Exact triangles in Seiberg-Witten Floer theory. Part II: geometric limits of flow lines

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Contents

1 Introduction

This work is the continuation of [4]. In the previous part we analyzed the geometric limits of solutions of the Seiberg–Witten equations on a 3-manifold $Y(r)$ with a long cylinder $T^2 \times [-r, r]$. We applied the result to the case of a homology sphere Y with an embedded knot K along which Dehn surgery with framing one or zero is performed to produce a homology sphere Y_1 or a manifold Y_0 with the homology of $S^1 \times S^2$. We proved that there is a suitable choice of metrics on Y and Y_1, and of a "surgery perturbation" μ of the equations on a tubular neighborhood of the knot in Y, such that the moduli spaces of solutions on Y, Y_1, and Y_0 are related by

$$\mathcal{M}_{Y,\mu} \cong \mathcal{M}_{Y_1} \cup \bigcup_k \mathcal{M}_{Y_0}(\mathbf{s}_k),$$

where the $\mathbf{s}_k$ are the possible choices of Spin^c structures on Y_0. We also analyzed the splitting of the spectral flow and the relative grading of the Floer complexes on Y, Y_1, and Y_0.

In this paper we concentrate on the analysis of the geometric limits of flow lines, that is, of solutions of the Seiberg–Witten equations on the four-manifold $Y(r) \times \mathbf{R}$, as $r \to \infty$. This is necessary for an understanding of how the boundary operators in the Floer complexes of Y, Y_1, and Y_0 are related.

We first consider the unperturbed equations on $Y(r) \times \mathbf{R}$. We derive estimates that control the convergence on compact sets. The essential phenomenon which regulates the behavior of solutions is a non-uniformity of the convergence in the non-compact time direction $t \in \mathbf{R}$, as we stretch the length of the cylinder $r \to \infty$. This can be seen as an effect of the presence of small eigenvalues of the linearization of the Seiberg-Witten equations on $Y(r)$ at the two asymptotic values corresponding to $t \to \pm\infty$. In fact, as already observed in [4], the linearization at these asymptotic values is a self-adjoint first order elliptic operator with small eigenvalues decaying like $1/r$, as $r \to \infty$.

We can decompose the manifold Y as $Y = V \cup_{T^2} \nu(K)$, where V is the knot complement and $\nu(K)$ is a tubular neighborhood of the knot. We use the same notation V and $\nu(K)$ in the following to indicate the manifolds completed with an infinite cylindrical end of the form $T^2 \times [0, \infty)$.

We first analyze the properties of the geometric limits obtained in the convergence of solutions $(\mathcal{A}_r, \Psi_r)$ on fixed compact sets independent of $r \geq r_0$. We show that on the two sides $V \times \mathbf{R}$ and $\nu(K) \times \mathbf{R}$ these limits on compact sets decay exponentially in radial gauge along the region $T^2 \times [0, \infty) \times \mathbf{R}$ to an asymptotic value that is (up to gauge) a constant flat connection on T^2.

We then analyze the convergence in asymptotic regions of the form $Y(r) \times ([T_r, \infty) \cup (-\infty, -T_r])$, "away from compact sets". To this purpose, we introduce a suitable rescaling of the coordinates in such a way as to report within a finite region the behavior in the complement of an arbitrarily large compact sets. After the rescaling, we obtain convergence to paths in $\mathcal{M}_V$ and $\mathcal{M}_{\nu(K)}$, and to holomorphic maps in $\chi_0(T^2, V)$ and $\chi_0(T^2, \nu(K))$.

These data match the geometric limits on the flat cylindrical region $T^2 \times [-r, r] \times \mathbf{R}$, which also consist of a flat connection, as limit uniformly on compact set, and a holomorphic map as limit after the rescaling. Thus, we can use the data of the geometric limits described above in order to form approximate solutions on $Y(r) \times \mathbf{R}$, for $r(T, \tau)$ depending on the rescaling parameters, used here as gluing parameters. We prove surjectivity of the linearization of the Seiberg–Witten equations at these approximate solutions. Thus, the approximate solutions can be deformed to actual solutions on $Y(r) \times \mathbf{R}$. In other words, we obtain a complete analogue of the gluing theorem for solutions of the 3-dimensional Seiberg–Witten monopole equations analyzed in [4].

The rescaling technique and radial gauge limits that we consider in this paper can be thought of as an analogue of the analysis of non-abelian ASD equations and holomorphic curves of [5].

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2 Energy and curvature estimates

Let Y be a compact oriented smooth three-manifold with a fixed trivialization of the tangent bundle, with a local basis $\{e_i\}$. This trivialization determines a *Spin*-structure on Y, with spinor bundle S. Twisting S with a line bundle L gives a Spin^c-structure $\mathbf{s}$, with spinor bundle W and determinant $\det(W) = \det(S) \otimes L^2$. For the purpose of this paper we shall be concerned with the case of a homology sphere Y, with the unique choice of the trivial Spin^c-structure, or with the case of Y_0, a 3-manifold with the homology of $S^1 \times S^2$, with the infinite family of Spin^c structures $\mathbf{s}_k$ with

$c_1(\det(W_k)) = 2k$, as discussed in [4].

On the manifold $Y(r)$, consider a solution (A_r, ψ_r) of the Seiberg–Witten equations

$$\begin{aligned} * F_A &= \sigma(\psi, \psi) \\ \partial_A \psi &= 0, \end{aligned} \tag{1}$$

given in terms of a $U(1)$-connection A on the line bundle L, and a section $\psi \in \Gamma(Y, W)$. The 1-form $\sigma(\psi, \psi)$ is given in local coordinates by $\sigma(\psi, \psi) = \sum_i \langle e_i.\psi, \psi\rangle e^i$. Solutions are critical points of the Chern–Simons–Dirac functional

$$CSD(A, \psi) = -\frac{1}{2}\int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \int_Y \langle \psi, \partial_A \psi\rangle dvol_Y.$$

For the moment, we only consider the unperturbed functional and the unperturbed equations (1). Later in this paper it will be necessary to formulate the results in the perturbed case, with the perturbations introduced in [4].

Let L_{A_r,ψ_r} be the linearization of the equations, namely the operator

$$L_{A_r,\psi_r}(\alpha, \phi) = \left(- * d\alpha + \sigma(\psi_r, \phi), \partial_{A_r}\phi + \frac{1}{2}\alpha.\psi_r\right).$$

This, together with the infinitesimal action of the gauge group and the gauge fixing condition, defines the extended Hessian of the Chern-Simons-Dirac functional, namely the first order elliptic operator

$$\begin{aligned} H_{A_r,\psi_r}(f, \alpha, \phi) = \quad & (- * d\alpha + \sigma(\psi_r, \phi) - df, \\ & \partial_{A_r}\phi + \tfrac{1}{2}\alpha.\psi_r + f\psi_r, \\ & -d^*\alpha - iIm\langle\psi_r, \phi\rangle). \end{aligned}$$

In [4] we proved the following gluing theorem. We assigned a metric on Y which has positive scalar curvature on a tubular neighborhood $\nu(K)$ of the knot. We proved that, with this choice of the metric, for large $r \geq R_0$, the solutions (A_r, ψ_r) can be written as a gluing

$$(A_r, \psi_r) = (A', \psi')\#_r(A'', 0).$$

Here (A', ψ') is a finite energy solution of (1) on the knot complement V endowed with an infinite cylindrical end $T^2 \times [0, \infty)$, and $(A'', 0)$ is a flat connection on $\nu(K)$. The length of the cylinder $r \geq R_0$ appears as the gluing parameter. The gluing happens at the asymptotic value a_∞ of (A', ψ') along the end $T^2 \times [0, \infty)$ of the knot complement V: a_∞ is a flat connection on T^2, which lives in a cyclic cover $\chi_0(T^2, V)$ of the character variety $\chi(T^2)$. Up to a gauge transformation, a_∞ agrees with the restriction to T^2 of the connection A''.

In this paper we formulate the analogous gluing theorem for the gradient flow lines of the Chern–Simons–Dirac functional, that is, for solutions of the Seiberg–Witten equations on $Y(r) \times \mathbf{R}$. The gluing theorem, in this case, is more complicated because of the lack of uniformity in the convergence as $r \to \infty$. This gives rise to more complicated geometric limits. We report here a statement from [4] which is the ultimate source of the interesting non-uniform limits of flow lines that we are going to discuss in the rest of this paper.

Proposition 2.1 *The dimension $N(r, o(1))$ of the span of eigenvectors of the operator H_{A_r,ψ_r} with eigenvalues satisfying $\mu(r) \to 0$ as $r \to \infty$ is given by*

$$N(r, o(1)) = \dim Ker_{L^2}(H_{(A',\psi')}) + \dim Ker_{L^2}(H_{(A'',0)}) + \dim Ker(Q_{a_\infty}),$$

where a_∞ is the asymptotic value of (A', ψ') as $s \to \infty$.

For any $\epsilon > 0$, the dimension $N(r, r^{-(1+\epsilon)})$ of the span of eigenvectors of the operator H_{A_r,ψ_r} with eigenvalues $\mu < r^{-(1+\epsilon)}$ is given by

$$N(r, r^{-(1+\epsilon)}) = \dim Ker_{L^2}(H_{(A',\psi')}) + \dim Ker_{L^2}(H_{(A'',0)}) + \dim \ell_1 \cap \ell_2,$$

where $\ell_1 \cap \ell_2$ is the intersection of the two Lagrangian submanifolds determined by the extended L^2 solutions of $H_{(A',\psi')}(\alpha, \phi) = 0$ and $H_{(A'',0)}(\alpha, \phi) = 0$ in the tangent space $H^1(T^2, i\mathbf{R}) = Ker(Q_{a_\infty})$ of $\chi_0(T^2)$.

Thus, under the assumption that

$$Ker_{L^2}(H_{(A',\psi')}) = Ker_{L^2}(H_{(A'',0)}) = 0,$$

we have $N(r, o(1)) = 2$, generated by the elements of $H^1(T^2, i\mathbf{R})$, and $N(r, r^{-(1+\epsilon)}) = 0$, for all $\epsilon > 0$.

The proof of Proposition 2.1 relies on the results of [3], cf. [4].

