The Monge-Ampère equation

The Monge-Ampère equation is the most important fully nonlinear partial differential equation, with various applications in geometry and physics. Many people made important contributions to the equation. We studied the Dirichlet problem of the real Monge-Ampère equation

$$
\begin{aligned}
\det D^2 u &= f(x) \quad \text{in } \Omega, \\
u &= \phi \quad \text{on } \partial\Omega.
\end{aligned} \tag{0.1}
$$

One of our main results is

Theorem (global regularity [ma2]). *Let Ω be a uniformly convex domain in $\mathbb{R}^n$, with boundary $\partial\Omega \in C^3$. Suppose $\phi \in C^3(\bar{\Omega})$, $\inf f > 0$, and $f \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1)$. Then (0.1) has a convex solution u which satisfies the a priori estimate*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \le C,$$

where C depends only on n, α, Ω, $\inf f$, $\|f\|_{C^\alpha(\bar{\Omega})}$, and $\|\phi\|_{C^3}$.

Using this theorem, we proved the global regularity for the second boundary value problem of the affine maximal surface equation [ma2].

All conditions in the theorem are sharp. In particular, the assumptions $\partial\Omega, \phi \in C^3$ are optimal, they cannot be weakened to $\partial\Omega \in C^{2,1}$ or $\phi \in C^{2,1}$ [ma6]. Under sufficiently smooth conditions on f, ϕ and $\partial\Omega$, this theorem was previously obtained independently by Caffarelli, Nirenberg, and Spruck, and by Krylov. Our proof used a key lemma in Caffarelli-Nirenberg-Spruck's paper. The interior $C^{2,\alpha}$-estimate for Hölder continuous f and $W^{2,p}$-estimate for continuous f were obtained by Caffarelli; see [ma3] for a detailed proof of the $C^{2,\alpha}$-estimate, where we also proved the continuity of D^2u for Dini continuous f.

• In [ma4] we established the global second derivative estimate for solutions to the Dirichlet problem of degenerate Monge-Ampère equations (i.e. $f \ge 0$), under the assumptions that $\partial\Omega, \phi \in C^{3,1}$ and $f^{1/(n-1)} \in C^{1,1}$. By an example in [ma7], the condition $f^{1/(n-1)} \in C^{1,1}$ is optimal. The assumptions $\partial\Omega, \phi \in C^{3,1}$ are also optimal. This estimate was previously proved by Krylov under the assumption $f^{1/n} \in C^{1,1}$.

• In [ma5] we proved the existence of infinitely many entire convex solutions to the Monge-Ampère equation (0.1) in $\mathbb{R}^n$, under the assumption $c_1 \le f \le c_2$ for positive constants $c_2 \ge c_1$. This assumption can be relaxed a little bit, but whether there exists an entire convex solution to (0.1) for any given positive function f is still an open problem. The result implies the existence of complete, noncompact solutions to the Minkowski problem and to the Gauss curvature flow. When $f \equiv 1$, an entire convex solution must be a quadratic polynomial (by Jörgens for $n = 2$, Calabi for $n \le 5$, and Pogorelov for all n).

References

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