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On the design of a reflector antenna II

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Abstract. The reflector antenna design problem requires to solve a second boundary value problem for a complicated Monge-Ampère equation, for which the traditional discretization methods fail. In this paper we reduce the problem to that of finding a minimizer or a maximizer of a linear functional subject to a linear constraint. Therefore it becomes an linear optimization problem and algorithms in linear programming apply.

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1. Introduction

The light reflection law, namely the normal of a reflecting surface bisects the angle formed by the incident ray and the reflected ray, is one of the basic principles we learnt in school. Reflective surfaces are widespread in our society. In applications we need to study the inverse problem, of which a prototype is the design of reflector antennas.

In this paper we continue our investigation of a reflector antenna system studied in [21]. The system consists of a detector located at the origin $\mathcal{O}$, a reflecting surface Γ which is a radial graph over a domain Ω in the north hemisphere $\{x = (x_1, x_2, x_3) \in S^2 : x_3 > 0\}$,

$$\Gamma = \{x\rho(x);\ \ x \in \Omega\} \quad \rho > 0, \tag{1.1}$$

and a target area in the outer space, from which we wish to receive signals, where $S^2 = \{x \in \mathbf{R}^3 : \ |x| = 1\}$ is the unit sphere. The target area is identified with a domain $\Omega^* \subset S^2$ in the way that a ray from the target area is regarded as a point in Ω^*.

The above model can also be interpreted as an illumination system, which consists of a point light source at $\mathcal{O}$, a reflecting surface Γ, and a target area Ω^* in a far field, such that all reflected rays fall in that field (namely the direction of reflected rays falls in Ω^*). Let f be the illumination on the input domain Ω, namely the distribution of the intensity of rays from $\mathcal{O}$, and let g be the illumination on the

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output domain Ω^*. Suppose there is no loss of energy in the reflection. Then we have, by the energy conservation law,

$$\int_\Omega f = \int_{\Omega^*} g. \tag{1.2}$$

For a ray from $\mathcal{O}$ to a point $z = x\rho(x) \in \Gamma$, where $x \in \Omega$, the direction of the reflected ray is, by the reflection law,

$$T(x) = T_\rho(x) = x - 2\langle x, n\rangle n, \tag{1.3}$$

where n is the outward normal of Γ at z, $\langle x, n\rangle$ denotes the inner product. By the energy conservation, T is a measure preserving mapping, that is

$$\int_{T^{-1}(E)} f = \int_E g \quad \forall \text{ Borel set } E \subset \Omega^*. \tag{1.4}$$

From (1.4) we obtain a partial differential equation for the reflector antenna system. Indeed, by (1.4) the Jacobi determinant of the mapping T at $x \in D$ is equal to $f(x)/g(T(x))$, which yields the equation

$$\mathcal{L}\rho = \eta^{-2}\det(-\nabla_i\nabla_j\rho + 2\rho^{-1}\nabla_i\rho\nabla_j\rho + (\rho-\eta)\delta_{ij}) = f(x)/g(T(x)) \tag{1.5}$$

in a local orthonormal coordinate system on S^2, where ∇ is the covariant derivative, $\eta = (|\nabla\rho|^2 + \rho^2)/2\rho$, and $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. This is an extremely complicated, fully nonlinear partial differential equation of Monge-Ampere type. A natural boundary condition is

$$T(\Omega) = \Omega^*. \tag{1.6}$$

Because of its applications in optics, electro-magnetics, and acoustics, reflector antenna is widely used in our society. Therefore it is highly desirable to find an efficient algorithm for the construction of the reflecting surface. In the last half century there were numerous papers in the literature, particularly in engineering literature, on numerical solutions of the problem [5,14,18,23,24]. However (1.5) is a strongly nonlinear PDE of mixed type, namely it may be elliptic in one area and hyperbolic in another area, depending on the geometric behavior of the function u. The traditional methods fail to create a satisfactory discretization scheme, except for some special cases, such as the radially symmetric case, which reduces the problem to an ordinary differential equation. In the general case an algorithm for numerical solutions is given in [3], see also Remark 1.4 in [21]. This algorithm, which uses approximation by the boundaries of convex bodies enclosed by paraboloids, depends on special characters of Monge-Ampère type equations, and is not straightforward for engineers.

