_l. \end{aligned}$$

Denote $F =: \Sigma_k c^{kk} c_{iik} c_{jjk} - c_{iijj}$. From the above formulae and noting that $i \neq j$, we obtain

$$\begin{aligned} F = &\frac{z_k^2}{\varphi'}\left[\varphi''(1 + 2\delta_{ik}) + (\varphi''' - 3\frac{(\varphi'')^2}{\varphi'})z_i^2\right]\left[\varphi''(1 + 2\delta_{jk}) + (\varphi''' - 3\frac{(\varphi'')^2}{\varphi'})z_j^2\right] \\ &-[\varphi'' + (\varphi''' - \frac{(\varphi'')^2}{\varphi'})(z_i^2 + z_j^2) + (\varphi'''' - 6\frac{\varphi'''\varphi''}{\varphi'} + 3\frac{(\varphi'')^3}{(\varphi')^2})z_i^2 z_j^2]. \end{aligned} \tag{6.6}$$

First let us check the cost function

$$c(x, y) = \sqrt{1 + |x - y|^2}. \tag{6.7}$$

The corresponding Monge-Ampère equation is

$$\det(-D^2u + (1 - |Du|^2)^{1/2}(\delta_{ij} - u_iu_j)) = (1 - |Du|^2)^{(n+2)/2} f(x)/g(T(x)). \tag{6.8}$$

The negative sign on the left hand side is due to the c-concavity of u. For the cost function (6.7), we have $\phi(t) = \sqrt{1+2t}$, and by direct computation,

$$\begin{aligned} &\varphi''' - 3\frac{(\varphi'')^2}{\varphi'} = 0, \\ &\varphi'''' - 6\frac{\varphi'''\varphi''}{\varphi'} + 3\frac{(\varphi'')^3}{(\varphi')^2} = 0. \end{aligned} \tag{6.9}$$

Hence

$$\begin{aligned} F &= -\varphi'' + \frac{(\varphi'')^2}{\varphi'}\left[\Sigma_k(1+2\delta_{ik})(1+2\delta_{jk})z_k^2 - 2(z_i^2 + z_j^2)\right] \\ &= -\varphi'' + |z|^2\frac{(\varphi'')^2}{\varphi'} \geq -\varphi'' > 0. \end{aligned} \tag{6.10}$$

The matrix $\mathcal{A}$ in (6.8) is given by

$$\mathcal{A}(x,z) = \mathcal{A}(z) = (1 - |z|^2)^{1/2}(\delta_{ij} - z_iz_j) \tag{6.11}$$

and our calculations above show that

$$A_{ij,kl}\xi_i\xi_j\eta_k\eta_l > 0 \tag{6.12}$$

for all $\xi, \eta \in \mathbf{R}^n$, $\xi \perp \eta$, $|z| < 1$. The condition (6.12) may also be verified directly from (6.11). We remark that (6.10) and (6.12) are not true for all $|z| < 1$ if $i = j$ in (6.3) or $\xi = \eta$ in (6.12).

Note also for (6.7) that (A1) only holds for $|z| < 1$. However this does not affect our proof on Theorems 2.1 and 3.1.

By a linear transformation, (A3) is also satisfied for

$$c(x,y) = [1 + a_{ij}(x_i - y_i)(x_j - y_j)]^{1/2},$$

where $\{a_{ij}\}$ is a positive matrix. Note that the graph of the function $[1 + a_{ij}x_ix_j]^{1/2}$ is a *hyperboloid*.

Next we verify (A3) for the cost function

$$c(x,y) = \sqrt{1 - |x-y|^2} \tag{6.13}$$

For this cost function, equation (2.17) takes the form

$$\det(-D^2u - (1 + |Du|^2)^{1/2}(\delta_{ij} + u_iu_j)) = (1 + |Du|^2)^{(n+2)/2} f(x)/g(T(x)). \tag{6.14}$$

For the cost function (6.13), we have $\varphi(t) = \sqrt{1-2t}$. Hence (6.9) holds and similarly to (6.10) we have

$$F = -\varphi'' + |z|^2\frac{(\varphi'')^2}{\varphi'}.$$

By (6.5) and noting that $\varphi' < 0$, we obtain

$$F \geq -\frac{\varphi'\varphi''}{\varphi' + r^2\varphi''} > 0, \tag{6.15}$$

so that (A3) is satisfied if $|x-y| < 1$ for all $x \in \overline{\Omega}$ and $y \in \overline{\Omega}^*$. Unlike the case of our previous example, (A3) holds without the restriction $\xi \perp \eta$. This is easily seen directly from (6.14) and is equivalent to the function $A_{ij}\xi_i\xi_j(x,z)$ being uniformly convex with respect to z.

