The affine maximal surface equation

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$$U^{ij} w_{x_i x_j} = 0, \tag{0.1}$$

where $w = \left[\det D^2 u\right]^{-\frac{n+1}{n+2}}$, (U^{ij}) is the co-factor matrix of D^2u, is the Euler equation of the affine volume functional $A(u) = \int_\Omega \left[\det D^2 u\right]^{1/(n+2)}$. It is the analogue of the minimal surface equation in affine geometry, and can be viewed as a system of two second order pdes, which are elliptic when u is convex. As the functional A is concave, a convex solution to (0.1) must be a maximizer of A under local perturbation, and so is called affine maximal.

1. Affine Bernstein problems. S.S. Chern conjectured that *A Euclidean complete, locally convex, affine maximal surface in* $\mathbb{R}^3$ *is an elliptic paraboloid.* A related problem, called Calabi's conjecture, asks whether *An affine complete, locally convex, affine maximal surface in* $\mathbb{R}^3$ *is an elliptic paraboloid.* Calabi himself proved the Bernstein theorem under both affine and Euclidean completeness. We proved that both conjectures hold true [am1, am2]. In [am2] we proved that affine completeness implies Euclidean completeness when $n > 1$, which is itself a long standing open problem in affine geometry. Therefore the Chern conjecture is stronger than the Calabi conjecture. A.M. Li and F. Jia proved Calabi's conjecture by a different method.

2. Affine Plateau problem. Also raised by Chern and Calabi, it is the affine invariant analogue of the classical Plateau problem for minimal surfaces. We reformulated it as a geometric variational problem for the affine area functional, and proved the existence of maximizers in all dimensions, and the interior regularity in 2-dim [am3].

A special case is the first boundary value problem, namely prescribing the function u and its gradient Du on the boundary $\partial\Omega$. We obtained the corresponding existence and regularity results for the variational first boundary value problem [am3].

We also established the global regularity of solutions to the second boundary value problem, that is prescribing u and w on $\partial\Omega$ [am4].

3. Affine mean curvature equation

$$U^{ij} w_{x_i x_j} = f(x). \tag{0.2}$$

By Caffarelli's $C^{2,\alpha}$ estimate for strict convex solutions to the Monge-Ampère equation, and Caffarelli-Gutierrez's Hölder continuity for the linearized Monge-Ampère equation, we have the following $C^{4,\alpha}$ and $W^{4,p}$ estimates [am3, am5].

$W^{4,p}$ estimate. Let $u \in C^4(\Omega)$ be a locally uniformly convex solution of (0.2). Then $\forall\ \Omega' \subset\subset \Omega$, $p \ge 1$,

$$\|u\|_{W^{4,p}(\Omega')} \le C,$$

where C depends on $n, p, \sup_\Omega |f|$, $\text{dist}(\Omega', \partial\Omega)$, and the modulus of convexity of u.

$C^{4,\alpha}$ estimate. Let $u \in C^4(\Omega)$ be a locally uniformly convex solution of (0.2) with $f \in C^\alpha(\bar{\Omega})$, $0 < \alpha < 1$. Then $u \in C^{4,\alpha}(\Omega)$ and for any $\Omega' \subset\subset \Omega$,

$$\|u\|_{C^{4,\alpha}(\Omega')} \le C,$$

where C depends on $n, \alpha, \|f\|_{C^\alpha(\Omega)}$, $\mathrm{dist}(\Omega', \partial\Omega)$, and the modulus of convexity of u.

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