In this paper we show that the above reflector antenna design problem is an optimal transportation problem, and a solution is a maximizer or minimizer of a linear functional.

Theorem A. *Suppose that Ω and Ω^* are connected domains contained respectively in the north and south hemispheres. Suppose f and g are bounded positive functions. Then there is a maximizer (φ_1, ψ_1), which is unique up to a constant, of the problem*

$$\sup\{I(u,v) : (u,v) \in K\}, \tag{1.7}$$

where

$$I(u,v) = \int_\Omega f(x)u(x) + \int_{\Omega^*} g(y)v(y), \tag{1.8}$$

$$K = \{(u,v) \in (C(\overline{\Omega}), C(\overline{\Omega}^*)) : u(x) + v(y) \leq c(x,y) \forall x \in \Omega, y \in \Omega^*\}, \tag{1.9}$$

$$c(x,y) = -\log(1 - x \cdot y), \tag{1.10}$$

and $x \cdot y$ is the inner product in $\mathbf{R}^3$, such that $\rho_1 = e^{\varphi_1}$ is a solution of (1.5) (1.6).

Theorem B. *Let Ω, Ω^*, f and g be as in Theorem A. Then there is a minimizer (φ_1, ψ_1), unique up to a constant, of the problem*

$$\inf\{I(u,v) : (u,v) \in K'\}, \tag{1.11}$$

where $K' = \{(u,v) \in (C(\overline{\Omega}), C(\overline{\Omega}^*) : u(x) + v(y) \geq c(x,y) \, \forall \, x \in \Omega, y \in \Omega^*\}$, such that $\rho_2 = e^{\varphi_2}$ is a solution of (1.5) (1.6).

Theorems A and B show that the reflector antenna design problem is indeed a linear programming problem. It therefore provides an efficient and stable algorithm for numerical solutions of the problem. See discussions in Sect. 6.

In the above theorems the maximizer or minimizer is Lipschitz continuous, but may not be C^1 smooth. A solution needs to be understood as a generalized solution. The notion of generalized solutions was introduced in [21], and also in [4,13], see Sect. 2 below. In [21] we proved the existence and uniqueness of generalized solutions to (1.5) (1.6), and that a generalized solution is smooth if the boundaries $\partial\Omega$ and $\partial\Omega^*$ satisfy certain geometric condition. We also showed that in general the geometric condition is necessary for the regularity of solutions. Note that the uniqueness of generalized solutions implies that a solution of (1.5) (1.6) must be a constant multiplication of ρ_1 or ρ_2 in Theorems A or B.

The above theorems are inspired by recent advances in the study of the optimal transportation problem. See [1,7,8,17,20] for discussions. Our proof of Theorems A and B is particularly inspired by, and follows from that in [9]. For completeness we will include a detailed proof. The main point in this paper is an observation on the relation between the problem (1.5) (1.6) and the optimal transportation problem (1.7). Theorems A and B were announced in March 2001 at Cambridge and also in [22].

This paper is arranged as follows. In Sect. 2 we introduce the notion of generalized solutions. In Sect. 3 we prove Theorems A and B. In Sect. 4 we show that the reflector antenna design problem is indeed an optimal transportation problem. In Sect. 5 we give a new proof of the uniqueness of generalized solutions, using the uniqueness of optimal mappings to the transportation problem. Finally in Sect. 6 we give some remarks on algorithms for the maximizers or minimizers in Theorems A and B.

2. Generalized solutions

To introduce the notion of generalized solutions, we need the notion of supporting paraboloids. A paraboloid with focus at the origin can be represented as $\Gamma_p = \{x\,p(x):\ x \in S^2\}$ with

$$p(x) = p_{y,C}(x) = \frac{C}{1 - x \cdot y} \tag{2.1}$$

for some constant $C > 0$, where y is the axial direction of the paraboloid.