By a linear transformation, one sees that (A3) is satisfied for

$$c(x,y) = [1 - a_{ij}(x_i - y_i)(x_j - y_j)]^{1/2},$$

where $\{a_{ij}\}$ is a positive matrix. The graph of the function $[1 - a_{ij}x_ix_j]^{1/2}$ is an *ellipsoid*. Note that the function $\sqrt{1+|x|^2}$ is the Legendre transform of $-\sqrt{1-|x|^2}$.

The cost function $c(x,y) = \sqrt{1-|x-y|^2}$ is defined only for $|x-y| < 1$. Therefore we need to assume $d =: \sup\{|x-y| \mid x \in \Omega, y \in \Omega^*\} < 1$. As noted above, this does not affect our proof on Theorems 2.1 and 3.1.

Next we consider an extension of the cost function (6.7). That is

$$c(x,y) = (1 + \frac{1}{2}|x-y|^2)^p, \quad \frac{1}{2} \leq p < 1. \tag{6.16}$$

When $p = 1$, it is the quadratic function. We have $\varphi(t) = (1+t)^p$. Hence

$$\begin{aligned}
\varphi''' - 3\frac{\varphi''^2}{\varphi'} &= p(p-1)(1-2p)(1+t)^{p-3},\\
\varphi''' - \frac{\varphi''^2}{\varphi'} &= -p(p-1)(1+t)^{p-3},\\
\varphi'''' - 6\frac{\varphi'''\varphi''}{\varphi'} + 3\frac{\varphi''^3}{\varphi'^2} &= p(p-1)(p-3)(1-2p)(1+t)^{p-4}.
\end{aligned}$$

By (6.6),

$$\begin{aligned}
\frac{(1+t)^{2-p}}{p(1-p)}F = (1-p)\frac{z_k^2}{1+t}&\left[(1+2\delta_{ik}) + (1-2p)\frac{z_i^2}{1+t}\right]\cdot\\
&\left[(1+2\delta_{jk}) + (1-2p)\frac{z_j^2}{1+t}\right]\\
&+\left[1 - \frac{z_i^2+z_j^2}{1+t} + (3-p)(2p-1)\frac{z_i^2z_j^2}{(1+t)^2}\right]\\
= (1-p)&\left[\frac{|z|^2}{1+t} + 2\frac{z_i^2+z_j^2}{1+t} + (1-2p)\frac{|z|^2(z_i^2+z_j^2)}{(1+t)^2}\right.\\
&\left.+4(1-2p)\frac{z_i^2z_j^2}{(1+t)^2} + (1-2p)^2\frac{|z|^2z_i^2z_j^2}{(1+t)^3}\right]\\
&+\left[1 - \frac{z_i^2+z_j^2}{1+t} + (3-p)(2p-1)\frac{z_i^2z_j^2}{(1+t)^2}\right]
\end{aligned}$$

$$= \left[1-(2p-1)\frac{z_i^2+z_j^2}{(1+t)}+(2p-1)(3p-1)\frac{z_i^2z_j^2}{(1+t)^2}\right]$$
$$+(1-p)\frac{|z|^2}{1+t}\left[1+(1-2p)\frac{z_i^2+z_j^2}{1+t}+(1-2p)^2\frac{z_i^2z_j^2}{(1+t)^2}\right].$$

Hence $F>0$.

By a dilation we also obtain the interior regularity of potential functions for the cost function

$$c(x,y)=(\varepsilon+|x-y|^2)^{p/2} \tag{6.17}$$

where $1\le p<2$ and $\varepsilon>0$ is a constant. By sending $\varepsilon\to 0$, the cost function in (6.17) converges to

$$c(x,y)=|x-y|^p \quad 1\le p<2. \tag{6.18}$$

For the cost function (6.18), the best possible regularity for potential functions is $C^{1,p-1}$ for $1<p<2$ and $C^{0,1}$ when $p=1$, see §7.5. By the regularity for the cost function (6.17), it is reasonable to expect the $C^{1,\alpha}$ regularity for the cost function in (6.18) with $1<p<2$. We hope to treat the problem in a separate work.