Let $Y(r)$ be a closed three-manifold with a long tube $[-r, r] \times T^2$, endowed with the flat product metric on the tube.

Let $(\mathcal{A}_r, \Psi_r)$ be a finite energy solution of the Seiberg-Witten equations on the four-manifold $Y(r) \times \mathbf{R}$, that is, $(\mathcal{A}_r, \Psi_r)$ satisfies the equations

$$\begin{aligned} F^+_{\mathcal{A}} &= \tau(\Psi, \Psi) \\ D_{\mathcal{A}}\Psi &= 0, \end{aligned} \tag{2}$$

with the condition

$$\mathcal{E}_r = \|\partial_t A_r\|^2_{L^2(Y(r)\times\mathbf{R})} + \|\partial_t \psi_r\|^2_{L^2(Y(r)\times\mathbf{R})} < \infty.$$

Here $(A_r(t), \psi_r(t))$ is a solution in a temporal gauge, gauge equivalent to the original $(\mathcal{A}_r, \Psi_r)$. The self-dual form $\tau(\Psi, \Psi)$ is given in local coordinates by $\tau(\Psi, \Psi) = \sum_{i,j} \langle e_i e_j \Psi, \Psi \rangle e^i \wedge e^j$. As proved in [11], the finite energy condition ensures the existence of asymptotic values $(A_r(\pm\infty), \psi_r(\pm\infty))$ as $t \to \pm\infty$, for any fixed $r \geq r_0$. The asymptotic values satisfy the Seiberg-Witten equations (1) on $Y(r)$.

If the asymptotic values are irreducibles, satisfying $\psi_r(\pm\infty) \neq 0$, then the convergence as $t \to \pm\infty$ is exponential,

$$\|(A_r(t), \psi_r(t)) - (A_r(\pm\infty), \psi_r(\pm\infty))\|_{L^2_1(Y(r)\times\{t\})} \leq C_r e^{-\delta_r |t|}, \tag{3}$$

for all $t \geq T_r$. The rate of decay δ_r is determined by the absolute value of the first non-trivial eigenvalue of the linearization $L_{A_r(\pm\infty),\psi_r(\pm\infty)}$ of the monopole equations (1) at $(A_r(\pm\infty), \psi_r(\pm\infty))$, hence, by Proposition 2.1, the rate of decay to the asymptotic values at $\pm\infty$ satisfies

$$\delta_r = \frac{c}{r}.$$

We are interested in studying the boundary operator of the Floer complex, hence we are only interested in flow lines that connect irreducible critical points. In the case of equivariant Floer theory [11] we need also consider flow lines connecting to the reducible point, but in the framed configuration space the reducible solution is also a smooth point and a non-degenerate critical point, hence we have the same exponential decay of flow lines, as proved in [11].

Lemma 2.2 *Let $(\mathcal{A}, \Psi)$ be a solution of the Seiberg-Witten equations (2) on $Y(r) \times \mathbf{R}$, for a fixed $r > 0$. Suppose that $(\mathcal{A}, \Psi)$ is in a temporal gauge on $Y(r) \times [T, \infty)$ and $Y(r) \times (-\infty, -T]$ and assume that it decays exponentially to asymptotic values $(A(\pm\infty), \psi(\pm\infty))$ as $t \to \pm\infty$, where the $(A(\pm\infty), \psi(\pm\infty))$ satisfy the Seiberg-Witten equations (1).*

Then we have the following identity

$$-\frac{1}{2}\int_{Y(r)}(A(-\infty)-A_0)\wedge(F_{A(-\infty)}+F_{A_0})+\frac{1}{2}\int_{Y(r)}(A(\infty)-A_0)\wedge(F_{A(\infty)}+F_{A_0})$$
$$=\frac{1}{2}\int_{Y(r)\times\mathbf{R}}F_{\mathcal{A}}\wedge F_{\mathcal{A}}.$$

The identity holds more generally when replacing $Y(r)$ with a smaller domain $\Omega\subset Y(r)$.

Proof. We have

$$-\frac{1}{2}\int_{Y(r)}(A(-\infty)-A_0)\wedge(F_{A(-\infty)}+F_{A_0})+\frac{1}{2}\int_{Y(r)}(A(\infty)-A_0)\wedge(F_{A(\infty)}+F_{A_0})$$

$$\begin{aligned}
&= -\tfrac{1}{2}\textstyle\int_{Y(r)}(A(-\infty)-A_0)\wedge(F_{A(-\infty)}+F_{A_0})\\
&\quad +\tfrac{1}{2}\textstyle\int_{Y(r)}(A(\infty)-A_0)\wedge(F_{A(\infty)}+F_{A_0})\\
&= \tfrac{1}{2}\textstyle\int_{\mathbf{R}}dt\frac{\partial}{\partial t}\int_{Y(r)}((A(t)-A_0)\wedge(F_{A(t)}+F_{A_0}))\\
&= -\tfrac{1}{2}\textstyle\int_{Y(r)}\int_{\mathbf{R}}\frac{\partial}{\partial t}F_{A(t)}\wedge(A(t)-A_0)\wedge dt\\
&\quad -\tfrac{1}{2}\textstyle\int_{Y(r)}\int_{\mathbf{R}}(F_{A(t)}+F_{A_0})\wedge\frac{\partial}{\partial t}A(t)\wedge dt\\
&= -\tfrac{1}{2}\textstyle\int_{Y(r)\times\mathbf{R}}d_{Y(r)}(\frac{\partial}{\partial t}A(t))\wedge(A(t)-A_0)\wedge dt\\
&\quad -\tfrac{1}{2}\textstyle\int_{Y(r)}\int_{\mathbf{R}}(F_{A(t)}+F_{A_0})\wedge\frac{\partial}{\partial t}A(t)\wedge dt\\
&= -\tfrac{1}{2}\textstyle\int_{Y(r)\times\mathbf{R}}(F_{A(t)}-F_{A_0})\wedge\frac{\partial}{\partial t}A(t)\wedge dt\\
&\quad -\tfrac{1}{2}\textstyle\int_{Y(r)\times\mathbf{R}}(F_{A(t)}+F_{A_0})\wedge\frac{\partial}{\partial t}A(t)\wedge dt\\
&= -\textstyle\int_{Y(r)\times\mathbf{R}}F_{A(t)}\wedge\frac{\partial A(t)}{\partial t}\wedge dt\\
&= \tfrac{1}{2}\textstyle\int_{Y(r)\times\mathbf{R}}F_{\mathcal{A}}\wedge F_{\mathcal{A}}.
\end{aligned}$$

This completes the proof.
◇

Lemma 2.3 *Let $(\mathcal{A}_r,\Psi_r)$ be a solution of the Seiberg-Witten equations (2) on $Y(r)\times\mathbf{R}$, such that the gauge transformed element*

$$\lambda_r(\mathcal{A}_r,\Psi_r)=(A_r(t),\psi_r(t))$$

in a temporal gauge satisfies the finite energy condition

$$\mathcal{E}_r=\|\partial_tA_r\|^2_{L^2(Y(r)\times\mathbf{R})}+\|\partial_t\psi_r\|^2_{L^2(Y(r)\times\mathbf{R})}=$$

$$\int_{\mathbf{R}} \|\nabla CSD(A_r(t),\psi_r(t))\|^2_{L^2(Y(r))} dt < \infty.$$

Then, for any interval $[t_0,t_1]$ *of length* $\ell = t_1 - t_0$*, we have estimates*

$$\|\Psi_r\|^4_{L^4(Y(r)\times[t_0,t_1])} \leq 8\mathcal{E}_r + 2s_0^2 vol(Y(r_0))l,$$

$$\int_{t_0}^{t_1} \|F_{A_r(t)}\|^2_{L^2(Y(r)\times\{t\})} dt \leq \mathcal{E}_r + s_0^2 vol(Y(r_0))l,$$

and

$$\int_{t_0}^{t_1} \|\nabla_{A_r(t)}\psi_r(t)\|^2_{L^2(Y(r)\times\{t\})} dt \leq \mathcal{E}_r + s_0^2 vol(Y(r_0))l,$$

for all $r \geq r_0$*.*

Moreover, the energy $\mathcal{E}_r$ *is uniformly bounded in* $r \geq r_0$*. The constant* $s_0 \geq 0$ *is defined by*

$$s_0 = \max_{Y(r_0)}\{-s(x),0\}, \tag{4}$$

with $s(x)$ *the scalar curvature.*

Proof. The argument follows 6.12 of [16]. Since the cylinder $[-r,r]\times T^2$ is endowed with the flat metric, we have

$$s(r) = \max_{Y(r)}\{-s(x),0\} = s_0,$$

for $r \geq r_0$ and s_0 as in (4). Moreover, we can estimate

$$\mathcal{E}_r \geq \textstyle\int_{t_0}^{t_1}(\frac{1}{2}\|F_{A_r(t)}\|^2_{L^2(Y(r))} + \|\nabla_{A_r(t)}\psi_r(t)\|^2_{L^2(Y(r))})dt +$$
$$\textstyle\int_{Y(r)\times[t_0,t_1]} \frac{1}{8}|\Psi_r|^4 dv_{Y(r)}dt - \frac{s_0}{4}\int_{Y(r_0)\times[t_0,t_1]} |\Psi_r|^2 dv_{Y(r_0)}dt.$$

At any local maximum, $|\Psi_r|^2$ is bounded by the scalar curvature, hence a non-trivial maximum can only occur away from $[-r,r]\times T^2$, i.e. on $Y(r_0)\times \mathbf{R}$. Thus, the estimate above gives

$$\|\Psi_r\|^4_{L^4(Y(r)\times[t_0,t_1])} \leq 8\mathcal{E}_r + 4s_0^2 vol(Y(r_0))l.$$

The other two estimates follow similarly.

The uniform bound on the energy $\mathcal{E}_r$ is obtained as follows. As we discussed before, for each fixed $r > 0$, the finite energy condition for $(\mathcal{A}_r,\Psi_r)$ forces the existence of asymptotic values $(A_r(\pm\infty),\psi_r(\pm\infty))$ as $t \to \pm\infty$

that satisfy the 3-dim Seiberg-Witten equations (1). Moreover, if the asymptotic values are non-degenerate critical points of the Chern-Simons-Dirac functional, then the temporal gauge representative $(A_r(t), \psi_r(t))$ decays sufficiently fast in t to the asymptotic values in the L^2_2 topology.

