

AFFINE MAXIMAL HYPERSURFACES

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ABSTRACT. This is a brief survey of recent works by Neil Trudinger and myself on the Bernstein and Plateau for affine maximal hypersurfaces

1. Introduction

The concept of affine maximal surface in affine geometry corresponds to that of minimal surface in Euclidean geometry. The affine Bernstein problem and affine Plateau problem, as proposed in [9,5,7], are two fundamental problems for affine maximal surfaces. We shall describe some recent advances, mostly obtained by Neil Trudinger and myself [21-24], on these two problems.

Given an immersed hypersurface $\mathcal{M} \subset \mathbf{R}^{n+1}$, one defines the *affine metric* (also called the Berwald-Blaschke metric) by $g = |K|^{-1/(n+2)}II$, where K is the Gauss curvature, II is the second fundamental form of \mathcal{M} . In order that the metric is positive definite, the hypersurface will always be assumed to be locally uniformly convex, namely it has positive principal curvatures. From the affine metric one has the *affine area functional*,

$$(1.1) \quad A(\mathcal{M}) = \int_{\mathcal{M}} K^{1/(n+2)},$$

which can also be written as

$$(1.2) \quad A(u) = \int_{\Omega} [\det D^2 u]^{1/(n+2)}$$

if \mathcal{M} is given as the graph of a convex function u over a domain $\Omega \subset \mathbf{R}^n$. The affine metric and affine surface area are invariant under unimodular affine transformations.

A locally uniformly convex hypersurface is called *affine maximal* if it is stationary for the functional A under interior convex perturbation. A convex function is called an affine maximal function if its graph is affine maximal. Traditionally such hypersurfaces were called affine minimal [1,9]. Calabi suggested using the terminology affine maximal as the second variation of the affine area functional is negative [5]. If the hypersurface is a graph of

convex function u , then u satisfies the *affine maximal surface equation* (the Euler-Lagrange equation of the functional A),

$$(1.3) \quad L[u] := U^{ij}w_{ij} = 0,$$

where $[U^{ij}]$ is the cofactor of the Hessian matrix D^2u ,

$$(1.4) \quad w = [\det D^2u]^{-(n+1)/(n+2)},$$

and the subscripts i, j denote partial derivatives with respect to the variables x_i, x_j . Note that for any given i or j , U^{ij} , as a vector field in Ω , is divergence free. The equation (1.3) is a nonlinear fourth order partial differential equation, which can also be written in the short form

$$(1.5) \quad \Delta_g h = 0,$$

where $h = (\det D^2u)^{-1/(n+2)}$, and Δ_g denotes the Laplace-Beltrami operator with respect to g .

The quantity

$$H_A(\mathcal{M}) = \frac{-1}{n+1}L(u)$$

is called the affine mean curvature of \mathcal{M} , and is also invariant under unimodular affine transformations. In particular it is invariant if one rotates the coordinates or adds a linear function to u . The affine mean curvature of the unit sphere is n .

The affine Bernstein problem concerns the uniqueness of entire convex solutions to the affine maximal surface equation, and asks whether an entire convex solution of (1.3) is a quadratic polynomial. The Chern conjecture [9] asserts this is true in dimension two. Geometrically, and more generally, it can be stated as that a Euclidean complete, affine maximal, locally uniformly convex surface in 3-space must be an elliptic paraboloid. Calabi proved the assertion assuming in addition that the surface is affine complete [5], see also [6,7]. A problem raised by Calabi, called the Calabi conjecture in [19], is whether affine completeness alone is enough for the Bernstein theorem. The Chern conjecture was proved true in [21] (see Theorem 3.1 below). The Calabi conjecture was resolved in [22], as a byproduct of our fundamental result that affine completeness implies Euclidean completeness for locally uniformly convex hypersurfaces of dimensions larger than one (Theorem 3.2). See also [14] for a different proof of the Calabi conjecture.

The affine Plateau problem deals with the existence and regularity of affine maximal hypersurfaces with prescribed boundary of which the normal bundles on the boundary coincide with that of a given locally uniformly convex hypersurface. The affine Plateau problem, which had not been studied before, is more complicated when compared with the affine Bernstein problem in 3-space. The first boundary value problem, namely prescribing the solution and its gradient on the boundary, is a special case of the affine Plateau problem. We need to impose two boundary conditions as the affine maximal surface equation is a

fourth order equation. We will formulate the Plateau problem as a variational maximization problem and prove the existence and regularity of maximizers to the problem in 3-space [24] (Theorem 5.1). For the existence we need a uniform cone property of locally convex hypersurfaces, proved in [23], which also led us to the proof of the conjecture by Spruck in [20] (Theorem 4.1), concerning the existence of locally convex hypersurfaces of constant Gauss curvature.