Let $\rho \in C(\Omega)$ be a positive function, and let $\Gamma_\rho = \{x\rho(x):\ x \in \Omega\}$ denote the radial graph of ρ. We say Γ_p, where $p = p_{y,C}$, is a supporting paraboloid of ρ at the point $x_0\rho(x_0) \in \Gamma_\rho$ if

$$\begin{cases} \rho(x_0) = p_{y,C}(x_0), \\ \rho(x) \le p_{y,C}(x) \quad \forall\, x \in \Omega. \end{cases} \tag{2.2}$$

We say ρ is *admissible* if there is a supporting paraboloid at any point on the graph Γ_ρ.

Let ρ be an admissible function. We define a set-valued mapping $T_\rho:\ \Omega \to S^2$, such that for any $x_0 \in \Omega$, $T_\rho(x_0)$ is the set of points y_0 such that for some $C > 0$, $p_{y_0,C}$ is a supporting paraboloid of ρ at x_0. For any subset $E \subset \Omega$, we denote $T_\rho(E) = \bigcup_{x\in E} T_\rho(x)$.

From the definition, we see that if $T_\rho(x)$ contains more than one point, then ρ is not differentiable at x; and T_ρ is single valued at any differentiable point. Since an admissible function has supporting paraboloid at any point on its graph, it is semi-convex and is twice differentiable almost everywhere. Hence T_ρ is a single valued mapping almost everywhere.

By the mapping T_ρ, we introduce a measure $\mu = \mu_{\rho,g}$ in Ω, where $g \in L^1(S^2)$ is a nonnegative measurable function, such that for any Borel set $E \subset \Omega$,

$$\mu(E) = \int_{T_\rho(E)} g(x)dx. \tag{2.3}$$

In [21] we proved that μ is a Radon measure, see also [15].

Definition 2.1. *An admissible function ρ is called a generalized solution of (1.5) if $\mu_{\rho,g} = f\,dx$ as measures, namely for any Borel set $E \subset \Omega$,*

$$\int_E f = \int_{T_\rho(E)} g. \tag{2.4}$$

If furthermore ρ satisfies

$$\Omega^* \subset T_\rho(\Omega), \quad |\{x \in \Omega:\ f(x) > 0 \text{ and } T_\rho(x) - \overline{\Omega}^* \ne \emptyset\}| = 0, \tag{2.5}$$

then ρ is a generalized solution of (1.5) (1.6).

The above definition was introduced in [21]. Obviously an admissible smooth solution is a generalized solution. Whether a smooth solution is admissible depends on the geometry of the domain Ω. For example for the Monge-Ampère equation

$$\det D^2 u = f(x) \text{ in } \Omega, \tag{2.6}$$

where $f \geq 0$, the admissibility of functions is equivalent to the global convexity, where a function u is globally (locally resp.) convex if for any point $x \in \Omega$, the graph Γ_u lies above the tangent plane of Γ_u at $(x, u(x))$ globally (locally resp.). A locally convex smooth solution may not be globally convex if Ω is not convex. For Eq. (1.5), in order that a smooth solution is admissible, we need to assume the domain Ω is c-convex, see [15].

Next we need a Legendre type transform introduced in [10], see also Lemma 1.1 in [21].

Definition 2.2. *Let ρ be an admissible function on Ω. The Legendre transform of ρ, with respect to the function $\frac{1}{1-x\cdot y}$, is a function η defined on S^2, given by*

$$\eta(y) = \inf_{x\in\Omega} \frac{1}{\rho(x)(1-x\cdot y)}. \tag{2.7}$$

Denote $\Omega^* = T_\rho(\Omega)$. For any fixed $y_0 \in \Omega^*$, let the infimum (2.7) be attained at $x_0 \in \Omega$. Then

$$\eta(y_0) = \frac{1}{\rho(x_0)(1-x_0\cdot y_0)}, \tag{2.8}$$

$$\eta(y) \leq \frac{1}{\rho(x_0)(1-x_0\cdot y)} \quad \forall\ y \in \Omega^*, \tag{2.9}$$

$$\eta(y_0) \leq \frac{1}{\rho(x)(1-x\cdot y_0)} \quad \forall\ x \in \Omega. \tag{2.10}$$