Thus, the energy $\mathcal{E}_r$ can be written also as the total variation of the Chern-Simons-Dirac functional along the path $(A_r(t), \psi_r(t))$,

$$\mathcal{E}_r = CSD(A_r(-\infty), \psi_r(-\infty)) - CSD(A_r(\infty), \psi_r(\infty)).$$

Moreover, since the elements $(A_r(\pm\infty), \psi_r(\pm\infty))$ satisfy the equations (1) on $Y(r)$, we have $\partial_{A_r(\pm\infty)}\psi_r(\pm\infty) = 0$ and the variation of the CSD functional is simply given by

$$\mathcal{E}_r = \frac{1}{2}\int_{Y(r)} (A_r(\infty) - A_0) \wedge (F_{A_r(\infty)} + F_{A_0})$$

$$-\frac{1}{2}\int_{Y(r)} (A_r(-\infty) - A_0) \wedge (F_{A_r(-\infty)} + F_{A_0})$$

The previous Lemma shows that this quantity can be rewritten as

$$\mathcal{E}_r = \frac{1}{2}\int_{Y(r)\times\mathbf{R}} F_{\mathcal{A}_r} \wedge F_{\mathcal{A}_r}.$$

This is a topological term, conformal and gauge invariant, hence it does not change when stretching the cylinder $[-r, r] \times T^2$.
$\diamond$

Lemma 2.4 *Let $(A_r(t), \psi_r(t))$ be a finite energy solution of the Seiberg-Witten equations (2) on $Y(r) \times \mathbf{R}$, in temporal gauge in the $\mathbf{R}$-direction. Then, for all $t \geq T_r$, we can write the estimate (3) as*

$$\|(A_r(t), \psi_r(t)) - (A_r(\pm\infty), \psi_r(\pm\infty))\|_{L^2(Y(r)\times\{t\})} \leq Ce^{-\frac{c|t|}{r}},$$

where the constant C is independent of r. Moreover, on any fixed compact set $K \subset Y(r)$, independent of $r \geq r_0$, we have

$$\|(A_r(t), \psi_r(t)) - (A_r(\pm\infty), \psi_r(\pm\infty))\|_{L^2_1(K'\times\{t\})} \leq C_K e^{-\frac{c|t|}{r}},$$

with $K' \subset int(K)$, and with C_K independent of r.

Proof. The L^2 bound follows by the exponential decay of the energy functional: for $t > T_r$ we have

$$\frac{d}{dt}\mathcal{E}_r(t) \leq \mathcal{E}_r(0)\exp(-\frac{ct}{r}),$$

where

$$\mathcal{E}_r(t) = \int_t^\infty \|\partial_t A_r\|^2_{L^2(Y(r))} + \|\partial_t \psi_r\|^2_{L^2(Y(r))} dt,$$

and $\mathcal{E}_r(0)$ uniformly bounded in $r \geq r_0$, by Lemma 2.3.

The L^2_1 bound follows by retracing the argument of Lemma 6.14 of [16]: we obtain that the constant C_r of our (3) depends on the underlying manifold $Y(r)$ through the constant of the Sobolev multiplication theorem, and the estimates of our Lemma 2.3, which are uniform in $r \geq r_0$. Consider a compact set K that is embedded in $Y(r)$ for all $r \geq r_0$, so that the metric on K does not change with $r \geq r_0$. If we restrict our estimate to K, and we choose the cutoff function ξ of Lemma 6.14 of [16] supported in K, we obtain a uniform C_K depending only on K.
$\diamond$

Lemma 2.5 *Suppose given $(\mathcal{A}_r, \Psi_r)$, a finite energy solution of the equations (2) on $Y(r) \times \mathbf{R}$, for $r \geq r_0$. There is a pointwise bound*

$$|\Psi_r(x,t)|^2 \leq s_0,$$

where $s_0 = \max_{Y(r_0)}\{-s(x), 0\}$, with $s(x)$ the scalar curvature. Moreover, for any $t_0 < t_1$, we have an estimate

$$\int_{Y(r)\times[t_0,t_1]} \langle \Psi_r, \nabla^*_{\mathcal{A}_r}\nabla_{\mathcal{A}_r}\Psi_r\rangle dv_{Y(r)}dt \leq \frac{s_0}{4}\|\Psi_r\|^2_{L^2(Y(r_0)\times[t_0,t_1])}.$$

Proof. The finite energy condition ensures the existence of asymptotic values $\psi_r(x, \pm\infty)$ for the spinor $\Psi_r(x,t)$. Thus, we can estimate

$$|\Psi_r(x,t)|^2 \leq \max\{|\psi_r(x,\pm\infty)|^2, -s_r(x,t)\},$$

with $s_r(x,t)$ the scalar curvature on $Y(r) \times \mathbf{R}$, and the pointwise estimate follows. By the Weitzenböck formula and the equations we have

$$0 = \nabla^*_{\mathcal{A}_r}\nabla_{\mathcal{A}_r}\Psi_r + \frac{s_r}{4}\Psi_r + \frac{|\Psi_r|^2}{4}\Psi_r.$$

By integrating and by using the vanishing of the scalar curvature on the cylinder $T^2 \times [-r, r]$, we obtain the desired estimate.
$\diamond$

In the following we use the notation

$$\mathcal{E}_r(t_0, t_1) = \int_{t_0}^{t_1} \|\nabla CSD(A_r(t), \psi_r(t))\|^2_{L^2(Y(r))} dt.$$

Proposition 2.6 *Assume, as before, that $(\mathcal{A}_r, \Psi_r)$ is a finite energy solution of the equations (2) on $Y(r) \times \mathbf{R}$. Suppose given any two real numbers $t_0 < t_1$. Then, there is a uniform bound on $\|F_{\mathcal{A}_r}\|_{L^2(Y(r)\times[t_0,t_1])}$ for $r \geq r_0$,*

$$\|F_{\mathcal{A}_r}\|^2_{L^2(Y(r)\times[t_0,t_1])} \leq C_\ell,$$

where C_ℓ is a constant independent of r and depending only on $\ell = t_1 - t_0$.

Proof. According to the previous Lemma we have a uniform bound on the L^4-norm

$$\|\Psi_r\|_{L^4(Y(r)\times[t_0,t_1])},$$

hence, by the equations, we obtain a uniform bound on $\|F^+_{\mathcal{A}_r}\|^2_{L^2(Y(r)\times[t_0,t_1])}$.

We have the identity

$$(2|F^+_{\mathcal{A}}|^2 - |F_{\mathcal{A}}|^2) dv = -F_{\mathcal{A}} \wedge F_{\mathcal{A}}. \tag{5}$$

Thus, we obtain an estimate

$$\begin{aligned} \|F_{\mathcal{A}_r}\|^2_{L^2(Y(r)\times[t_0,t_1])} \leq 2 \quad & \|F^+_{\mathcal{A}}\|^2_{L^2(Y(r)\times[t_0,t_1])} + \mathcal{E}_r(t_0, t_1) \\ & + \textstyle\int_{Y(r)} \langle \psi_r(t_1), \partial_{A_r(t_1)} \psi_r(t_1) \rangle dv_{Y(r)} \\ & - \textstyle\int_{Y(r)} \langle \psi_r(t_0), \partial_{A_r(t_0)} \psi_r(t_0) \rangle dv_{Y(r)}. \end{aligned}$$

In the right hand side we can estimate

$$\mathcal{E}_r(t_0, t_1) \leq \mathcal{E}_r,$$

which is uniformly bounded in $r \geq r_0$, by Lemma 2.3. We need to estimate the term

$$|\int_{Y(r)} \langle \psi_r(t_1), \partial_{A_r(t_1)} \psi_r(t_1) \rangle dv_{Y(r)} - \int_{Y(r)} \langle \psi_r(t_0), \partial_{A_r(t_0)} \psi_r(t_0) \rangle dv_{Y(r)}|.$$

We estimate

$$
\begin{gathered}
|\int_{t_0}^{t_1} \frac{d}{dt} \int_{Y(r)} \langle \psi_r(t), \partial_{A_r(t)} \psi_r(t) \rangle dv_{Y(r)} dt| \leq \\
\int_{t_0}^{t_1} (2\|\partial_{A_r(t)} \psi_r(t)\|^2_{L^2(Y(r))} + 2|\langle F_{A_r(t)}, *\sigma(\psi_r(t), \psi_r(t)) \rangle|) dt \leq \\
\int_{t_0}^{t_1} 2\|\nabla_{A_r(t)} \psi_r(t)\|^2_{L^2(Y(r))} + \\
(\int_{t_0}^{t_1} \|F_{A_r(t)}\|^2_{L^2(Y(r))} dt)^{1/2} (\int_{t_0}^{t_1} \int_{Y(r)} \|\psi_r(t)\|^4 dv_{Y(r)} dt)^{1/2} + \\
\frac{s_0}{2} \|\Psi_r\|^2_{L^2(Y(r_0)\times[t_0,t_1])} + \frac{1}{2} \|\Psi_r\|^4_{L^4(Y(r)\times[t_0,t_1])}.
\end{gathered}
$$

Now the estimates of Lemma 2.3 give the required bound on $\|F_{\mathcal{A}_r}\|$.

◇

3 Convergence on compact sets

In the process of stretching the neck in $Y(r) \times \mathbf{R}$, the spinor tends to vanish pointwise on the long cylinder, as the following Lemma proves.

Lemma 3.1 *Assume, as before, that $(\mathcal{A}_r, \Psi_r)$ is a finite energy solution of the equations (2) on $Y(r) \times \mathbf{R}$. Given any compact set of the form $T^2 \times [-r_0, r_0] \times [t_0, t_1]$, we have pointwise convergence*

$$
|\Psi_r(x,t)| \to 0,
$$

for all $(x,t) \in T^2 \times [-r_0, r_0] \times [t_0, t_1]$, as $r \to \infty$.

Proof. The claim follows from the uniform bound

$$
\int_{T^2\times[-r,r]\times[t_0,t_1]} |\Psi_r(x,t)|^4 dv dt \leq C_\ell
$$

derived previously.