Equation (1.3) can be decomposed as a system of two second order partial differential equations, one of which is a linearized Monge-Ampère equation and the other is a Monge-Ampère equation, see (2.6) and (2.7) below. This formulation enables us to establish the regularity for equation (1.3) (Theorem 2.1), using the regularity theory for Monge-Ampère type equations [2,3]. A crucial assumption in Theorem 2.1 is the strict convexity of solutions, which is the key issue for both the affine Bernstein and affine Plateau problems. We succeeded in proving the necessary convexity estimates only in dimension two.

2. A priori estimates

Instead of the homogeneous equation (1.3), we consider here the non-homogeneous (prescribed affine mean curvature) equation

$$(2.1) \quad L(u) = f \quad \text{in } \Omega,$$

where f is a bounded measurable function, and Ω is a normalized convex domain in \mathbf{R}^n . A convex domain is called normalized if its minimum ellipsoid, that is the ellipsoid with minimum volume among all ellipsoids containing the domain, is a unit ball.

Let u be a smooth, locally uniformly convex solution of (2.1) which vanishes on $\partial\Omega$. First we need positive upper and lower bounds for the determinant $\det D^2u$. For the upper bound we have, by constructing appropriate auxiliary function, for any subdomain $\Omega' \subset\subset \Omega$, the estimate

$$(2.2) \quad \sup_{x \in \Omega'} \det D^2u(x) \leq C,$$

where C depends only on n , $\text{dist}(\Omega', \partial\Omega)$, $\sup_{\Omega} |Du|$, $\sup_{\Omega} f$, and $\sup_{\Omega} |u|$.

For the lower bound we need a key assumption, namely a control on the strict convexity of solutions, which can be measured by introducing the *modulus of convexity*. Let v be a convex function in Ω . For any $y \in \Omega$, $h > 0$, denote

$$S_{h,v}(y) = \{x \in \Omega \mid v(x) = v(y) + Dv(y)(x - y) + h\}.$$

The modulus of convexity of v is a nonnegative function, defined by

$$\rho_v(r) = \inf_{y \in \Omega} \rho_{v,y}(r), \quad r > 0,$$

where

$$\rho_{v,y}(r) = \sup\{h \geq 0 \mid S_{h,v}(y) \subset B_r(y)\}$$

if there exists $h \geq 0$ such that $S_{h,v}(y) \subset B_r(y)$, otherwise we define $\rho_{v,y}(r) = 0$. We have $\rho_v(r) > 0$ for all $r > 0$ if v is strictly convex.

Let u be a smooth, locally uniformly convex solution of (2.1). Then we have the following lower bound estimate, for any $\Omega' \subset\subset \Omega$,

$$(2.3) \quad \inf_{x \in \Omega'} \det D^2 u(x) \geq C,$$

where C depends on n , $\text{dist}(\Omega', \partial\Omega)$, $\sup_{\Omega} |Du|$, $\inf_{\Omega} f$, and ρ_u . The proof again can be achieved by introducing an appropriate auxiliary function.

From the a priori estimates (2.2) and (2.3) we then have

Theorem 2.1. *Let $u \in C^4(\Omega) \cap C^0(\bar{\Omega})$ be a locally uniformly convex solution of (2.1). Then for any subdomain $\Omega' \subset\subset \Omega$, we have:*

(i) $W^{4,p}$ estimate,

$$(2.4) \quad \|u\|_{W^{4,p}(\Omega')} \leq C,$$

where $p \in [1, \infty)$, C depends on $n, p, \sup_{\Omega} |f|$, $\text{dist}(\Omega', \partial\Omega)$, $\sup_{\Omega} |u|$, and ρ_u .

(ii) Schauder estimate,

$$(2.5) \quad \|u\|_{C^{4,\alpha}(\Omega')} \leq C,$$

where $\alpha \in (0, 1)$, C depends on $n, \alpha, \|f\|_{C^\alpha(\bar{\Omega})}$, $\text{dist}(\Omega', \partial\Omega)$, $\sup_{\Omega} |u|$, and ρ_u .