From (2.8) and (2.9) we see that $p_{x_0,c}(y) = \frac{C}{1-x_0\cdot y}$ $(C = 1/\rho(x_0))$ is a supporting paraboloid of η at y_0. Hence η is admissible. From (2.8) and (2.10) we see that $p_{y_0,C}(x) = \frac{C}{1-x\cdot y_0}$ $(C = 1/\eta(y_0))$ is a supporting paraboloid of ρ at x_0. Hence $y_0 \in T_\rho(x_0)$ if and only if $x_0 \in T_\eta(y_0)$. In particular the Legendre transform of η, when restricted in Ω, is ρ itself. If furthermore ρ is smooth and satisfies (1.5), then T_η is the inverse of T_ρ, and so η satisfies the equation

$$\mathcal{L}\eta = g(y)/f(T_\eta(x)), \tag{2.11}$$

where $\mathcal{L}$ is the operator in (1.5). The graph Γ_η is called the dual reflector antenna. See [10] for discussions on the Legendre transform.

The admissibility introduced above is called upper admissibility in [21]. Alternatively we can introduce lower admissibility in the same way. We need only to change the direction of the inequality in (2.2). For the corresponding Legendre transform, we need to change inf to sup in (2.7), and change the direction of the inequalities in (2.9) (2.10) accordingly. The upper and lower admissibilities correspond respectively to the convexity and concavity of functions when compared with the Monge-Ampère equation (2.6).

3. Proof of Theorems A and B

Lemma 3.1. *There exists a Lipschitz continuous maximizer* $(\varphi, \psi) \in K$ *for the supremum* $\sup_{(u,v)\in K} I(u, v)$.

Proof. For any given pair $(u, v) \in K$, let

$$v^*(y) = \inf_{x\in\Omega}[c(x, y) - u(x)] \quad \forall\ y \in \Omega^*. \tag{3.1}$$

Then for any $y \in \Omega^*$, by the continuity of c and u, there exists $x \in \overline{\Omega}$ such that

$$v^*(y) = c(x, y) - u(x) \geq v(y).$$

The last inequality is because $(u, v) \in K$. Hence $v^* \geq v$. Next for any $y_1 \neq y_2 \in \Omega^*$, let $x_1 \in \overline{\Omega}$ such that $v^*(y_1) = c(x_1, y_1) - u(x_1)$. Then

$$\begin{aligned} v^*(y_1) - v^*(y_2) &\geq [c(x_1, y_1) - u(x_1)] - [c(x_1, y_2) - u(x_1)] \\ &= c(x_1, y_1) - c(x_1, y_2) \\ &\geq -\beta|y_1 - y_2|, \end{aligned}$$

where $\beta = \sup\{|Dc(x, y)| :\ x \in \Omega, y \in \Omega^*\}$. Similarly we have

$$v^*(y_2) - v^*(y_1) \geq -\beta|y_2 - y_1|.$$

Hence v^* is Lipschitz continuous, with Lipschitz constant β.

Let

$$u^*(x) = \inf_{y\in\Omega^*}[c(x, y) - v^*(y)] \quad \forall\ x \in \Omega. \tag{3.2}$$

Then as above we have $u^* \geq u$ and u^* is Lipschitz continuous, with Lipschitz constant β. From (3.2) we have also that $(u^*, v^*) \in K$. Hence $I(u, v) \leq I(u^*, v^*)$ as f, g are positive.

Choose a sequence $(u_k, v_k) \in K$ such that $I(u_k, v_k) \to \sup_K I(u, v)$. Then $(u_k^*, v_k^*) \in K$ and $I(u_k^*, v_k^*) \to \sup_K I(u, v)$. Observing that

$$I(u, v) = I(u + C, v - C) \tag{3.3}$$

for any constant C, by the energy conservation condition (1.2), we may suppose by adding a constant that $v_k^*(y_0) = 0$ for some fixed point $y_0 \in \Omega^*$. By the Lipschitz continuity, it follows that $\{v_k^*\}$ is uniformly bounded. Hence $\{u_k^*\}$ is also uniformly bounded. Therefore by choosing a subsequence we may suppose that (u_k^*, v_k^*) converges uniformly to (φ, ψ). Then φ and ψ are Lipschtz continuous and (φ, ψ) is a maximizer. □