◇

Thus, the self-dual part of the curvature is also converging to zero pointwise on the long cylinder.

Corollary 3.2 *If $(\mathcal{A}_r, \Psi_r)$ is a finite energy solution of the equations (2) on $Y(r) \times \mathbf{R}$, then, on compact sets of the form $T^2 \times [-r_0, r_0] \times [t_0, t_1]$, the connection $\mathcal{A}_r$ converges to a finite energy solution $\mathcal{A}$ of the abelian self dual equation $F^+_{\mathcal{A}} = 0$.*

Proposition 3.3 *Inside the manifold $Y(r) \times \mathbf{R}$, for all $r \geq r_0$, consider fixed compact sets of the form $K_V \subset V \times \mathbf{R}$, $K_\nu \subset \nu(K) \times \mathbf{R}$, or $K_0 \subset T^2 \times [-r_0, r_0] \times \mathbf{R}$. We assume that the metric on $\nu(K)$ has non-negative scalar curvature, and that it is flat on the cylinder $T^2 \times [-r, r]$. Let $(\mathcal{A}_r, \Psi_r)$ be finite energy solutions of the equations (2) on $Y(r) \times \mathbf{R}$. Then there exists a subsequence $(\mathcal{A}_{r'}, \Psi_{r'})$ and gauge transformations $\lambda_{r'}$ on $Y(r') \times \mathbf{R}$, such that the sequence $\lambda_{r'}(\mathcal{A}_{r'}, \Psi_{r'})$ satisfies the following properties. It converges smoothly on the compact sets K_V to a finite energy solution $(\mathcal{A}, \Psi)$ of the four-dimensional Seiberg-Witten equations on $V \times \mathbf{R}$. On the compact sets K_ν it converges to a finite energy solution $(\mathcal{A}, 0)$ of the abelian ASD equation on $\nu(K) \times \mathbf{R}$, and on the compact sets K_0 it converges to a finite energy solution of the abelian ASD equation on $T^2 \times \mathbf{R}^2$.*

Proof. We can assume that $(\mathcal{A}_r, \Psi_r)$ is in $L^2_{k,\delta_r}(Y(r') \times \mathbf{R})$, with $\delta_r \sim 1/r$, by the exponential decay to the asymptotic values. On a fixed compact set K the uniform pointwise bound on the spinor implies the L^2-bound

$$\|\Psi_r\|_{L^2(K)} \leq Vol(K) s_0.$$

The uniform bound on the curvature provides a uniform L^2_1-bound on the connection (up to a gauge transformation), as in Lemma 5.3.1 of [14]. The uniform pointwise bound on the spinor, together with the uniform bound

$$\|\nabla_{\mathcal{A}_r} \Psi_r\|_{L^2(Y(r)\times[t_0,t_1])} \leq \frac{1}{4} s_0 \|\Psi_r\|_{L^2(Y(r_0)\times[t_0,t_1])}$$

provides a uniform L^2-bound for $dF^+_{\mathcal{A}_r}$, as in Lemma 5.3.3 of [14]. A bootstrapping argument then bounds the higher Sobolev norms.
◇

3.1 Asymptotics of finite energy ASD connections

In this subsection we study explicitly the solutions of the abelian ASD equation on $T^2 \times \mathbf{R}^2$ and on an asymptotic end of the form $T^2 \times [0, \infty) \times \mathbf{R}$. The first case will provide the geometric limit on compact sets on the cylinder $T^2 \times [-r, r] \times \mathbf{R}$, as $r \to \infty$, and the second case will give the geometric limit on compact sets on $\nu(K) \times \mathbf{R}$, and will serve as a model for the asymptotics of the limits on $V \times \mathbf{R}$, as we discuss later in this work.

If we represent the connection as

$$\mathcal{A} = a(w, s, t) + f(w, s, t)ds + h(w, s, t)dt,$$

the abelian ASD equation can be written as the pair of equations

$$\partial_t a - dh + *(\partial_s a - df) = 0$$

$$\partial_t f - \partial_s h + *F_a = 0.$$

With a change of variables $z = s + it$ and $z = e^{\rho+i\theta}$, we can write

$$a(w, \rho, \theta) = a(w, e^{\rho+i\theta})$$

$$f(w, \rho, \theta) = e^{-\rho}\cos\theta \; h(w, e^{\rho+i\theta}) - e^{-\rho}\sin\theta \; f(w, e^{\rho+i\theta})$$
$$h(w, \rho, \theta) = e^{-\rho}\cos\theta \; f(w, e^{\rho+i\theta}) + e^{-\rho}\sin\theta \; h(w, e^{\rho+i\theta}).$$

The equations become

$$\partial_\rho a - dh - *(\partial_\theta a - df) = 0$$

$$\partial_\rho f - \partial_\theta h - e^{2\rho} * F_a = 0.$$

Up to a gauge transformation we can assume that the equations are in radial gauge, that is $h \equiv 0$ for ρ large enough. Thus, we get

$$\partial_\rho a - *(\partial_\theta a - df) = 0 \tag{6}$$

$$\partial_\rho f - e^{2\rho} * F_a = 0. \tag{7}$$

Notice that the equation in this form can be interpreted as the abelian ASD equation on $T^2 \times S^1 \times \mathbf{R}$, where the metric on T^2 has a conformal factor depending on $\rho \in \mathbf{R}$, of the form $e^{-\rho} g_{T^2}$, where g_{T^2} is the standard metric on the flat torus. In fact the $*$ operator on p-forms rescales like $*_\rho = (e^{-\rho})^{2-2p} *$.

We are considering finite energy solutions, that is, we impose the condition

$$\int_{-\infty}^{\infty} \int_0^{2\pi} \left(\|\partial_\rho a\|^2_{L^2(T^2), g_{T^2}} + e^{2\rho} \|F_a\|^2_{L^2(T^2), g_{T^2}} \right) d\theta d\rho < \infty.$$

Under a change of variables $\alpha = e^\rho a$ we rewrite equations (6) and (7) as

$$\partial_\rho \alpha = \alpha + *\partial_\theta \alpha - e^\rho * df \tag{8}$$

$$\partial_\rho f = e^\rho * d\alpha. \tag{9}$$

We can write $\alpha = u(w,\rho,\theta)dx + v(w,\rho,\theta)dy$, where $w=(x,y)$ are the coordinates on the torus T^2. We can expand u, v, and f in Fourier series in the variables θ, x, and y.

We get

$$u(w,\theta,\rho) = \sum u_n(\rho)_{lk} \frac{e^{in\theta}}{(2\pi)^{1/2}} \frac{e^{ilx}}{(2\pi)^{1/2}} \frac{e^{iky}}{(2\pi)^{1/2}},$$

$$v(w,\theta,\rho) = \sum v_n(\rho)_{lk} \frac{e^{in\theta}}{(2\pi)^{1/2}} \frac{e^{ilx}}{(2\pi)^{1/2}} \frac{e^{iky}}{(2\pi)^{1/2}},$$

$$f(w,\theta,\rho) = \sum f_n(\rho)_{lk} \frac{e^{in\theta}}{(2\pi)^{1/2}} \frac{e^{ilx}}{(2\pi)^{1/2}} \frac{e^{iky}}{(2\pi)^{1/2}}.$$

The system of equations becomes the ODE

$$\frac{d}{d\rho}\begin{pmatrix} u_n(\rho)_{lk} \\ v_n(\rho)_{lk} \\ f_n(\rho)_{lk} \end{pmatrix} = \begin{pmatrix} 1 & -in & ike^{\rho} \\ in & 1 & -ile^{\rho} \\ ike^{\rho} & -ile^{\rho} & 0 \end{pmatrix} \begin{pmatrix} u_n(\rho)_{lk} \\ v_n(\rho)_{lk} \\ f_n(\rho)_{lk} \end{pmatrix}. \qquad (10)$$

In the case of the asymptotics on the end $T^2 \times [0,\infty) \times \mathbf{R}$, we consider the same equations and restrict solutions to the domain $\theta \in [-\pi/2, \pi/2]$, in fact, we are not imposing any boundary conditions at $\theta = -\pi/2$ and $\theta = \pi/2$, other than the functions being smooth across $\theta = -\pi/2$ and $\theta = \pi/2$.

The assumption that $\mathcal{A}$ is a $U(1)$ connection imposes the constraint on the coefficients

$$u_n(\rho)_{lk} = -\overline{u_{-n}(\rho)_{-l,-k}},$$
$$v_n(\rho)_{lk} = -\overline{v_{-n}(\rho)_{-l,-k}},$$
$$f_n(\rho)_{lk} = -\overline{f_{-n}(\rho)_{-l,-k}}.$$

A direct analysis of this system of ODE's (cf. [9], §X) proves the following Proposition.

Proposition 3.4 *The only finite energy solutions of the abelian ASD equation on $T^2 \times \mathbf{R}^2$ are flat connections on T^2, constant in the $\mathbf{R}^2$ directions. On the asymptotic end $T^2 \times [r_0,\infty) \times \mathbf{R}$ all the finite energy solutions are* *of the form*

$$
\begin{aligned}
a(w,\rho,\theta) = \quad & (u_0 + \textstyle\sum_{n\neq 0} u_n e^{-|n|\rho + in\theta} \\
& + \textstyle\sum_{n,l,k} l c_{nlk} e^{i(lx+ky+n\theta)})dx + \\
& (v_0 + \textstyle\sum_{n\neq 0} v_n e^{-|n|\rho + in\theta} \\
& + \textstyle\sum_{n,l,k} k c_{nlk} e^{i(lx+ky+n\theta)})dy \\
f(w,\rho,\theta) = \quad & f_0 + \textstyle\sum_{n,l,k} n c_{nlk} e^{i(lx+ky+n\theta)}
\end{aligned} \tag{11}
$$

with the coefficients that satisfy

$$\sum_n |n u_n|^2 + \sum_{n,l,k} |(n^2+l^2+k^2) c_{nlk}|^2 < \infty,$$

and

$$c_{nlk} = -\overline{c_{-n,-l,-k}} \quad \textit{and} \quad u_n = -\overline{u_{-n}}.$$

In radial gauge, the limit in the $\rho \to \infty$ direction is given by

$$
\begin{aligned}
a_\infty(w,\theta) = \quad & (u_0 + \textstyle\sum_{n,l,k} l c_{nlk} e^{i(lx+ky+n\theta)})dx + \\
& (v_0 + \textstyle\sum_{n,l,k} k c_{nlk} e^{i(lx+ky+n\theta)})dy \\
f_\infty(w,\theta) = \quad & f_0 + \textstyle\sum_{n,l,k} n c_{nlk} e^{i(lx+ky+n\theta)}.
\end{aligned}
$$

Proof.