Note that the gradient of u is locally controlled by ρ_u , the modulus of convexity of u . To prove Theorem 2.1, we write (2.1) as a second order partial differential system

$$(2.6) \quad U^{ij} w_{ij} = f \quad \text{in } \Omega,$$

$$(2.7) \quad \det D^2 u = w^{-(n+2)/(n+1)} \quad \text{in } \Omega,$$

where (2.6) is regarded as a second order elliptic equation for w . By (2.2) and (2.3), and the Hölder continuity of linearized Monge-Ampère equation [3], we have the interior a priori Hölder estimate for w . We note that the Hölder continuity in [3] is proved for the homogeneous equation, but the argument there can be easily carried over to the non-homogeneous case under (2.2) and (2.3). By the interior Schauder estimate for the Monge-Ampère equation [2], we obtain the interior a priori $C^{2,\alpha}$ estimate for u . It follows that (2.6) is a linear uniformly elliptic equation with Hölder coefficients. Hence Theorem 2.1 follows.

The control on strict convexity is a key condition in Theorem 2.1. One cannot expect the strict convexity of solutions when $n \geq 3$. Indeed, there are convex solutions to the Monge-Ampère equation

$$(2.8) \quad \det D^2 u = 1$$

which are not strictly convex, and so not smooth [17]. Note that any non-smooth convex solution of (2.8) can be approximated by smooth ones, and a smooth solution of (2.8) is obviously a solution of (2.1), with $f = 0$.

An interesting problem is to find appropriate conditions to estimate the strict convexity of solutions of (2.1). For the affine Bernstein problem it suffices to prove convexity estimate for solutions vanishing on the boundary. We succeeded only in dimension two, see §5.

3. The affine Bernstein problem

We say a hypersurface \mathcal{M} , immersed in \mathbf{R}^{n+1} , is Euclidean complete if it is complete under the metric induced from the standard Euclidean metric.

Theorem 3.1. *A Euclidean complete, affine maximal, locally uniformly convex surface in \mathbf{R}^3 is an elliptic paraboloid.*

Theorem 3.1 extends Jorgens' theorem [11], which asserts that an entire convex solution of (2.8) in \mathbf{R}^2 must be a quadratic function. Jorgens' theorem also leads to the Bernstein theorem for minimal surfaces in dimension two [11]. Jorgens' theorem was extended to higher dimensions by Calabi [4] for $2 \leq n \leq 5$ and Pogorelov [17] for $n \geq 2$. See also [8]. Observe that the Chern conjecture follows from Theorem 3.1 immediately.

The proof of Theorem 3.1 uses the affine invariance of equation (1.3) and the a priori estimates in §2. First note that a Euclidean complete locally uniformly convex hypersurface must be a graph. Suppose the surface in Theorem 3.1 is the graph of a nonnegative convex function u with $u(0) = 0$. For any constant $h > 1$, let T_h be the linear transformation which normalizes the section $S_{h,u}^0 = \{u < h\}$, and let $v_h(x) = h^{-1}u(T_h^{-1}(x))$. By the convexity estimate in dimension two, the modulus of convexity of v_h is independent of h . Hence there is a uniform positive distance from the origin to the boundary $\partial T_h(S_{h,u}^0)$. By Theorem 2.1, we infer that the largest eigenvalue of T_h is controlled by the least one of T_h , which implies that u is defined in the entire \mathbf{R}^2 . By Theorem 2.1 again, $D^3v_h(0)$ is bounded. Hence for any given $x \in \mathbf{R}^2$,

$$|D^2u(x) - D^2u(0)| \leq Ch^{-1/2} \rightarrow 0$$

as $h \rightarrow 0$, namely $D^2u(x) = D^2u(0)$.

Note that the dimension two restriction is used only for the strict convexity estimate. The affine Bernstein problem was investigated by Calabi in a number of papers [5,6,7]. Using the result that a nonnegative harmonic function (i.e. h in (1.5)) defined on a complete manifold with nonnegative Ricci curvature must be a constant, he proved that, among others, the Bernstein theorem in dimension two, under the additional hypothesis that the surface is also complete under the affine metric.

Instead of the Euclidean completeness as in the Chern conjecture, Calabi asks whether affine completeness alone is sufficient for the Bernstein theorem. This question was recently answered affirmatively in [22]. See also [14] for a different treatment based on the result in [16]. In [22] we proved a much stronger result. That is

Theorem 3.2. *An affine complete, locally uniformly convex hypersurface in \mathbf{R}^{n+1} , $n \geq 2$, is also Euclidean complete.*

The converse of Theorem 3.2 is not true [12], nor is it for $n = 1$. For the proof, which uses the Legendre transform and Lemma 4.1 below, we refer the reader to [22] for details.