Finally we consider cost functions determined by the distance between points on graphs of functions over $\mathbf{R}^n$. Suppose that f and g are C^2 functions defined on Ω and Ω^* respectively with graphs $\mathcal{M}_f$, $\mathcal{M}_g$ and let $X=(x,f(x))$, $Y=(y,g(y))$ denote points in $\mathcal{M}_f$ and $\mathcal{M}_g$. Then the cost function

$$\tilde{c}(x,y):=|X-Y|^2 \tag{6.19}$$

is equivalent to the cost function

$$c(x,y)=-(x\cdot y+f(x)g(y)). \tag{6.20}$$

By direct computation, we obtain

$$\begin{aligned}
c_{i,j} &= -\delta_{ij}-f_ig_j,\\
\det c_{i,j} &= (-1)^n(1+\nabla f\cdot\nabla g)\ne 0 \quad \text{if } \nabla \mathrm{f}\cdot\nabla \mathrm{g}\ne -1,\\
c^{i,j} &= -\big(\delta_{ij}-\frac{f_ig_j}{1+\nabla f\cdot\nabla g}\big),\\
c_{ij,k} &= -f_{ij}g_k,\\
c_{l,st} &= -f_lg_{st},\\
c_{ij,st} &= -f_{ij}g_{st},\\
\Sigma_{k,l}c^{k,l}c_{ij,k}c_{l,st}-c_{ij,st} &= \frac{f_{ij}g_{st}}{1+\nabla f\cdot\nabla g}
\end{aligned}$$

Consequently if f and g are uniformly convex or concave, with bounded gradients satisfying $\nabla f\cdot\nabla g>-1$, then (A3) is satisfied (for all $\xi,\eta\in\mathbf{R}^n$). Moreover, if

$$f_{ij}\xi_i\xi_j,\, g_{ij}\xi_i\xi_j\ge\lambda|\xi|^2 \quad \forall\,\xi\in\mathbf{R}^n,$$

and some $\lambda > 0$, we obtain, for fixed i, j,

$$\begin{aligned} F &= c^{s,j} c^{t,j} [c^{k,l} c_{ii,k} c_{l,st} - c_{ii,st}] \\ &= \frac{f_{ii} g_{st}}{1 + \nabla f \cdot \nabla g} c^{s,j} c^{t,j} \\ &\geq \frac{\lambda^2}{(1 + M_1^2)^3}. \end{aligned} \tag{6.21}$$

where $M_1 \geq \sup_\Omega |\nabla f|, \sup_{\Omega^*} |\nabla g|$. An example of a function satisfying the above conditions for arbitrary bounded $\Omega \subset \mathbf{R}^n$ is given by

$$f(x) = \sqrt{1 + |x|^2}, \tag{6.22}$$

in which case $\tilde{c}$ becomes the square of the distance between points on the hyperboloid, $x_{n+1}^2 = 1 + |x|^2$. Another example will be the paraboloid

$$f(x) = \varepsilon |x|^2 \tag{6.23}$$

for sufficiently small $\varepsilon > 0$, depending on Ω. In particular (A3) is thus satisfied for cost functions

$$c_\varepsilon(x, y) = -x \cdot y - \varepsilon^2 |x|^2 |y|^2$$

for sufficiently small ε, which perturbs the standard quadratic cost function $c(x, y) = -x \cdot y$.

There are other functions satisfying (A3). For example, by the computation in [18,32], the cost function for the reflector antenna design problem, see (7.7), satisfies a related condition on the sphere. If one projects it to the Euclidean space, it satisfies assumption (A3).

7. Remarks

7.1. The cost function $c(x, y) = |x - y|^2$

For this special cost function, the problem (2.17) (2.18) is reduced to

$$\begin{aligned} g(Du) \det D^2 u &= f(x), \\ Du(\Omega) &= \Omega^*, \end{aligned} \tag{7.1}$$

where $\frac{1}{2}|x|^2 - u$ is the potential function. The problem (7.1), namely prescribing the normal mapping image, was proposed in [20]. The global smooth *convex* solution in dimension 2 was first obtained in [12]. In high dimensions the existence and regularity of solutions were established in [7,8] and later in [30]. We remark that there is a simpler proof for the existence and interior regularity of solutions to (7.1) [11], which was used in [34] for the oblique boundary value problem for the Monge-Ampère equation.