It is easy to find a special family of solutions of the form

$$c_{nlk} \begin{pmatrix} l\, e^\rho \\ k e^\rho \\ n \end{pmatrix},$$

with $-\overline{c_{-n,-l,-k}} = c_{nlk}$. These correspond to the condition $d\alpha = 0$ in the equation (9). In the original variables, that is multiplying the connection terms by $e^{-\rho}$, these are solutions of (6) and (7) constant in the ρ-direction.

We assume for the moment that $(k,l) \neq (0,0)$. Following the standard methods of ODE theory, we use these solutions in order to reduce the system to a system of two equations. If we assume $l \neq 0$, we can consider the matrix

$$Z = \begin{pmatrix} le^\rho & 0 & 0 \\ ke^\rho & 1 & 0 \\ n & 0 & 1 \end{pmatrix}.$$

The change of variables

$$\begin{pmatrix} u \\ v \\ f \end{pmatrix} = Z \begin{pmatrix} \omega \\ \nu \\ \phi \end{pmatrix}$$

gives the new system of equations

$$\frac{d}{d\rho}\begin{pmatrix} \omega \\ \nu \\ \phi \end{pmatrix} = Z^{-1}A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega \\ \nu \\ \phi \end{pmatrix},$$

where A is the matrix

$$A = \begin{pmatrix} 1 & -in & 0 \\ in & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This system consists of the equation

$$\omega' = \frac{-in}{l} e^{-\rho}\nu + \frac{ik}{l}\phi$$

and the two by two system

$$\frac{d}{d\rho}\begin{pmatrix} l\nu \\ l\phi \end{pmatrix} = \begin{pmatrix} ink + l & -i(k^2+l^2)e^{\rho} \\ i(n^2e^{-\rho} - l^2e^{\rho}) & -ink \end{pmatrix} \begin{pmatrix} \nu \\ \phi \end{pmatrix}.$$

Since we are interested in the large $\rho >> 0$ behavior of the system we can isolate a leading term and treat the rest of the system as a perturbation.

Let us define

$$L_\rho = \begin{pmatrix} 0 & -i(k^2+l^2)e^{\rho} \\ -il^2e^{\rho} & 0 \end{pmatrix}$$

to be the leading term and

$$P_\rho = \begin{pmatrix} ink + l & 0 \\ in^2e^{-\rho} & -ink \end{pmatrix}$$

to be the perturbation.

The unperturbed system has eigenvalues

$$\lambda_{\pm}(\rho) = \pm ie^{\rho}(k^2+l^2)^{1/2},$$

with eigenvectors

$$U_{\pm} = \begin{pmatrix} 1 \\ \frac{\pm l}{(k^2+l^2)^{1/2}} \end{pmatrix}.$$

So, upon diagonalizing the matrix we obtain solutions $\exp(\pm i e^{\rho}(k^2+l^2)^{1/2})$, hence in the original system of coordinates we have solutions

$$\begin{aligned} \nu(\rho) = \quad & \tfrac{(k^2+l^2)^{1/2}}{2l} c_1 \exp(i e^{\rho}(k^2+l^2)^{1/2}) \\ & + \tfrac{(k^2+l^2)^{1/2}}{2l} c_2 \exp(-i e^{\rho}(k^2+l^2)^{1/2}), \end{aligned} \tag{12}$$

and

$$\begin{aligned} \phi(\rho) = \quad & -\tfrac{1}{2} c_1 \exp(i e^{\rho}(k^2+l^2)^{1/2}) \\ & + \tfrac{1}{2} c_2 \exp(-i e^{\rho}(k^2+l^2)^{1/2}). \end{aligned} \tag{13}$$

with the remaining equation that gives

$$\begin{aligned} & \omega(\rho) = \textstyle\int_0^{\rho} \left(\tfrac{-in(k^2+l^2)^{1/2}}{l^2} (c_1 \exp(i e^{\tau}) + c_2 \exp(-i e^{\tau})) - \right. \\ & \left. \tfrac{ik}{2l} e^{\tau} (c_1 \exp(i e^{\tau}) - c_2 \exp(-i e^{\tau})) \right) d\tau. \end{aligned} \tag{14}$$

The perturbed system has eigenvalues

$$\tilde{\lambda}_{\pm}(\rho) = \frac{1 \pm \sqrt{1 - 4(e^{2\rho}(k^2+l^2) - n^2 - \frac{ink}{l})}}{2}.$$

The matrix can be diagonalized so that the solutions are of the form

$$\exp\left(\int_0^{\rho} \tilde{\lambda}_{\pm}(\tau) d\tau \right).$$

The long distance $\rho >> 0$ behavior of these solutions is given by the asymptotics $e^{\rho \pm i e^{\rho}}$. On the other hand the eigenvectors become asymptotically ρ-independent and approach the eigenvectors $U_{\pm}$ of the unperturbed system. Thus, the asymptotic behavior of the solutions will be of the form

$$\nu(\rho) \sim \frac{(k^2+l^2)^{1/2}}{2l} \left(c_1 e^{\rho + i e^{\rho}} + c_2 e^{\rho - i e^{\rho}} \right) \tag{15}$$

and

$$\phi(\rho) \sim \frac{-1}{2} \left(c_1 e^{\rho + i e^{\rho}} - c_2 e^{\rho - i e^{\rho}} \right) \tag{16}$$

The third variable is then obtained as

$$\begin{array}{l}\omega(\rho) \sim \int_0^\rho \left(\frac{-in(k^2+l^2)^{1/2}}{l^2}(c_1 \exp(ie^\tau) + c_2 \exp(-ie^\tau)) - \right. \\ \left. \frac{ik}{2l} e^\tau (c_1 \exp(ie^\tau) - c_2 \exp(-ie^\tau))\right) d\tau. \end{array} \tag{17}$$

Then the original solutions will be of the form

$$\begin{pmatrix} u \\ v \\ f \end{pmatrix} = \begin{pmatrix} le^\rho \omega(\rho) \\ ke^\rho \omega(\rho) + \nu(\rho) \\ n\omega(\rho) + \phi(\rho) \end{pmatrix}.$$

The case with $l = 0$ and $k \neq 0$ is analogous. In fact, in that case we can use a similar reduction by considering the matrix

$$Z = \begin{pmatrix} 0 & 1 & 0 \\ ke^\rho & 0 & 0 \\ n & 0 & 1 \end{pmatrix}.$$

The change of variables

$$\begin{pmatrix} u \\ v \\ f \end{pmatrix} = Z \begin{pmatrix} \omega \\ \nu \\ \phi \end{pmatrix}$$

gives the new system of equations

$$\frac{d}{d\rho} \begin{pmatrix} \omega \\ \nu \\ \phi \end{pmatrix} = Z^{-1} A \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega \\ \nu \\ \phi \end{pmatrix},$$

which is of the form

$$\omega' = \frac{in}{k} e^{-\rho} \nu$$

and the remaining two by two system

$$\frac{d}{d\rho} \begin{pmatrix} \nu \\ \phi \end{pmatrix} = \begin{pmatrix} 1 & ike^\rho \\ \frac{-in^2}{k} e^{-\rho} + ike^\rho & 0 \end{pmatrix} \begin{pmatrix} \nu \\ \phi \end{pmatrix}.$$

Again we can isolate a leading term

$$L_\rho = \begin{pmatrix} 0 & ike^\rho \\ ike^\rho & 0 \end{pmatrix}$$

and a perturbation

$$P_\rho = \begin{pmatrix} 1 & 0 \\ \frac{-in^2}{k} e^{-\rho} & 0 \end{pmatrix}.$$

The unperturbed system has eigenvalues $\lambda_\pm = \pm ike^\rho$ and ρ-independent eigenvectors

$$U_\pm = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$

The perturbed system has eigenvalues

$$\tilde{\lambda}_\pm = \frac{1 \pm \sqrt{1 - 4(k^2 e^{2\rho} - n^2)}}{2}.$$

The asymptotics of the solutions is $\exp(\rho \pm ike^\rho)$, with eigenvectors that asymptotically approach the ρ-independent eigenvectors $U_\pm$. Thus, we obtain asymptotics

$$\omega(\rho) \sim \frac{in}{2k} \int_0^\rho (c_1 e^{+ike^\tau} + c_2 e^{-ike^\tau}) d\tau,$$

$$\nu(\rho) \sim \frac{1}{2}(c_1 e^{\rho + ike^\rho} + c_2 e^{\rho - ike^\rho})$$

and

$$\phi(\rho) \sim \frac{1}{2}(c_1 e^{\rho + ike^\rho} - c_2 e^{\rho - ike^\rho}).$$

Among these families of solutions, the only elements that satisfy the finite energy condition are written in the original variables as solutions of (6) and (7) of the form

$$\begin{aligned} a(w,\rho,\theta) = \quad & (u_0 + \textstyle\sum_{n,l,k} l c_{nlk} e^{i(lx+ky+n\theta)}) dx + \\ & (v_0 + \textstyle\sum_{n,l,k} k c_{nlk} e^{i(lx+ky+n\theta)}) dy \\ f(w,\rho,\theta) = \quad & f_0 + \textstyle\sum_{n,l,k} n c_{nlk} e^{i(lx+ky+n\theta)}. \end{aligned} \tag{18}$$

The remaining solutions satisfy $(k,l) = (0,0)$. These satisfy the conditions $df = 0$ and $d\alpha = 0$ in the equations (8) and (9). These are solutions of the form

$$\begin{pmatrix} c_1(n,0,0)e^{(1+n)t} - ic_2(n,0,0)e^{(1-n)t} \\ c_1(n,0,0)e^{(1+n)t} + ic_2(n,0,0)e^{(1-n)t} \\ c_3(n,0,0) \end{pmatrix}$$

to the system with constant coefficients with matrix

$$A = \begin{pmatrix} 1 & -in & 0 \\ in & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives solutions of the original equations (6) and (7) of the form

$$\begin{aligned} a(w,\rho,\theta) = & \left(u_0 + \sum_{n\neq 0} u_n e^{-|n|\rho + in\theta}\right) dx \\ & + \left(v_0 + \sum_{n\neq 0} v_n e^{-|n|\rho + in\theta}\right) dy \\ f(w,\rho,\theta) = & \ f_0 \end{aligned} \qquad (19)$$

This proves the Proposition. In fact, we can write the function

$$f_\infty - f_0 = -i\frac{d}{d\theta}\sum_{n,l,k} c_{nlk} e^{i(lx+ky+n\theta)},$$

so that, if we define

$$\gamma(\theta) = \sum_{l,k} -ic_{nlk} e^{i(lx+ky+n\theta)},$$

the flat connection a_∞ can be written as

$$a_\infty - (u_0 dx + v_0 dy) = d\gamma(\theta).$$

This means that the path of asymptotic flat connections in the covering $\chi_0(T^2, \nu(K))$ of the character variety $\chi(T^2)$ can be written as

$$a_\infty(\theta) = \lambda(\theta)\cdot a_0,$$

where $\lambda : T^2 \times [0,\pi] \to U(1)$ satisfies

$$\lambda^{-1}(\theta) d_{T^2}\lambda(\theta) = d_{T^2}\gamma(\theta).$$

$\diamond$

Thus we have obtained the following.