4. Locally convex hypersurfaces with boundary

In this section we present some results in [23], which guarantee the sub-convergence of bounded sequences of locally convex hypersurfaces with prescribed boundary.

Recall that a hypersurface $\mathcal{M} \subset \mathbf{R}^{n+1}$ (not necessarily smooth) is called locally convex if it is a locally convex immersion of a manifold \mathcal{N} and there is a continuous vector field on the convex side of \mathcal{M} , transversal to \mathcal{M} everywhere. Let T denote the immersion, namely $\mathcal{M} = T(\mathcal{N})$. For any given point $x \in \mathcal{M}$, $T^{-1}(x)$ may contain more than one point. To avoid confusion when referring to a point $x \in \mathcal{M}$ we understand a pair (x, p) for some point $p \in \mathcal{N}$ such that $x = T(p)$. We say $\omega_x \subset \mathcal{M}$ is a neighborhood of $x \in \mathcal{M}$ if it is the image of a neighborhood of p in \mathcal{N} . The r -neighborhood of x , $\omega_r(x)$, is the connected component of $\mathcal{M} \cap B_r(x)$ containing the point x . In [23] we proved the following fundamental lemma for locally convex hypersurfaces.

Lemma 4.1. *Let \mathcal{M} be a compact, locally convex hypersurface in \mathbf{R}^{n+1} , $n > 1$. Suppose the boundary $\partial\mathcal{M}$ lies in the hyperplane $\{x_{n+1} = 0\}$. Then any connected component of $\mathcal{M} \cap \{x_{n+1} < 0\}$ is convex.*

A locally convex hypersurface \mathcal{M} is called convex if it lies on the boundary of the convex closure of \mathcal{M} itself. From Lemma 4.1 it follows that a (Euclidean) complete locally convex hypersurface with at least one strictly convex point is convex, and that a closed, locally convex hypersurface is convex. Lemma 4.1 also plays a key role in the proof of Theorem 3.2.

An application we will use here is the uniform cone property for locally convex hypersurfaces. Let $\mathcal{C}_{x,\xi,r,\alpha}$ denote the cone

$$\mathcal{C}_{x,\xi,r,\alpha} = \{y \in \mathbf{R}^{n+1} \mid |y - x| < r, \langle y - x, \xi \rangle \geq \cos \alpha |y - x|\}.$$

We say that $\mathcal{C}_{x,\xi,r,\alpha}$ is an inner contact cone of \mathcal{M} at x if this cone lies on the concave side of $\omega_r(x)$. We say \mathcal{M} satisfies the *uniform cone condition* with radius r and aperture α if \mathcal{M} has an inner contact cone at all points with the same r and α .

Lemma 4.2. *Let $\mathcal{M} \subset B_R(0)$ be a locally convex hypersurface with boundary $\partial\mathcal{M}$. Suppose \mathcal{M} can be extended to $\widetilde{\mathcal{M}}$ such that $\partial\mathcal{M}$ lies in the interior of $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}} - \mathcal{M}$ is locally strictly convex. Then there exist $r, \alpha > 0$ depending only on n, R , and the extended part $\widetilde{\mathcal{M}} - \mathcal{M}$, such that the r -neighborhood $\omega_r(x)$ is convex for any $x \in \mathcal{M}$, and \mathcal{M} satisfies the uniform cone condition with radius r and aperture α .*

In [23] we have shown that if $\partial\mathcal{M}$ is smooth and \mathcal{M} is smooth and locally uniformly convex near $\partial\mathcal{M}$, then \mathcal{M} can be extended to $\widetilde{\mathcal{M}}$ as required in Lemma 4.2. The main

point of Lemma 4.2 is that r and α depend only on n, R and the extended part $\widetilde{\mathcal{M}} - \mathcal{M}$. Therefore it holds with the same r and α for a family of locally convex hypersurfaces, which includes all locally uniformly convex hypersurfaces with boundary $\partial\mathcal{M}$, contained in $B_R(0)$, such that its Gauss mapping image coincides with that of \mathcal{M} . For any sequence of locally convex hypersurfaces in this family, the uniform cone property implies the sequence converges subsequently and no singularity develops in the limit hypersurface. This property is the key for the existence proof of maximizers to the affine Plateau problem. It also plays a key role for our resolution of the Plateau problem for prescribed constant Gauss curvature (as conjectured in [20]), see [23]. We state the result as follows.