Theorem 7.1. *Let Ω^* be a convex domain in $\mathbf{R}^n$. Suppose f and g are positively pinched functions satisfying (2.1). Then there is a generalized solution of (7.1), in the sense of Aleksandrov.*

Proof. Let Ψ_ε denote the set of convex sub-solutions w of the equation

$$g(Dw)\det D^2 w = f(x)e^{\varepsilon w}, \tag{7.2}$$

where ε is a small positive constant, such that $Dw(\Omega) \subset \Omega^*$. Let

$$u_\varepsilon(x) = \sup_{w\in\Psi_\varepsilon} w(x).$$

Then u_ε is a generalized solution of (7.2). By the convexity of Ω^*, we have $Du_\varepsilon(\Omega) \subset \overline{\Omega}^*$. By assumption (2.1), we have $\sup_\Omega u_\varepsilon > 0 > \inf_\Omega u_\varepsilon$. Hence u_ε converges as $\varepsilon \to 0$ to a generalized solution u of (7.1). By $Du_\varepsilon(\Omega) \subset \overline{\Omega}^*$ and the convexity of Ω^*, we have $Du(\Omega) \subset \overline{\Omega}^*$. The assumption (2.1) implies that $Du(\Omega) = \Omega^*$. □

It was also proved in [11] by constructing proper barriers that if $\partial\Omega$ is uniformly convex, then $Du(x) \to \partial\Omega^*$ as $x \to \partial\Omega$ [11]. Hence for any point $x_0 \in \Omega$, there is a linear function ℓ such that the set $\omega = \{x \in \Omega \mid u(x) < \ell(x)\}$ is contained in Ω and $x_0 \in \omega$. Hence if $f, g \in C^2$, by Pogorelov's interior second order derivative estimate [24], (7.1) becomes uniformly elliptic and so $u \in C^{3,\alpha}(\Omega)$ for any $\alpha \in (0, 1)$.

In the case when $\Omega = \mathbf{R}^n$, the existence of generalized solutions was proved by Pogorelov [23], see also [3,11]. For the existence result in Theorem 7.1, we remark that by approximation, it suffices to assume that f and g are nonnegative integrable functions satisfying (2.1). The convexity of Ω^* can also be dropped by first extending g to a bounded convex domain $\tilde\Omega^*$ such that $g = 0$ outside $\tilde\Omega^* - \Omega^*$ and then choosing a sequence of positive functions $g_m \to g$ uniformly and using Corollary 3.1, see [32]. In this case a generalized solution is defined as in (3.7). Recall that for the Monge-Ampère equation (7.1), the generalized solution of Aleksandrov is usually defined by

$$|\partial^+ u(E)| = \int_E f(x)/g(Du) \tag{7.3}$$

for any Borel set $E \subset \Omega$, where $\partial^+ u$ is the normal mapping of u. If $g > 0$, the integration in (7.3) is well defined as a convex function is differentiable almost everywhere. If Ω^* is convex and f, g are bounded positive functions, both definitions (3.7) and (7.3) are equivalent by Lemma 5.1. However if Ω^* is not convex, $Du(\Omega)$ may not be contained in $\overline{\Omega}^*$ completely. In this case we need to use definition (3.7) as $g = 0$ outside Ω^* by the mass balance condition (2.1).

7.2. The reflector antenna design problem

An important application of the optimal transportation theory is the design of the reflector antenna [33]. The problem is to construct a reflecting surface, which is a radial graph

$$\Gamma = \{x \cdot \rho(x) \mid \ x \in \Omega\}, \tag{7.4}$$

where Ω is a domain in the unit sphere S^2, such that a detector located at the origin can receive reflected rays from a given region Ω^* in the outer space, which is regarded as another domain in the sphere S^2. This problem is equivalent to solving the boundary value problem

$$\begin{gathered}\eta^{-2}\det(-\nabla_i\nabla_j\rho + 2\rho^{-1}\nabla_i\rho\nabla_j\rho + (\rho-\eta)\delta_{ij}) = f(x)/g(T(x)), \\ T(\Omega) = \Omega^*,\end{gathered} \tag{7.5}$$

where ∇ denotes the covariant derivative in a local orthonormal frame on S^2, $\eta = (|\nabla\rho|^2 + \rho^2)/2\rho$, and

$$T(x) = T_\rho(x) = x - 2\langle x, n\rangle n \tag{7.6}$$

with n the unit outward normal of Γ.

