Optimal transportation

In optimal transportation we wish to transport one mass distribution to another one such that the total cost is minimized. The problem was first studied by Monge in his paper in 1781, and a major breakthrough was made by Kantorovich. Due to various applications, in the last two decades there were renewed interests in the topic, and many significant works have been obtained. Our main contribution is about the regularity of the optimal mappings.

Regularity in optimal transportation.
By Kantorovich's dual functional, the optimal mapping $T: \Omega \to \Omega^*$ is determined by a potential function u which satisfies the Monge-Ampère type equation

$$\det\left(D^2 u - c_{xx}(x, T(x)\right) = f(x)/g(T(x)), \tag{0.1}$$

subject to the natural boundary condition $T(\Omega) = \Omega^*$, where $c(x, y)$ is the cost function, f, g are the mass distributions, Ω, Ω^* are two domains in $\mathbb{R}^n$.

For the special cost function $c(x, y) = |x - y|^2$, (0.1) is the standard Monge-Ampère equation, the regularity was obtained by Caffarelli under sharp conditions, and also by Delanoe ($n = 2$) and Urbas ($n \geq 2$) when f, g and the domains Ω, Ω^* are sufficiently smooth.

We studied the regularity of solutions for general cost functions.
(i) In [ot2] we proved the interior C^3 regularity if $f, g > 0, \in C^2$, Ω^* is c-convex, and the cost function c satisfies a condition A3.
(ii) In [ot5] we obtained the interior $C^{2,\alpha}$ regularity for Hölder continuous f, g.
(iii) In [ot4] we proved the global C^3 regularity if $f, g > 0, \in C^2$, both Ω and Ω^* are c-convex, and c satisfies A3w (a degenerate form of A3).

The c-convexity of domains in the above papers is necessary [ot2]. Leoper proved A3w is also necessary for the regularity. He also proved the $C^{1,\alpha}$ regularity for measurable f, g, and the optimal exponent α was later found by Jiakun Liu. In the above papers we also assume two natural conditions A1 and A2 for the existence and uniqueness of optimal mappings (which are due to Brenier, Caffarelli, Gangbo and McCann).

• Monge's problem. In Monge's paper the cost function is the natural one, $c(x, y) = |x - y|$, namely the cost is proportional to the distance the mass is moved. This cost function does not satisfy the conditions A1 and A2 mentioned above, and optimal mapping is not unique. There was a very complicated proof for the existence of optimal mappings by Sudakov in 1970s and his proof contains a gap. In [ot1] we gave a much shorter proof for Monge's problem. A similar proof was simultaneously found by Caffarelli-Feldman-McCann. A proof using the p-Laplace operator was given earlier by Evans and Gangbo, under some mild assumptions on the mass distributions. The gap in Sudakov's paper has also been fixed by Ambrosio.

References

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