Corollary 3.5 *A finite energy solution of the abelian ASD equation on an asymptotic end of the form $T^2 \times [0,\infty) \times \mathbf{R}$ decays exponentially along the radial direction. The asymptotic values for different angles θ are all within the same gauge class of flat connections on T^2. Thus, we have decay to a point $a_\infty = [a_\infty(\theta)]$.*

Remark 3.6 *Up to a global change of gauge, the only solutions of the form (11) that extend to all of $\nu(K) \times \mathbf{R}$ are flat connections on T^2 constant in the s and t directions.*

In the light of Remark 3.6, the explicit analysis of the asymptotics (11) seems a somewhat useless complication. However, the analysis above will be crucial in the following section, in order to describe, with a perturbative analysis, the asymptotic behavior of finite energy solutions of the Seiberg-Witten equations on $V \times \mathbf{R}$ with an end $T^2 \times [0, \infty) \times \mathbf{R}$.

3.2 Asymptotics of monopoles on $V \times \mathbf{R}$

We now analyze the asymptotics of solutions of the four-dimensional Seiberg-Witten equations on the end $T^2 \times [r_0, \infty) \times \mathbf{R}$ of the manifold $V \times \mathbf{R}$, where V is the knot complement in V, endowed with an infinite cylindrical end.

We write the connection $\mathcal{A} = a(w, s, t) + f(w, s, t)ds + h(w, s, t)dt$, and the spinor section $\Psi = (\alpha, \beta)$ with $\alpha(s, t) \in \Lambda^{0,0}(T^2)$ and $\beta(s, t) \in \Lambda^{0,1}(T^2)$.

The Seiberg-Witten equations can be written in the form

$$\partial_t a - dh + *(\partial_s a - df) = *i(\bar{\alpha}\beta + \alpha\bar{\beta})$$

$$\partial_t f - \partial_s h + *F_a = \frac{i}{2}(|\alpha|^2 - |\beta|^2),$$

for the curvature equation, and

$$\partial_t \alpha + h\alpha + i\partial_s \alpha + if\alpha + \bar{\partial}_a^* \beta = 0$$

$$\partial_t \beta + h\beta - i\partial_s \beta - if\beta + \bar{\partial}_a \alpha = 0,$$

for the Dirac equation.

As before, we introduce the variables $w = (x, y) \in T^2$, and $z = s + it$, $z = e^{\rho + i\theta}$ on $[r_0, \infty) \times \mathbf{R}$, and the change of coordinates

$$\begin{aligned} a(w, \rho, \theta) &= a(w, e^{\rho+i\theta}) \\ f(w, \rho, \theta) &= e^{-\rho} \cos\theta \; h(w, e^{\rho+i\theta}) - e^{-\rho} \sin\theta \; f(w, e^{\rho+i\theta}) \\ h(w, \rho, \theta) &= e^{-\rho} \cos\theta \; f(w, e^{\rho+i\theta}) + e^{-\rho} \sin\theta \; h(w, e^{\rho+i\theta}) \\ \alpha(w, \rho, \theta) &= \alpha(w, e^{\rho+i\theta}) \\ \beta(w, \rho, \theta) &= \beta(w, e^{\rho+i\theta}). \end{aligned} \tag{20}$$

The Seiberg-Witten equations in radial gauge (i.e. with $h \equiv 0$ for large ρ) are then written in the form

$$\begin{aligned}
\partial_\rho a &= *(\partial_\theta a - df + i(\bar\alpha\beta + \alpha\bar\beta)) \\
\partial_\rho f &= e^{2\rho} * (F_a + \tfrac{i}{2}(|\alpha|^2 - |\beta|^2)\omega) \\
\partial_\rho \alpha &= i(\partial_\theta \alpha + f\alpha + e^{\rho+i\theta}\bar\partial_a^* \beta) \\
\partial_\rho \beta &= -i(\partial_\theta \beta + f\beta + e^{\rho-i\theta}\bar\partial_a \alpha).
\end{aligned} \tag{21}$$

In order to study the asymptotic behavior of these solutions, we can again use Fourier transform. We first analyze the asymptotic behavior of a linear system and then introduce the non-linear terms in a sequence of successive approximations [9]. In order to simplify the expression of the quadratic terms in (21), we use the notation

$$A \cdot B = \sum (ab)_{nlk} \frac{e^{i(n\theta+lx+ky)}}{(2\pi)^{3/2}},$$

with the coefficients given by

$$(ab)_{nlk} = \sum \frac{1}{(2\pi)^{3/2}} a_{ijh} b_{n-i\ l-j\ k-h},$$

for

$$A = \sum a_{nlk} \frac{e^{i(n\theta+lx+ky)}}{(2\pi)^{3/2}}$$

and

$$B = \sum b_{nlk} \frac{e^{i(n\theta+lx+ky)}}{(2\pi)^{3/2}}.$$

With this notation, the equation (21) becomes

$$\begin{aligned}
u'_{nlk} = &\quad -inv_{nlk} + ikf_{nlk} + 2iRe(\bar\alpha\beta)_{nlk} \\
v'_{nlk} = &\quad inu_{nlk} - ilf_{nlk} + 2iIm(\bar\alpha\beta)_{nlk} \\
f'_{nlk} = &\quad ike^{2\rho}u_{nlk} - ile^{2\rho}v_{nlk} + e^{2\rho}(\bar\alpha\alpha)_{nlk} - (\bar\beta\beta)_{nlk} \\
\alpha'_{nlk} = &\quad -n\alpha_{nlk} + ie^{\rho}\tfrac{1}{2}(il-k)\bar\beta_{-n-1\ -l\ -k} + \\
&\quad ie^{\rho}\tfrac{1}{2}(a\bar\beta)_{nlk} + i(f\alpha)_{nlk} \\
\beta'_{nlk} = &\quad n\beta_{nlk} - ie^{\rho}(il-k)\alpha_{n+1\ lk} \\
&\quad -ie^{\rho}(a\alpha)_{nlk} - i(f\beta)_{nlk}.
\end{aligned} \tag{22}$$

Consider the linear system of equations

$$
\begin{aligned}
\partial_\rho a &= *(\partial_\theta a - df) \\
\partial_\rho f &= e^{2\rho} * F_a \\
\partial_\rho \alpha &= i(\partial_\theta \alpha + e^{\rho+i\theta}\bar{\partial}^* \beta) \\
\partial_\rho \beta &= -i(\partial_\theta \beta + e^{\rho-i\theta}\bar{\partial}\alpha).
\end{aligned}
\tag{23}
$$

This is the uncoupled system of the abelian ASD equation and the linear Dirac equation. We analyzed the first two equations in the previous section.

In the Dirac equations, since again we are interested in the large $\rho >> 0$ behavior of the finite energy solutions, we can isolate a leading term and a perturbation.

The leading term gives a system of the form

$$
\begin{aligned}
\alpha'_{nlk} &= ie^\rho \tfrac{1}{2}(il-k)\bar{\beta}_{-n-1\ -l\ -k} \\
\beta'_{nlk} &= -ie^\rho(il-k)\alpha_{n+1\ lk},
\end{aligned}
\tag{24}
$$

Whereas we identify perturbation all the terms of the form

$$
\begin{pmatrix} -n & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} \alpha_{nlk} \\ \beta_{nlk} \end{pmatrix}.
\tag{25}
$$

We analyze solutions to the unperturbed system (24). If we introduce the notation $\alpha_{nlk} = \eta_{nlk} + i\xi_{nlk}$ and $\beta_{nlk} = p_{nlk} + iq_{nlk}$, the system (24) uncouples in the independent systems of eight equations:

$$
\begin{aligned}
p'_{nlk} &= e^\rho(l\eta_{n+1\ lk} - k\xi_{n+1\ lk}) \\
q'_{nlk} &= e^\rho(l\xi_{n+1\ lk} + k\eta_{n+1\ lk}) \\
\eta'_{n+1\ lk} &= \tfrac{e^\rho}{2}(-lp_{-n-2\ -l\ -k} - kq_{-n-2\ -l\ -k}) \\
\xi'_{n+1\ lk} &= \tfrac{e^\rho}{2}(lp_{-n-2\ -l\ -k} - kq_{-n-2\ -l\ -k}) \\
p'_{-n-2\ -l\ -k} &= e^\rho(-l\eta_{-n-1\ -l\ -k} + k\xi_{-n-1\ -l\ -k}) \\
q'_{-n-2\ -l\ -k} &= e^\rho(-l\eta_{-n-1\ -l\ -k} - k\xi_{-n-1\ -l\ -k}) \\
\eta'_{-n-1\ -l\ -k} &= \tfrac{e^\rho}{2}(lp_{nlk} + kq_{nlk}) \\
\xi'_{-n-1\ -l\ -k} &= \tfrac{e^\rho}{2}(-lp_{nlk} + kq_{nlk})
\end{aligned}
\tag{26}
$$