From the proof we see that (φ, ψ) satisfies

$$\begin{aligned} \varphi(x) &= \inf\{c(x, y) - \psi(y) :\ y \in \Omega^*\}, \\ \psi(y) &= \inf\{c(x, y) - \varphi(x) :\ x \in \Omega\}. \end{aligned} \tag{3.4}$$

Let $\rho = e^{\varphi}$ and $\eta = e^{\psi}$. From (3.4) we see that ρ and η are admissible functions, and ρ is the Legendre transform of η, and η is the Legendre transform of ρ.

Lemma 3.2. *Let (φ, ψ) be a maximizer in Lemma 3.1. Then the equation*

$$\varphi(x) + \psi(t(x)) = c(x, t(x)) \tag{3.5}$$

is uniquely solvable for $t(x) \in \overline{\Omega}^*$ at any differentiable point of φ. Furthermore the mapping t is Borel measurable and is determined by

$$t(x) = T_\rho(x) = x - 2\langle x, n\rangle n \tag{3.6}$$

at any differentiable point of $\rho = e^{\varphi}$, where n is the outward normal of the radial graph of ρ at the point $x\rho(x)$.

Proof. Let $x_0 \in \Omega$ be a differentiable point of φ. Let $y_0 \in \overline{\Omega}^*$ such that

$$\begin{aligned} \varphi(x_0) &= c(x_0, y_0) - \psi(y_0), \\ \varphi(x) &\le c(x, y_0) - \psi(y_0) \quad \forall\, x \in \Omega. \end{aligned} \tag{3.7}$$

We claim that y_0 is uniquely determined and is given by the right hand side of (3.6).

Indeed, denote

$$p(x) = exp(c(x, y_0) - \psi(y_0)) = \frac{C}{1 - x \cdot y_0}, \tag{3.8}$$

where $C = exp(-\psi(y_0))$. Then ρ and p are positive functions on Ω, and by (3.7), p is a supporting paraboloid of Γ_ρ at x_0 with axial direction y_0. Observe that at differentiable points of ρ, the supporting paraboloid is uniquely determined. Hence y_0 is unique. By the reflecting property of paraboloid, we have

$$y_0 = T_p(x_0) = x_0 - 2\langle x_0, n\rangle n,$$

where n is the normal of the paraboloid Γ_p at the point $x_0 p(x_0)$. Hence y_0 is given by (3.6).

Now we define the mapping t by $t(x_0) = y_0$. Then t is well defined a.e., and by Rademacher's theorem, t is Borel measurable. □

Lemma 3.3. *The mapping t in Lemma 2.2 is a measure preserving mapping.*

Proof. We need to prove that t satisfies (1.4), which is equivalent to proving that for any continuous function $h \in C(\overline{\Omega}^*)$,

$$\int_\Omega h(t(x)) f(x) = \int_{\Omega^*} h(y) g(y). \tag{3.9}$$

Let $h \in C(\overline{\Omega}^*)$ and $\varepsilon \in (-1, 1)$ be a small constant. Let

$$\begin{aligned} \psi_\varepsilon(y) &= \psi(y) + \varepsilon h(y) \quad y \in \Omega^*, \\ \varphi_\varepsilon(x) &= \inf_{y \in \overline{\Omega}^*} \{c(x, y) - \psi_\varepsilon(y)\} \quad x \in \Omega, \end{aligned} \tag{3.10}$$

where (φ, ψ) is the maximizer in Lemma 3.1. Then $(\varphi_\varepsilon, \psi_\varepsilon) \in K$. We claim that if φ is differentiable at x, then

$$\varphi_\varepsilon(x) = \varphi(x) - \varepsilon h(t(x)) + o(\varepsilon). \tag{3.11}$$

Indeed, suppose the infimum in (3.10) is attained at $y_\varepsilon \in \overline{\Omega}^*$. Since $t(x)$ is uniquely determined, we have $y_\varepsilon \to t(x)$ as $\varepsilon \to 0$. Hence (3.11) follows from (3.4), (3.10), and the continuity of h.