Theorem 4.1. *Let $\Gamma = (\Gamma_1, \dots, \Gamma_n) \subset \mathbf{R}^{n+1}$ be a smooth disjoint collection of closed co-dimension two embedded submanifolds. Suppose Γ bounds a locally strictly convex hypersurface S with Gauss curvature $K(S) > K_0 > 0$. Then Γ bounds a smooth, locally uniformly convex hypersurface of Gauss curvature K_0 .*

If S is a (multi-valued) radial graph over a domain in S^n which does not contain any hemi-spheres, Theorem 4.1 was established in [10]. Theorem 4.1 has been extended to more general curvature functions in [18].

5. The affine Plateau problem

First we formulate the affine Plateau problem as a variational maximization problem. Let \mathcal{M}_0 be a compact, connected, locally uniformly convex hypersurface in \mathbf{R}^{n+1} with smooth boundary $\Gamma = \partial\mathcal{M}_0$. Let $S[\mathcal{M}_0]$ denote the set of locally uniformly convex hypersurfaces \mathcal{M} with boundary Γ such that the image of the Gauss mapping of \mathcal{M} coincides with that of \mathcal{M}_0 . Then any two hypersurfaces in $S[\mathcal{M}_0]$ are diffeomorphic. Let $\overline{S}[\mathcal{M}_0]$ denote the set of locally convex hypersurfaces which can be approximated by smooth ones in $S[\mathcal{M}_0]$. Our variational affine Plateau problem is to find a smooth maximizer to

$$(5.1) \quad \sup_{\mathcal{M} \in \overline{S}[\mathcal{M}_0]} A(\mathcal{M}).$$

To study (5.1) we need to extend the definition of the affine area functional to non-smooth convex hypersurfaces. Different but equivalent definitions can be found in [13]. Here we adopt a new definition introduced in [21, 24], which is also more straightforward. Observe that the Gauss curvature K can be extended to a measure on a non-smooth convex hypersurface, and the measure can be decomposed as the sum of a singular part and a regular part, $K = K_s + K_r$, where the singular part K_s is a measure supported on a set of Lebesgue measure zero, and the regular part K_r can be represented by an integrable function. We extend the definition of affine area functional (1.1) to

$$(5.2) \quad A(\mathcal{M}) = \int_{\mathcal{M}} K_r^{1/(n+2)}.$$

The affine area functional is upper semi-continuous [13,15]. See also [21,24] for different proofs.

A necessary condition for the affine Plateau problem is that the Gauss mapping image of \mathcal{M}_0 cannot contain any semi-spheres. Indeed if \mathcal{M} is affine maximal such that its Gauss mapping image contains, say, the south hemi-sphere, then the pre-image of the south hemi-sphere is a graph of a convex function u over a domain Ω such that $|Du(x)| \rightarrow \infty$ as $x \rightarrow \partial\Omega$. Then necessarily $\det D^2u = \infty$ and so $w = 0$ on $\partial\Omega$. It follows that $w \equiv 0$ in Ω , a contradiction.

Theorem 5.1. *Let \mathcal{M}_0 be a compact, connected, locally uniformly convex hypersurface in \mathbf{R}^3 with smooth boundary $\Gamma = \partial\mathcal{M}_0$. Suppose the image of the Gauss mapping of \mathcal{M}_0 does not contain any semi-spheres. Then there is a smooth maximizer to (5.1).*

To prove the existence we observe that by the necessary condition, there exists a positive constant R such that $\mathcal{M} \subset B_R(0)$ for any $\mathcal{M} \in \overline{S}[\mathcal{M}_0]$. Hence by the uniform cone property, Lemma 4.2, any maximizing sequence in $\overline{S}[\mathcal{M}_0]$ is sub-convergent. The existence of maximizers then follows from the upper semi-continuity of the affine area functional. Note that the existence is true for all dimensions.

To prove the regularity we need to show that

- (i) \mathcal{M} can be approximated by smooth affine maximal surfaces; and
- (ii) \mathcal{M} is strictly convex.

The purpose of (i) is such that the a priori estimate in Section 2 is applicable. Note that (i) also implies the Bernstein Theorem 3.1 holds for non-smooth affine maximal surfaces.

By the penalty method we proved (i) for all dimensions, using the following classical solvability of the second boundary value problem for the affine maximal surface equation.