In [33] it was proved that problem (7.4) can be reduced to an optimal transportation problem with cost function

$$c(x, y) = -\log(1 - x \cdot y), \quad x \in \Omega, y \in \Omega^*. \tag{7.7}$$

More precisely, we have

Theorem 7.2. *Suppose Ω and Ω^* are disconnected, and f and g are bounded positive functions. Let (φ, ψ) be a maximizer of $\sup_{(u,v)\in K} I(u, v)$, where I and K are as (2.4) (2.5), with the cost function given in (7.7) above. Then $\rho = e^\varphi$ is a solution of (7.4).*

The existence and uniqueness of generalized solutions and the a priori estimates, corresponding respectively to Theorems 3.1 and 4.1, were established in [32]. In this case verification of conditions (A1) to (A3) is superfluous as the mapping T is already known by (7.6) and the Monge-Ampère equation (7.5) is in the form (4.22) automatically.

7.3. An example

In this subsection we show that if the domain Ω^* is not c^*-convex, then there exist smooth, positive functions f and g such that the generalized solution to (2.17) and (2.18) in Theorem 3.1 is not smooth. The argument here is similar to that in [32], p. 362, where it is shown that the solution to the reflector antenna design problem may not be smooth if the domain does not satisfy a geometric condition, which corresponds to the c^*-convexity introduced in this paper. The existence of non-smooth solutions can be proved in the following steps.

(i) If u is a smooth c-concave solution of (2.17) (2.18), then $T_u(\Omega) = \Omega^*$. This follows from (2.1) immediately.

(ii) If Ω^* is not c^*-convex, then there exists an open subset set, say a ball, $\omega \subset \Omega$, and two points $y_1, y_2 \in \Omega^*$ such that any c^*-segment (relative to any points in ω) connecting y_1 and y_2, or more generally connecting any two points respectively in $B_r(y_1), B_r(y_2) \subset \Omega^*$ for some sufficiently small $r > 0$, does not completely fall in Ω^*.

(iii) Let $\{f_i\}$ be a sequence of smooth, positive functions defined on Ω which converges to the function $f_0 = \chi_\omega$, the characteristic function of the domain ω. Let $\{g_i\}$ be a sequence of smooth, positive functions defined on Ω^* which converges to $g_0 = C(\chi_{B_r(y_1)} + \chi_{B_r(y_2)})$, where $C > 0$ is a positive constant such that the necessary condition (2.1) holds. Let u_i be the corresponding solutions to (2.17) (2.18) with $f = f_i$ and $g = g_i$. If u_i is smooth, then by (i), $T_{u_i}(\Omega) \subset \overline{\Omega}^*$.

Let $u_0 = \lim_{i\to\infty} u_i$. Then u_0 is a generalized solution of (2.17) (2.18) with $\Omega = \omega$, $\Omega^* = B_r(y_1) \cup B_r(z_2)$, $f = f_0$, and $g = g_0$. By the mass balance condition, we have for a.e. $x \in \omega$,

$T_{u_0}(x) \in B_r(y_1) \cup B_r(y_2)$. Hence there exists a point $x_0 \in \omega$ such that $T_{u_0}(x_0)$ contains at least two points p_1 and p_2 with $p_1 \in B_r(y_1)$ and $p_2 \in B_r(z_2)$. Since u_i is smooth, for any point $x \in \omega$, there is a unique c-supporting function of u_i at x. For any given $x_0 \in \omega$, and any $y \in \mathbf{R}^n$ such that $D_x c(x_0, y) \in \partial^+ u_0(x_0)$, by vertical translation of the graph of $c(\cdot, y)$ and noting that $u_0 = \lim u_i$, we see that there is a constant α such that $c(\cdot, y) + \alpha$ is a c-supporting function of u_0 at x_0. That is

$$T_{u_0}(x_0) = \{y \in \mathbf{R}^n \mid \ D_x c(x_0, y) \in \partial^+ u_0(x_0)\}. \tag{7.8}$$

Note that (7.8) is not true in general for non-smooth solutions, see Remark 3.1 or Section 7.5 below. Recalling that $\partial^+ u_0(x_0)$ is a convex set, we see that $T_{u_0}(x_0)$ is c^*-convex with respect to x_0. Hence it contains a c^*-segment ℓ (relative to x_0) connecting p_1 and p_2. Namely $\ell \subset \Omega^*$. But since Ω^* is not c^*-concave, by our choice of y_1 and y_2 above, ℓ does not fall in Ω^* completely. We reach a contradiction.

7.4. Convexity of domains relative to cost functions

In this subsection we check the c^*-convexity of Ω^* as assumed in Theorem 2.1. We will assume both Ω and Ω^* are topological balls with smooth boundaries. By definition 2.3, Ω^* is c^*-convex if any c^*-segment (relative to points in Ω) intersects $\partial\Omega^*$ at most two points.