Next, since (φ, ψ) is a maximizer, we have

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[I(\varphi_\varepsilon, \psi_\varepsilon) - I(\varphi, \psi)] \\ &= \lim_{\varepsilon\to 0} \{ \int_\Omega \frac{\varphi_\varepsilon(x) - \varphi(x)}{\varepsilon} f(x) + \int_{\Omega^*} h(y)g(y) \} \\ &= -\int_\Omega h(t(x))f(x) + \int_{\Omega^*} h(y)g(y). \end{aligned}$$

Hence (1.10) holds. □

Proof of Theorem A. Let (φ, ψ) be a maximizer of (1.7). Denote $\rho = e^\varphi$ and $\eta = e^\psi$. Then by (3.4), ρ and η are admissible, and η is the Legendre transform of ρ (restricted to Ω^*). In Sect. 2 we showed that if $y \in T_\rho(x_1) \cap T_\rho(x_2)$, where $x_1 \neq x_2$, then $x_1, x_2 \in T_\eta(y)$, and so η is not differentiable at y. Since the set on which η is not differentiable has measure zero, by Lemma 3.3 we see that the mapping t is one to one almost everywhere from Ω to Ω^*. Hence by the measure preserving condition (1.4) and note that $T_\rho = t$ a.e., we see that T_ρ satisfies (2.4), namely ρ is a generalized solution of (1.5). By Lemma 3.2, we have $t(x) \in \overline{\Omega}^*$ at any differentiable point of ρ. By the energy conservation condition (1.2) and the assumption that $f, g > 0$, we see that (2.5) holds. Hence ρ is a generalized solution of (1.5) (1.6). The uniqueness of maximizer of (1.7) will be proved in Lemma 4.2. □

Theorem B can be proved in the same way as above, using the lower admissibility to replace the upper admissibility. We leave the proof to the reader.

We also remark that the assumption in Theorems A and B that Ω and Ω^* are contained in the north and south hemispheres is natural in applications but not necessary for the mathematical treatment. We need only to assume that $\inf\{|x-y| : x \in \Omega, y \in \Omega^*\} > 0$, such that the function $\frac{1}{1-x\cdot y}$ is uniformly bounded for $x \in \Omega$ and $y \in \Omega^*$.

4. An optimal transportation problem

In this section we show that the reflector antenna design problem is indeed an optimal transportation problem, namely the Monge-Kantorovich mass transfer problem, with the cost function $c(x, y)$ given in (1.10).

First we briefly introduce the optimal transportation problem. Let Ω and Ω^* be two domains on a manifold or in $\mathbf{R}^n$, together with two mass distributions f and g

with equal total mass. The optimal transportation problem concerns the existence of optimal mappings which minimizes the cost functional

$$\mathcal{C}(s) = \int_\Omega c(x, s(x)) f(x) \tag{4.1}$$

among all measure preserving mappings. A mapping s is called measure preserving if it is Borel measurable and satisfies (1.4) or (3.9). The functional $\mathcal{C}$ measures the cost to transfer a mass (energy) distribution $f \in L^1(\Omega)$ to another one $g \in L^1(\Omega^*)$.

The optimal transportation problem was first studied by Monge [16] with the special cost function $c(x, y) = |x - y|$. A breakthrough was made by Kantorovich [11,12] who introduced the dual functional (1.8). For Monge's cost function the existence of optimal mappings was not proved until recently in [2,8,19]. If the cost function $c(x, y)$ is a uniformly convex function of $x - y$, or a concave function of $|x - y|$, the existence of optimal mappings was proved in [9]. In our case the cost function is the special one given in (1.10).

Lemma 4.1. *Let $c(x, y) = -\log(1 - x \cdot y)$ be the cost function in (1.10). Then the mapping t in Lemma 3.2 is the unique minimizer of the functional $\mathcal{C}$. Moreover we have*

$$\inf_{s \in \mathcal{S}} \mathcal{C}(s) = \sup_{(u,v) \in K} I(u, v), \tag{4.2}$$

where $\mathcal{S}$ denotes the set of all measure preserving mappings from Ω to Ω^*.