Consider the matrices

$$
A = \begin{pmatrix} l & -k \\ l & k \end{pmatrix}
$$

and

$$B = \frac{1}{2}\left(\begin{array}{cc} -l & -k \\ l & -k \end{array}\right).$$

Up to a reparametrization in the variable $\tau = e^{\rho}$, we obtain the autonomous linear system with the 8×8 matrix

$$M = \left(\begin{array}{cccc} 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & -A \\ -B & 0 & 0 & 0 \end{array}\right).$$

If we write $\eta_0 = (k^2 + 6kl + l^2)^{1/2}$, $\eta_1 = k - l$, and $\eta_2 = k + l$, then the eigenvalues of M are of the form $\pm\lambda_i^{nlk}$, for $i = 1, \ldots 4$, with

$$\begin{array}{ll} \lambda_1^{nlk} = \frac{1}{2}(\eta_2^2 + \eta_1\eta_0)^{1/2} & \lambda_2^{nlk} = \frac{1}{2}(\eta_2^2 - \eta_1\eta_0)^{1/2} \\ \lambda_3^{nlk} = \frac{1}{2}(-\eta_2^2 - \eta_1\eta_0)^{1/2} & \lambda_4^{nlk} = \frac{1}{2}(-\eta_2^2 + \eta_1\eta_0)^{1/2}. \end{array} \tag{27}$$

It is easy to see that, in the basis $U_i^{\pm}(nlk)$ of eigenvectors, the solutions corresponding to λ_1^{nlk} and λ_2^{nlk} are saddles for all non-trivial (n, l, k) and solutions corresponding to λ_3^{nlk} and λ_4^{nlk} are centers for all non-trivial (n, l, k). Thus, the finite energy solutions will be of the form

$$\left(\begin{array}{c} p_{nlk} \\ q_{nlk} \\ \eta_{n+1\ lk} \\ \xi_{n+1\ lk} \\ p_{-n-2\ -l\ -k} \\ q_{-n-2\ -l\ -k} \\ \eta_{-n-1\ -l\ -k} \\ \xi_{-n-1\ -l\ -k} \end{array}\right) = \sum_{i=1}^{2} c_i^{nlk} \exp(-|\lambda_i^{nlk}|\tau) U_i^{-}(nlk), \tag{28}$$

for $\tau = e^{\rho}$.

If we add the perturbation terms (25) we can proceed as in the case of the ASD equation and observe that, for large $\rho \to \infty$ the eigenvalues and eigenvectors of the perturbed system converge to the eigenvalues and eigenvectors $\pm\lambda_i^{nlk}$ and $U_i^{\pm}(nlk)$ described above.

In the variables $\alpha_{nlk} = \eta_{nlk} + i\xi_{nlk}$ and $\beta_{nlk} = p_{nlk} + iq_{nlk}$, the perturbed system can be written as the non-autonomous linear system with the 8×8

matrix

$$\tilde{M} = \begin{pmatrix} nI & e^\rho A & 0 & 0 \\ 0 & -(n+1)I & e^\rho B & 0 \\ 0 & 0 & -(n+2)I & -e^\rho A \\ -e^\rho B & 0 & 0 & (n+1)I \end{pmatrix}, \tag{29}$$

where I is the 2×2 identity matrix.

The eigenvalues have a term independent of ρ which reduces to the diagonal entries of the matrix $\tilde{M}$ as $\rho \to -\infty$, and terms containing e^ρ, of which the highest order term reduces to the eigenvalues of the matrix M. Thus, the rate of decay as $\rho \to \infty$ in the directions corresponding to eigenvalues of negative real part is at least given by $\exp(-n\rho)$ and at most by $\exp(-C_{nlk}e^\rho)$, for some positive constants $C_{nlk} > 0$, or by mixed terms $\exp(n\rho - C_{nlk}e^\rho)$.

With this analysis in place, we know the asymptotic behavior as $\rho \to \infty$ of the finite energy solutions of the linear system. Now we want to analyze the asymptotic behavior of solutions of the original non-linear equations.

Suppose given a finite energy solution $\Xi_0 = (u_0, v_0, f_0, \alpha_0, \beta_0)$ of the linear system. Consider a perturbed linear system with matrix L_{Ξ_0} as follows:

$$\begin{aligned} u'_{nlk} = & \quad -inv_{nlk} + ikf_{nlk} + 2iRe(\bar{\alpha}_0\beta)_{nlk} + 2iRe(\bar{\alpha}\beta_0)_{nlk} \\ v'_{nlk} = & \quad inu_{nlk} - ilf_{nlk} + 2iIm(\bar{\alpha}_0\beta)_{nlk} + 2iIm(\bar{\alpha}\beta_0)_{nlk} \\ f'_{nlk} = & \quad ike^{2\rho}u_{nlk} - ile^{2\rho}v_{nlk} \\ & \quad +2e^{2\rho}(Re(\bar{\alpha}_0\alpha)_{nlk} - (\bar{\beta}_0\beta)_{nlk}) \\ \alpha'_{nlk} = & \quad -n\alpha_{nlk} + ie^\rho\tfrac{1}{2}(il-k)\bar{\beta}_{-n-1\ -l\ -k} \\ & \quad +ie^\rho\tfrac{1}{2}(a_0\bar{\beta})_{nlk} + ie^\rho\tfrac{1}{2}(a\bar{\beta}_0)_{nlk} + i(f_0\alpha)_{nlk} + i(f\alpha_0) \\ \beta'_{nlk} = & \quad n\beta_{nlk} - ie^\rho(il-k)\alpha_{n+1\ lk} - ie^\rho(a_0\alpha)_{nlk} - ie^\rho(a\alpha_0)_{nlk} \\ & \quad -i(f_0\beta)_{nlk} - i(f\beta_0). \end{aligned} \tag{30}$$

Similarly, we can define inductively solutions $\Xi_{\nu+1}$ satisfying the linear system of equations

$$\Xi'_{\nu+1} = L_{\Xi_\nu}\Xi_{\nu+1},$$

with initial condition $\Xi_{\nu+1}(0) = \Xi_\nu(0)$.

Proposition 3.7 *All the solutions $\Xi_{\nu+1}$ are finite energy. Moreover, we have*

$$\Xi_\nu \to \Xi$$

uniformly on compact sets along with all derivative, where Ξ is a solution of the original non-linear system (21)

$$\Xi' = L_{\Xi}\Xi.$$

The solution Ξ has the same asymptotic behavior as $\rho \to \infty$ as Ξ_0, up to terms that decay faster as $\rho \to \infty$.

Proof. Inductively, the system of equations satisfied by $\Xi_{\nu+1}$ can be separated into a leading term and a perturbation. The rate of decay of the solutions to the system with the leading term only will determine the asymptotics of $\Xi_{\nu+1}$.

In the case of the system (30), we have the coefficients of Ξ_0 decaying at least like $e^{-n\rho}$. Thus, for large ρ, the leading terms in the system (30) will be the same as the leading terms in the original linear system.

Inductively, at the ν-th stage, we have the original linear system

$$\begin{aligned}
u'_{nlk} = \;& -inv_{nlk} + ikf_{nlk} \\
& 2iRe(\bar{\alpha}_\nu\beta)_{nlk} + 2iRe(\bar{\alpha}\beta_\nu)_{nlk} \\
v'_{nlk} = \;& inu_{nlk} - ilf_{nlk} \\
& 2iIm(\bar{\alpha}_\nu\beta)_{nlk} + 2iIm(\bar{\alpha}\beta_\nu)_{nlk} \\
f'_{nlk} = \;& ike^{2\rho}u_{nlk} - ile^{2\rho}v_{nlk} \\
& 2e^{2\rho}(Re(\bar{\alpha}_\nu\alpha)_{nlk} - (\bar{\beta}_\nu\beta)_{nlk}) \\
\alpha'_{nlk} = \;& -n\alpha_{nlk} + ie^{\rho}\tfrac{1}{2}(il-k)\bar{\beta}_{-n-1\ -l\ -k} \\
& ie^{\rho}\tfrac{1}{2}(a_\nu\bar{\beta})_{nlk} + ie^{\rho}\tfrac{1}{2}(a\bar{\beta}_\nu)_{nlk} + i(f_\nu\alpha)_{nlk} + i(f\alpha_\nu) \\
\beta'_{nlk} = \;& n\beta_{nlk} - ie^{\rho}(il-k)\alpha_{n+1\ lk} \\
& -ie^{\rho}(a_\nu\alpha)_{nlk} - ie^{\rho}(a\alpha_\nu)_{nlk} - i(f_\nu\beta)_{nlk} - i(f\beta_\nu).
\end{aligned} \tag{31}$$

Under the inductive hypothesis that the coefficients of the solutions Ξ_ν decay at a rate at least $e^{-n\rho}$ as $\rho \to \infty$, we see that the leading terms in the system (31) are the same as in the original unperturbed system. The eigenvalues $\pm\tilde{\lambda}_i^{nkl}(\rho)$ of the perturbed system (31) and the corresponding eigenvectors $\tilde{U}_i^{\pm}(\rho)$ converge asymptotically to the ρ-independent eigenvalues and eigenvectors of the matrix M. Thus, the solutions Ξ_ν are of finite energy and maintain the same rate of decay as the original solution Ξ_0.
$\diamond$

3.3 Uniform convergence

Notice that, in the particular case where the pointwise convergence proved in Lemma 3.1 extends uniformly to an infinite strip $T^2 \times [-r_0, r_0] \times \mathbf{R}$, in the sense specified below, then we have uniform convergence of the $(\mathcal{A}_r, \Psi_r)$ in the smooth topology on the infinite strip to a flat connection. In particular, if we consider the geometric limits of solutions on (1) on $Y(r)$ as $r \to \infty$, as analysed in [4], we see that, in this special case, the asymptotic values $(A_r(\pm\infty), \psi_r(\pm\infty))$ break through the same flat connection on T^2, when $r \to \infty$.