7.4.1. The cost function $c(x, y) = |x - y|^2$, or equivalently $c(x, 0) = -x \cdot y$. In this case a c^*-segment is a line segment in the classical sense, and the c^*-convexity of domains is the same as convexity in the usual sense.

7.4.2. The reflector antenna design problem. Then we have the cost function $c(x, y) = -\log(1 - x \cdot y)$ (see (7.7)), where x, y are points on the unit sphere S^2. Recall that a c^*-segment is the image of the mapping T_u (defined in (2.12)) of a line segment. For the reflection problem, the mapping T is given by (7.6) and a line segment on the sphere is a part of a great circle, namely the intersection of S^2 with a plane passing through the origin. Therefore a c^*-segment is given by $\ell =: \{y \in S^2 \mid \ y = x - 2\langle x, n\rangle n\}$ for $n \in S^2$ on a plane passing through the origin. It is easy to see that the set $\ell \subset S^2$ is contained in a plane passing through the origin. Therefore Ω^* is c^*-convex if and only if *any plane $P \subset \mathbf{R}^3$ which passes through the origin intersects with $\partial\Omega^*$ at most two points*. In other words, Ω^* is c^*-convex if and only if it is convex as a domain in the sphere in the usual sense.

Remark 7.1. In [32] the c^*-convexity was phrased as condition (C) in page 360, and the condition was mistakenly stated as "any plane $P \subset \mathbf{R}^3$ which passes through a point in Ω intersects with $\partial\Omega^*$ at most two points".

7.4.3. The cost function $c(x, y) = \sqrt{1 + |x - y|^2}$. For simplicity, we will assume in the following that the dimension $n = 2$. By definition 2.2, a c^*-segment is the solution set $\{y\}$ to $D_x c(x_0, y) = z$ for z on a line segment in $\mathbf{R}^n$. By a translation we assume that $x_0 = 0$. By a rotation of the coordinates we assume that the line segment is in $\{x_1 = t\}$, which is parallel to the x_2-axis. Then we have the equation

$$\frac{-y_1}{\sqrt{1+|y|^2}} = t, \qquad \frac{-y_2}{\sqrt{1+|y|^2}} = s \in \mathbf{R}^1. \tag{7.9}$$

Noting that $|Dc| < 1$, we have $t^2 + s^2 < 1$, and hence

$$y = (y_1, y_2) = \frac{-(t,s)}{\sqrt{1-t^2-s^2}}. \tag{7.10}$$

From the first equation we also have

$$y_1 = \frac{-t}{\sqrt{1-t^2}}\sqrt{1+y_2^2}, \tag{7.11}$$

which is simply a hyperbola.

From (7.11) we see that if the origin is contained in Ω^*, then Ω^* is (uniformly) c^*-convex (relative to the origin) if it is convex. If the origin is not contained in Ω^*, the curvature of the hyperbola (7.11) attains its maximum at the point $(\frac{-t}{\sqrt{1-t^2}}, 0)$. The maximum value $\frac{|t|}{\sqrt{1-t^2}}$ is exactly the transport distance from the origin to the point y given by (7.10) at $s = 0$. Therefore Ω^* is c^*-convex (relative to the origin) if

$$\inf_{\partial\Omega^*} \kappa > \inf\{|y| \mid \ y \in \Omega^*\}. \tag{7.12}$$

where $\kappa(y)$ denotes the curvature of $\partial\Omega^*$ at y (with respect to the inner normal). Condition (7.12) can be relaxed, indeed on the far side of $\partial\Omega^*$ it suffices to assume $\partial\Omega^*$ is convex.

It follows that if $\Omega \subset \Omega^*$, then Ω^* is c^*-convex (relative to Ω) if it is convex. If $\Omega \not\subset \Omega^*$, then Ω^* is c^*-convex (relative to Ω) if

$$\inf_{\partial\Omega^*} \kappa > \sup\{d_x \mid \quad x \in \partial\Omega\}, \tag{7.13}$$

where $d_x = dist(x, \Omega^*)$.

7.4.4. Next we consider the cost function $c(x,y) = \sqrt{1-|x-y|^2}$ given in (6.13). Since the graph of this cost function is a hemisphere, geometrically the center of all spheres tangential to a fixed line is a unit circle in $\mathbf{R}^3$, whose projection on the plane $\{x_3 = 0\}$ is an ellipse, which can be represented as (after a rotation of axes)

$$y_1^2 + \frac{y_2^2}{a^2} = 1, \tag{7.14}$$

where $a \in [0,1]$. When $a = 0$, the ellipse becomes a line segment.