Proof. For any $(u, v) \in K$ and $s \in \mathcal{S}$, we have

$$\begin{aligned} \int_\Omega u(x) f(x) + \int_{\Omega^*} v(y) g(y) &= \int_\Omega u(x) f(x) + \int_\Omega v(s(x)) f(x) \\ &\le \int_\Omega c(x, s(x)) f(x). \end{aligned} \tag{4.3}$$

(4.4)

Hence

$$I(u, v) \le \mathcal{C}(s) \quad \forall \ (u, v) \in K, \ s \in \mathcal{S}. \tag{4.5}$$

By (3.5), the equality in (4.3) holds when $(u, v) = (\varphi, \psi)$ and $s = t$. Hence t is a minimizer of $\inf_{s \in \mathcal{S}} \mathcal{C}$.

Suppose there exists another minimizer $\bar{t} \in \mathcal{S}$. Then $\mathcal{C}(\bar{t}) = \mathcal{C}(t) = I(\varphi, \psi)$. Since the equality holds in (4.3) when $(u, v) = (\varphi, \psi)$ and $\varphi(x) + \psi(y) \le c(x, y)$ for any $x \in \Omega$, $y \in \Omega^*$, we must have

$$\varphi(x) + \psi(\bar{t}(x)) = c(x, \bar{t}(x))$$

for almost all $x \in \Omega$. By lemma 3.2, we must have $\bar{t}(x) = t(x)$ for almost all $x \in \Omega$. This completes the proof. □

Lemma 4.2. *Suppose Ω and Ω^* are connected domains. Then the maximizer (φ, ψ) of $\sup_K I(u, v)$ is unique up to a constant.*

Proof. Let $(\varphi', \psi') \in K$ be another maximizer. Let $\rho' = e^{\varphi'}$. Then by Lemma 4.1 and (3.6), the mapping $T_{\rho'}$ is also an optimal mapping and $T_{\rho'} = T_\rho$ almost everywhere. It follows $D\varphi' = D\varphi$ a.e.. As φ and φ' are Lipschitz continuous, we see that $\varphi - \varphi'$, as a function in the Sobolev space $W^{1,2}(\Omega)$, has vanishing gradient. Hence $\varphi = \varphi' + C$. By (3.4) we also have $\psi = \psi' - C$. □

Mathematically one can also consider the problem of maximizing the cost functional $\mathcal{C}$. Then correspondingly we have

$$\sup_{s\in\mathcal{S}} \mathcal{C}(s) = \inf_{(u,v)\in K} I(u,v). \tag{4.6}$$

We have similar results for the minimizers of $\inf_K I(u,v)$ as Lemmas 4.1 and 4.2.

5. Uniqueness of generalized solutions

In this section we give a new proof of the uniqueness of generalized solutions. We will consider only (upper) admissible generalized solutions introduced in Definition 2.1.

Theorem C. *Let ρ be a generalized solution of (1.5) (1.6). Then $\rho = C\rho_1$ for some positive constant C, where ρ_1 is the function in Theorem A.*

Proof. Let η be the Legendre transform of ρ. Let E and E^* denote respectively the sets on which ρ and η are not twice differentiable. Then $|E| = |E^*| = 0$. Observe that $x \in T_\eta(y)$ if and only if $y \in T_\rho(x)$, and that $T_\rho(x)$ is a single point in E^* if $x \in T_\eta(E^*)$ is a differentiable point of ρ. Hence by (2.4) we have

$$\int_{T_\eta(E^*)} f = \int_{E^*} g = 0. \tag{5.1}$$

The above formula implies that T_ρ is one to one, and T_η is the inverse of T_ρ, almost everywhere on $\{x \in \Omega : \ f(x) > 0\}$. It follows that T_ρ is a measure preserving mapping, namely it satisfies (1.4).

