# The $k$-Hessian equation

For a function $u \in C^2(\Omega)$, where $\Omega$ is a domain in $\mathbb{R}^n$, the $k$-Hessian operator $F_k[u]$ is the $k$-trace ($k^{th}$ elementary symmetric polynomials of the eigenvalues) of the Hessian matrix $D^2u$. It is respectively the Laplacian and the Monge-Ampère operator when $k = 1$ and $k = n$. It is fully nonlinear (when $k \geq 2$), of divergent form, and is elliptic when $u$ is $k$-admissible.

The global regularity of $k$-admissible solutions to the Dirichlet problem was established by Caffarelli, Nirenberg, and Spruck, and also by Ivochkina in some cases. The global regularity for more general equations, including the Hessian quotient equations, was obtained by Trudinger. In the degenerate case, the global second derivative estimate for the Dirichlet problem was obtained by Krylov. Bo Guan and John Urbas have also made important contributions to the equation.

We did the following works:
1. Proved a class of Sobolev type inequalities.
2. Developed the theory of Hessian measures.

By the Sobolev type inequality, we proved the existence of Mountain-Pass solutions to the corresponding variational problems, in both the sub-critical and the critical growth cases [he4, he3]. We also proved an interior second derivative estimate of Pogorelov type for the $k$-Hessian equation [he4] (higher regularity follows from Evans and Krylov's regularity theory).

By our Hessian measures, a Wolff potential estimate for the $k$-Hessian equation was obtained by D. Labutin. Recently N.C. Phuc and I.E. Verbitsky also obtained some interesting results on the $k$-Hessian equation, including a different proof of the Sobolev type inequality by using the Wolff potential estimate.

**Theorem 1 (Sobolev type inequalities [he1]).**
*Let $\Phi_0^k(\Omega)$ denote the set of $k$-admissible functions vanishing on $\partial\Omega$. Denote*

$$\|u\|_{\Phi_0^k(\Omega)} = \Big[\int_\Omega (-u)F_k[u]\Big]^{\frac{1}{k+1}}.$$

*(i) If $1 \leq k < \frac{n}{2}$, then $\forall\ u \in \Phi_0^k(\Omega)$*

$$\|u\|_{L^{p+1}(\Omega)} \leq C\|u\|_{\Phi_0^k} \quad \forall\ \ p+1 \in [1, k^*],$$

*where $k^* = \frac{n(k+1)}{n-2k}$, $C$ depends only on $n, k, p$, and $|\Omega|$.*
*(ii) If $k = \frac{n}{2}$, then $\forall\ u \in \Phi_0^k(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{\Phi_0^k}$$

*for any $p < \infty$, where $C$ depends only on $n, p$, and $diam(\Omega)$.*
*(iii) If $\frac{n}{2} < k \leq n$, then $\forall\ u \in \Phi_0^k(\Omega)$,*

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{\Phi_0^k},$$

*where $C$ depends on $n, k,$ and $diam(\Omega)$.*

**Remarks**

(i) When $k < \frac{n}{2}$, the exponent $k^*$ above is optimal. When $p+1 = k^*$ and $\Omega = \mathbb{R}^n$, the best constant is achieved by the function $u(x) = [1+|x|^2]^{(2k-n)/2k}$.
(ii) When $k = \frac{n}{2}$, we have the following Moser-Trudinger type inequality [he2]. For any $u \in \Phi_0^k(\Omega)$,

$$\sup\Big\{\int_\Omega exp\big(\alpha\big(\frac{u}{\|u\|_{\Phi_0^k}}\big)^\beta\big) : \quad u \in \Phi_0^k(\Omega)\Big\} < C,$$

where $\alpha = n\big[\frac{\omega_{n-1}}{k}\binom{n-1}{k-1}\big]^{2/n}$, $\beta = \frac{n+2}{n}$, $\omega_n$ is the surface area of the unit sphere $S^n$, and $C$ depends only on $n$ and $\text{diam}(\Omega)$.
(iii) When $k > \frac{n}{2}$, any $k$-admissible function is Hölder continuous with the optimal exponent $\alpha = 2 - \frac{n}{k}$ [he6].
(iv) By a gradient flow argument we proved [he5] that for any $u \in \Phi_0^k(\Omega)$, $\|u\|_{\Phi_0^l(\Omega)} \le C\|u\|_{\Phi_0^k(\Omega)}$, where $1 \le l < k \le n$ and $C$ depends on $n, k, l,$ and $\Omega$.
(v) The domain $\Omega$ must be $(k-1)$-convex if $\Phi_0^k \ne \emptyset$. Theorem 1 was proved by a gradient flow method.

For Hessian measures, we extend the notion of $k$-admissibility to upper-semicontinuous functions. One of the main ingredients in the Hessian measures is

**Theorem 2 (weak convergence of Hessian measures [he6]).**
*For any $k$-admissible function $u$, there is an associated Borel measure $\mu_k[u]$ such that*
*(i) $\mu_k[u] = F_k[u]$ when $u \in C^2$, and*
*(ii) if $\{u_m\}$ is a sequence of $k$-admissible functions converging to $u$ in $L^1$, then $\mu_k[u_m]$ converges to $\mu_k[u]$ weakly.*

The proof was built upon sharp integral estimates for $k$-admissible functions and their gradients. The above weak convergence has been used to prove various properties of $k$-admissible functions, like those in the Newton potential theory for subharmonic functions.

Applying techniques in Hessian measures to quasilinear sub-elliptic equations, we obtained the corresponding weak continuity result and Wolff potential estimate [he7].

## References


[he1] X.-J. Wang, A class of fully nonlinear elliptic equations and related functionals, Indiana Univ. Math. J., 43(1994), 25-54.
[he2] G.J. Tian, X.-J. Wang, Moser-Trudinger type inequalities for the k-Hessian equation, preprint.
[he3] X.-J. Wang, The k-Hessian equation, Lectures Notes in Math. Vol.1977, Springer, 2009, 70pp.
[he4] K.S. Chou and X.-J. Wang, A variational theory of the Hessian equation, Comm. Pure and Applied Math., 54(2001), 1029-1064.
[he5] N.S. Trudinger, X.-J. Wang, Poincare type inequalities for Hessian integrals, Calc. Var. PDE, 6(1998), 315-328.
[he6] N.S. Trudinger, X.-J. Wang, Hessian measures
I, Topolo. Methods in Nonlinear Anal., 10(1997), 225-239;
II, Annals of Math., 150(1999), 579-604; III, J. Functional Anal., 193(2002), 1-23.

[he7] N.S. Trudinger, X.-J. Wang, On the weak continuity of elliptic operators and applications to potential theory, Amer. J. Math., 124 (2002), 369-410.