Theorem 5.2. *Consider the problem*

$$(5.3) \quad \begin{aligned} L(u) &= f(x, u) \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega, \\ w &= \psi \quad \text{on } \partial\Omega, \end{aligned}$$

where w is given in (1.4), Ω is a uniformly convex domain with $C^{4,\alpha}$ boundary, $0 < \alpha < 1$, f is Hölder continuous, non-decreasing in u , $\varphi, \psi \in C^{4,\alpha}(\overline{\Omega})$, and ψ is positive. Then there is a unique uniformly convex solution $u \in C^{4,\alpha}(\overline{\Omega})$ to the above problem.

To prove Theorem 5.2 we first prove that u satisfies (2.2) and (2.3), and that w is Lipschitz continuous on the boundary, namely $|w(x) - w(y)| \leq C|x - y|$ for any $x \in \Omega$, $y \in \partial\Omega$. Theorem 5.2 is then reduced to the boundary $C^{2,\alpha}$ estimate for the Monge-Ampère equation. The proof for the boundary $C^{2,\alpha}$ estimate involves a delicate iteration scheme. We refer to [24] for details.

The interior $C^{2,\alpha}$ estimate for the Monge-Ampère equation was proved by Caffarelli [2], using a perturbation argument. The boundary $C^{2,\alpha}$ estimate, which also uses a similar perturbation argument, contains substantial new difficulty, as that the sections

$$S_{h,v}^0(y) = \{x \in \Omega \mid v(x) < v(y) + Dv(y)(x - y) + h\}$$

can be normalized for the interior estimate but not for the boundary estimate. We need to prove that $S_{h,v}^0(y)$ has a good shape for sufficiently small $h > 0$ and $y \in \partial\Omega$.

Finally we would like to mention our idea of proving the strict convexity, namely (ii) above. Note that for both the affine Bernstein and affine Plateau problem, the dimension two assumption is only used for the proof of the strict convexity. To prove the strict convexity we suppose to the contrary that \mathcal{M} contains a line segment. Let P be a tangent plane of \mathcal{M} which contains the line segment. Then the contact set F , namely the connected set of $P \cap \mathcal{M}$ containing the line segment, is a convex set. If F has an extreme point which is an interior point of \mathcal{M} , by rescaling and choosing appropriate coordinates we obtain a sequence of affine maximal functions which converges to a convex function v , such that $v(0) = 0$ and $v(x) > 0$ for $x \neq 0$ in an appropriate coordinate system, and v is not C^1 at the origin 0. In dimension two this means $\det D^2v$ is unbounded near 0, which is in contradiction with the estimate (2.2).

If all extreme points of F are boundary points of \mathcal{M} , we use the Legendre transform to get a new convex function which is a maximizer of a variational problem similar to (5.1), and satisfies the properties as v above, which also leads to a contradiction.

6. Remarks

We proved the affine Bernstein problem in dimension two. In high dimensions ($n \geq 10$) a counter-example was given in [21], where we proved that the function

$$(6.1) \quad u(x) = (|x'|^9 + x_{10}^2)^{1/2}$$

is affine maximal, where $x' = (x_1, \dots, x_9)$.

The function u in (6.1) contains a singular point, namely the origin. The graph of u is indeed an *affine cone*, that is all the level sets $S_{h,u} = \{u = h\}$ are affine self-similar, in the sense that there is an affine transformation T_h such that $T_h(S_{h,u}) = S_{1,u}$. The above counter-example shows that there is an affine cone in dimensions $n \geq 10$ which is affine maximal but is not an elliptic paraboloid. We have not been successful in finding smooth counter-examples. Little is known for dimensions $3 \leq n \leq 9$.

For the affine Plateau problem, an interesting problem is whether the maximizer satisfies the boundary conditions. If \mathcal{M}_0 is the graph of a smooth, uniformly convex function φ , defined in a bounded domain $\Omega \subset \mathbf{R}^n$, the Plateau problem becomes the first boundary value problem, that is equation (1.3) subject to the boundary conditions:

$$(6.2) \quad u = \varphi \text{ on } \partial\Omega,$$

$$(6.3) \quad Du = D\varphi \text{ on } \partial\Omega.$$

In this case we also proved the uniqueness of maximizers of (5.1). Obviously the maximizer u satisfies (6.2). Whether u satisfies (6.3) is still unknown. Recall that the Dirichlet problem of the minimal surface equation is solvable for any smooth boundary values if and

only if the boundary is mean convex. Therefore an additional condition may be necessary in order that (6.3) is fulfilled.

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