By definition 2.2, it is easy to see that a c^*-segment must be given by (7.14). Note that the cost function c is defined only when $|x-y| < 1$. We assume that $\sup\{|x-y| \mid x \in \Omega, y \in \Omega^*\} < R^2$ for some $R < 1$. In order that Ω^* is c^*-convex, the curvature of $\partial\Omega^*$ must be less than that of the ellipse (7.14) where $|y| < R$. Direct computation shows that the curvature of (7.14) in $\{|y| \le R\}$ is bounded by $\frac{a}{(1-(R^2-a^2))^{3/2}}$, which attains its maximum when $a = \frac{1}{2}(1-R^2)$. Therefore Ω^* is c^*-convex if $\sup\{|x-y| \mid x \in \Omega, y \in \Omega^*\} < R^2$ and the curvature of $\partial\Omega^*$ satisfies

$$\inf_{\partial\Omega^*} \kappa > \frac{\sqrt{2}}{3^{3/2}} \frac{1}{1-R^2}. \tag{7.15}$$

7.4.5. For cost functions in (6.19), or equivalently in (6.20), a c^*-segment is the solution set to the equation

$$(D_x c(x,y) =)\ -y - g(y) Df(x) = z$$

for z on a line segment. By a translation and rotation of coordinates we assume that $x = 0$ and $z \in \{x_1 = t\}$. Then we have

$$y_1 + g(y_1, y_2) f_1(0) = -t. \tag{7.16}$$

Denote $\dot{y}_1 = \frac{dy_1}{dy_2}$ and $\ddot{y}_1 = \frac{d^2 y_1}{dy_2^2}$. Differentiating equation (7.16) we have

$$\begin{aligned} \dot{y}_1 + f_1(g_1 \dot{y}_1 + g_2) = 0, \\ \ddot{y}_1 + f_1(g_1 \ddot{y}_1 + g_{11}\dot{y}_1^2 + 2g_{12}\dot{y}_1 + g_{22}) = 0, \end{aligned}$$

where $g_1 = \frac{\partial}{\partial y_1} g$, $g_{12} = \frac{\partial^2}{\partial y_1 \partial y_2} g$ etc. The curvature of the c^*-segment at y is equal to

$$\frac{\ddot{y}_1}{(1+\dot{y}_1^2)^{3/2}} = \frac{-f_1}{1 + f_1 g_1} \frac{(\dot{y}_1, 1)\,(D^2 g)\,(\dot{y}_1, 1)^T}{(1+\dot{y}_1^2)^{3/2}},$$

where the superscript T means transpose. Hence

$$\begin{aligned} \left|\frac{\ddot{y}_1}{(1+\dot{y}_1^2)^{3/2}}\right| &\le \frac{|f_1|}{(1+f_1 g_1)(1+\dot{y}_1^2)^{1/2}} \sup |D^2 g| \\ &= \frac{|f_1|}{\sqrt{(1+f_1 g_1)^2 + f_1^2 g_2^2}} \sup |D^2 g| \\ &\le \frac{|\nabla f|}{1 - |\nabla f|\,|\nabla g|} \sup |D^2 g|. \end{aligned}$$

Therefore Ω^* is c^*-convex (relative to any set) if

$$\inf_{\partial \Omega^*} \kappa > \frac{|\nabla f|}{1 - |\nabla f|\,|\nabla g|} \sup |D^2 g|. \tag{7.17}$$

Therefore in comparison with the cost function $c(x,y) = |x-y|^2$, if we choose the uniformly convex or concave functions f and g such that the right hand side of (7.17) is small, then we have the advantage of interior regularity of the potential functions, without sacrificing much of the convexity assumption for the domains.

Recall that if $\Omega \subset\subset \Omega^*$, the c^*-convexity for the cost function $c(x,y) = \sqrt{1+|x-y|^2}$ is weaker than that for the quadratic cost function.

7.5. Remarks on the structure condition (A3)

For the Monge-Ampère equation (4.22), condition (A3) is satisfied if the form $A_{ij}\xi_i\xi_j$ is uniformly concave with respect to z. As we saw from example (6.7) the extension of this condition to $\xi \perp \eta$ in (4.21) is crucial. If we only assume the concavity of $A_{ij}\xi_i\xi_j$ or allow $c_0 = 0$ in condition (A3), then interior regularity need not hold as evidenced by the celebrated example of Pogorelov for the Monge-Ampère equation [24]. Under the degenerate form of (A3), one would at most expect interior regularity under some boundary conditions, as for the Monge-Ampère equation [17,24].