Let $\varphi = \log\rho$ and $\psi = \log\eta$. Since η is the Legendre transform of ρ and ρ is the Legendre transform of η, (3.4) holds for φ and ψ. By Lemma 3.2, we have

$$\varphi(x) + \psi(T_\rho(x)) = c(x, T_\rho(x)) \quad a.e. \tag{5.2}$$

Hence

$$\begin{aligned} \int_\Omega \varphi(x)f(x) + \int_{\Omega^*} \psi(y)g(y) &= \int_\Omega \varphi(x)f(x) + \int_\Omega \psi(T_\rho(x))f(x) \\ &= \int_\Omega c(x, T_\rho(x))f(x). \end{aligned} \tag{5.3}$$

Namely (φ, ψ) is a maximizer by (4.2). By the uniqueness in Sect. 4, we conclude that $\varphi = \varphi_1 + C$ for some constant C, where φ_1 is the maximizer in Theorem A. □

6. Remarks on algorithms

In this section we discuss algorithms for the maximization problem (1.7). The minimization problem (1.11) can be changed to a maximization problem by replacing (u, v) by $(-u, -v)$.

To find a numerical solution for the maximization problem (1.7)-(1.10), we choose a set of points $\{x_i \in \Omega, y_j \in \Omega^*,\ i = 1, \cdots, I, j = 1, \cdots, J\}$. Then we need to find a solution to the problem

$$(6.1) \qquad \max\{ \sum_{i=1}^{I} a_i u_i + \sum_{j=1}^{J} b_j v_j \},$$
$$u_i + v_j \le c_{i,j} \quad i = 1, \cdots, I,\ j = 1, \cdots, J,$$

where $a_i = f(x_i), b_j = g(y_j)$, and $c_{i,j} = c(x_i, y_j)$ are coefficients, and $u_i = u(x_i), v_j = v(y_j)$ are variables.

(6.1) is a standard linear programming problem. The canonical form is usually written in the form

$$(6.2) \qquad \min \quad c \cdot x,$$
$$Ax \ge b,$$

where $x = (x_1, \cdots, x_m)$ are variables, $c = (c_1, \cdots, c_m)$ and $b = (b_1, \cdots, b_n)$ are constants, $A = (a_{ij})$ $(i = 1, \cdots, n,\ j = 1, \cdots, m)$ is a matrix, and $c \cdot x = \sum c_i x_i$. The constraint $Ax \ge b$ is understood as

$$\sum_{j=1}^{m} a_{ij} x_j \ge b_i \ \ \forall \ i = 1, \cdots, n.$$

The problem (6.2) has a dual problem, which can also be written in the canonical form

$$(6.3) \qquad \max \quad b \cdot y,$$
$$A^T y \le c,$$

where $y = (y_1, \cdots, y_n)$ are variables, A^T is the transpose of A, and $b \cdot y = \sum b_i y_i$.

It is easy to see that the dual of the dual problem (6.3) is the primal problem (6.2). Mathematically (6.2) is equivalent to (6.3) by replacing x by $-x$. It is also known that the maximum in (6.3) is equal to the minimum in (6.2), see [29], Theorem 23.

Remark. Our problem (6.1) is exactly the dual problem (6.3). By (3.3), if $\{u_i\}$ and $\{v_j\}$ is a solution, so is $\{u_i + C\}$ and $\{v_j - C\}$ for any constant C. Therefore we may fix $u_1 = 0$. We remark that if $\{u_i\}$ and $\{v_j\}$ is a solution of (6.1), then $\{\rho_i\} = \{e^{u_i}\}$ is a (positive) numerical solution to (1.5) (1.6). Note also that if ρ is a solution of (1.5) (1.6), so is $C\rho$ for any constant $C > 0$.

There are two well known algorithms in linear programming, namely the *ellipsoid algorithm* introduced by L.G. Khachiyan [26] and the *projective algorithm* introduced by N. Karmarkar [27]. Both algorithms *converges polynomially*, and both can be found in many linear programming books. We refer the reader to [29, 30], where the reader can find a detailed introduction of the algorithms. In particular sample computer programs, implemented in MATLAB, have been provided in [30]. For problem (6.1), we refer to the program in [30], pp. 297-299.

Finally we note that there are even fast polynomially convergent algorithm if all the entries of the constraint matrix A, such as in our problem (6.1), are either 0 or ± 1. See, e.g., [28].

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