When condition (A3) does not hold, (even in the degenerate form), the problem of reduced regularity such as in C^1 or $C^{1,\alpha}$ for some $\alpha > 0$, is reasonable, as for example in the case of the function

$$c(x, y) = \frac{1}{p}|x - y|^p, \quad p > 1. \tag{7.18}$$

For these and other cost functions, we believe that issue of regularity is related to the existence of c-concave functions φ whose contact sets $E = \{x \in \mathbf{R}^n \mid \varphi(x) = c(x, y) + a\}$ are disconnected, where $v(x) = c(x, y) + a$ is a c-support function. Locally the connectedness of E involves fourth order derivatives of the cost function c.

To construct a c-concave function for (7.18) (for $p > 2$) such that E is disconnected, we assume that $n = 2$ and $\Omega = B_r(0)$ for some small $r > 0$. Let

$$\eta_t(x) = \hat{c}(x_1 + t, x_2) + \alpha_t,$$

where $\hat{c}(x) = c(x, 0)$, α_t is a constant, $\alpha_{-t} = \alpha_t$, and $t \in (-1, 1)$. Let

$$\varphi(x) = \inf\{\eta_t(x) \mid \quad t \in (-1, 1)\}.$$

Then φ is c-concave. It is even, and smooth except possibly on the x_2-axis.

When $p > 2$, it is easy to verify that the second derivative $\partial_2^2 \hat{c}(t, 0)$ is increasing as $|t|$ increases (i.e. $\partial_1 \partial_2^2 c(t, 0) > 0$ for $t > 0$). Hence there exists α_t such that as $|t|$ increases, we have (i) $\eta_t(0)$ decreases and (ii) $\eta_t(0, \pm r)$ increases. The function φ is not smooth at $x_1 = 0$, when restricting to the line $\{x_2 = 0\}$.

By a vertical translation, there exists a constant α^* such that $c(x, 0) + \alpha^*$ is a c-supporting function. By (ii) above, we must have $\alpha^* \geq \alpha_0$. By (i), the set E can not be connected.

The above argument also applies to the cost functions $c(x, y) = -\sqrt{1 + |x - y|^2}$ and $c(x, y) = -\sqrt{1 - |x - y|^2}$, that is the negatives of our first two examples in Section 6, as well as the extensions

$$c(x, y) = (\varepsilon + |x - y|^2)^{p/2}$$

for $p > 2$, (cf example (6.16)).

For the cost function in (7.18) with $p \in (1, 2)$, one can show directly by examples that there is no $C^{1,1}$ regularity for solutions of (2.17). Indeed, we have

$$\begin{aligned} D_{x_i} c(x, y) &= |x - y|^{p-2}(x_i - y_i) \ = -D_{y_i} c(x, y), \\ D^2_{x_i x_j} c(x, y) &= |x - y|^{p-2}(\delta_{ij} + (p - 2)\frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2}) \end{aligned} \tag{7.19}$$

Hence

$$\det [c_{x_i y_j}(x, y)] = (p - 1)|x - y|^{n(p-2)}.$$

Let (u, v) be the corresponding potential functions. By (2.12), $D_x c(x, y) = Du$ at $y = T_u(x)$. Hence from (7.19), we have

$$x - y = |Du|^{\frac{2-p}{p-1}} Du$$

and

$$D^2_{x_i x_j} c(x, y) = |Du|^{\frac{p-2}{p-1}} (\delta_{ij} + (p-2)\frac{u_i u_j}{|Du|^2}).$$

The equation (2.17) becomes

$$\det(I + (p-2)\frac{Du \otimes Du}{|Du|^2} - |Du|^{\frac{2-p}{p-1}} D^2 u) = (p-1)\frac{f(x)}{g(y)}, \tag{7.20}$$

where I is the unit matrix. Direct computation shows that if $u(x) = C - \frac{\theta}{p}|x|^p$, where $C, \theta > 0$ are constants, the left hand side of (7.20) is equal to $(p-1)(1 + \theta^{\frac{1}{p-1}})^n$. Hence there is no $C^{1,1}$ regularity for the cost function (7.18) when $p \in (1, 2)$.

Finally we note, by considering radial solutions, that the potential functions relative to the cost function $c(x, y) = |x - y|$ are not C^1 smooth in general.

In a future work, we shall investigate connection between (A3) and the connectivity of contact sets as well as Hölder regularity issues.

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