By the results of the previous section, on any fixed compact set K in $T^2 \times [-r, r] \times \mathbf{R}$, or in $\nu(K) \times \mathbf{R}$, we have

$$\|F_{\mathcal{A}_r}\|_{L^2(K)} \to 0.$$

Proposition 3.8 *Suppose given a family of finite energy solutions* $\Xi_r = (\mathcal{A}_r, \Psi_r)$*, for all* $r \geq r_0$*. Suppose there exists a strip* $T^2 \times [r_0, r_1] \times \mathbf{R}$ *in* $T^2 \times [-r, r] \times \mathbf{R}$ *such that we have*

$$\|F_{\mathcal{A}_r}\|_{L^2(T^2 \times [r_0, r_1] \times \mathbf{R})} \to 0 \qquad (32)$$

as $r \to \infty$*. Then there exists a flat connection* a_{T^2} *on* T^2 *such that (up to gauge transformations) the sequence* Ξ_r *converges in the* L^2_2*-topology on* $T^2 \times [r_0, r_1] \times \mathbf{R}$ *to* a_{T^2}*. In particular this implies that, under the splitting of the critical points* $\mathcal{M}_{Y(r)} = \mathcal{M}_V \#_{\chi_0(T^2)} \chi(\nu(K))$*, for large enough* r*, the asymptotic values* $(A_r(\pm\infty), \psi_r(\pm\infty))$ *can be written as the gluing*

$$(A_r(\pm\infty), \psi_r(\pm\infty)) = (A'(\pm\infty), \psi'(\pm\infty)) \#_{a_{T^2}} (\lambda^{\pm} a_{T^2}, 0),$$

that is, they split through the same asymptotic flat connection on T^2*. Moreover, suppose that* $(\mathcal{A}', \Psi')$ *and* $(\mathcal{A}, 0)$ *are the limits on compact sets on* $V \times \mathbf{R}$ *and* $\nu(K) \times \mathbf{R}$ *respectively, with asymptotic values* $[a_V] \in \chi_0(T^2, V)$ *and* $[a_\nu] \in \chi_0(T^2, \nu(K))$*, as described in the previous sections. Then, in this particular case, there are gauge transformations* λ' *and* λ *on* V *and* $\nu(K)$ *respectively such that we have*

$$\lambda' a_V = a_{T^2} = \lambda a_\nu.$$

Proof. By Uhlenbeck weak convergence [8] [18], and the result of Proposition 8.3 of [8], the condition

$$\|F_{\mathcal{A}_r}\|_{L^2(T^2\times[r_0,r_1]\times\mathbf{R})} \to 0$$

implies that there exists a flat connection $\mathcal{A}$ on $T^2 \times [r_0, r_1] \times \mathbf{R}$ such that, up to gauge transformations λ_r in L^2_3 and up to passing to a subsequence, we have

$$\|\lambda_r\mathcal{A}_r - \mathcal{A}\|_{L^2_2(T^2\times[r_0,r_1]\times\mathbf{R})} \leq C\|F_{\mathcal{A}_r}\|_{L^2(T^2\times[r_0,r_1]\times\mathbf{R})} \to 0.$$

We can now show that we can further gauge transform the connections $\mathcal{A}_k$ so that this Uhlenbeck limit on $T^2 \times [r_0, r_1] \times \mathbf{R}$ is a flat connection on T^2, constant in the $t \in \mathbf{R}$ and $s \in [r_0, r_1]$ directions.

Claim: Let $\mathcal{A}$ be an $L^2_{2,loc}$ flat connection on $T^2 \times [r_0, r_1] \times \mathbf{R}$. Then, there exist and $L^2_{3,loc}$ gauge transformation λ on $T^2 \times [r_0, r_1] \times \mathbf{R}$ such that $\lambda\mathcal{A}$ is a flat connection a_{T^2} on T^2, constant in the t and s directions.

proof of Claim: We can write $\mathcal{A}$ in the form $\mathcal{A} = a_{T^2} + if\ dt + ih\ ds$, where a_{T^2} is a connection on T^2, and $f(x, y, t, s)$ and $h(x, y, t, s)$ are real valued functions. Then the equation $F_{\mathcal{A}} = 0$ becomes

$$\begin{array}{c} F_{a^2_T} = 0 \\ \partial_t a_{T^2} = d_{T^2} if \\ \partial_s a_{T^2} = d_{T^2} ih \\ \partial_t h = \partial_s f. \end{array} \tag{33}$$

Define the gauge transformation

$$\lambda = \exp\left(-i\int_{r_0}^s h(x,y,t,\sigma)d\sigma - i\int_0^t f(x,y,\tau,r_0)d\tau\right).$$

This satisfies

$$\begin{aligned} \lambda^{-1}d\lambda = &\ -ih(x,y,t,s)ds - if(x,y,t,r_0)dt \\ &+(-i\textstyle\int_{r_0}^s \partial_t h(x,y,t,\sigma)d\sigma)dt \\ &-id_{T^2}\left(\textstyle\int_{r_0}^s h(x,y,t,\sigma)d\sigma + \int_0^t f(x,y,\tau,r_0)d\tau\right) \\ = &\ -ih(x,y,t,s)ds - if(x,y,t,s)dt \\ &-id_{T^2}\left(\textstyle\int_{r_0}^s h(x,y,t,\sigma)d\sigma + \int_0^t f(x,y,\tau,r_0)d\tau\right), \end{aligned}$$

where the equality follows from the last equation of (33). The gauge transformed connection $\lambda\mathcal{A}$ is of the form

$$\lambda\mathcal{A} = a_{T^2} - id_{T^2}\left(\int_{r_0}^{s} h(x,y,t,\sigma)d\sigma + \int_0^t f(x,y,\tau,r_0)d\tau\right) = \tilde{a}_{T^2}.$$

This satisfies the equations

$$\begin{array}{c} F_{\tilde{a}_{T^2}} = 0 \\ \partial_t \tilde{a}_{T^2} = \partial_t a_{T^2} - id_{T^2}(f(x,y,t,r_0) + \int_{r_0}^s \partial_t h(x,y,t,\sigma)d\sigma) = \\ = \partial_t a_{T^2} - id_{T^2} f = 0 \\ \partial_s \tilde{a}_{T^2} = \partial_s a_{T^2} - id_{T^2} h = 0. \end{array}$$

This completes the proof of the Claim.

The asymptotic limits $a_\infty^\pm$ of $(A'(\pm\infty), \psi'(\pm\infty))$ then satisfy

$$[a_\infty^+] = [a_\infty^-] = [a_{T^2}]$$

in the universal cover $\chi_0(T^2, Y)$ of $\chi(T^2)$. This completes the proof of the Proposition.

$\diamond$

Notice, however, that in general the uniform convergence of Proposition 3.8 should not be expected. In fact, we have seen that the estimate (3) and Lemma 2.4 give

$$\|F_{A_r(t)}\|^2_{L^2(T^2\times[r_0,r_1]\times\{t\})} \leq Ce^{-\frac{c|t|}{r}}.$$

This allows for non-uniformity of the convergence: there may be sequences $T_r \to \infty$ for which (32) is violated, and an estimate like

$$\|F_{A_r(t)}\|^2_{L^2(T^2\times[r_0,r_1]\times\{t\})} \geq c > 0$$

holds for all $|t| \in [T_r - \ell/2, T_r + \ell/2]$, for some $\ell > 0$ independent of $r \geq r_0$. This is clearly compatible with the constraint (3).

In the next subsection we analyze the geometric limits of monopoles $(\mathcal{A}_r, \Psi_r)$ on the domains $T^2 \times [-r,r] \times [T_r, \infty)$ and $T^2 \times [-r,r] \times (-\infty, -T_r]$. This is where we encounter the non-uniform convergence which is not detected by limits on compact sets.

3.4 Holomorphic disks in $\chi(T^2)$

Consider the Seiberg-Witten equations in on $T^2 \times [-r, r] \times \mathbf{R}$. Let $(\mathcal{A}_r, \Psi_r)$ be a family of solutions, where we write the connections as

$$\mathcal{A}_r = a_r(w, s, t) + f_r(w, s, t)ds + h_r(w, s, t)dt$$

and the spinors as $\Psi_r = (\alpha_r(w, s, t), \beta_r(w, s, t))$.

Recall again that gauge equivalent solutions $(A_r(t), \psi_r(t))$, in a temporal gauge in the $t \in \mathbf{R}$ direction, decay exponentially at the asymptotic ends $t \to \pm\infty$ to critical points $(A_r(\pm\infty), \psi_r(\pm\infty))$ of the Chern-Simons-Dirac functional.

Suppose given a fixed compact set $T^2 \times [r_0, r_1]$ in $Y(r)$. Choose a sequence $T_r \to \infty$, for which the estimate (3) gives

$$\|a_r(w, s, \pm T_r) - a^{\pm}(w)\|^2 + \|f_r(w, s, \pm T_r) - f^{\pm}(w, s)\|^2 \leq Ce^{-cr}, \quad (34)$$

in the L^2_1-norm on $T^2 \times [r_0, r_1]$, where $a^{\pm}$ are the limits of $(A_r(\pm\infty), \psi_r(\pm\infty))$ as $r \to \infty$ on $T^2 \times [r_0, r_1]$. In the first part [4] of this work we showed that, in the gluing of solutions to the 3-dimensional Seiberg-Witten equations (1), we are only interested in critical points $(A_r(\pm\infty), \psi_r(\pm\infty))$ of the Chern-Simons-Dirac functional that break through smooth points $a^{\pm} \neq \vartheta$ in the character variety $\chi(T^2)$, as $r \to \infty$. Thus, we can assume that the elements $f^{\pm}(w, s)$ are exponentially small on $T^2 \times [r_0, r_1]$, for large enough $r_0 > 0$,

$$\|f^{\pm}(w, s)\|^2_{L^2(T^2 \times \{s\})} \leq Ce^{-\delta s}.$$

This exponential estimate follows from the analysis of the asymptotics of monopoles on a 3-manifold with an end modeled on $T^2 \times [0, \infty)$, as in [4].

This simple observation implies that we should expect more complicated geometric limits than the limits on compact sets analyzed in the previous section. In fact, for T_r growing sufficiently fast, we have convergence to the limits $a^{\pm}$ on $T^2 \times [r_0, r_1] \times \{t\}$, with $|t| \geq T_r$. Thus, whenever $a^+ \neq a^-$, we must have geometric limits which are not just constant flat connections on T^2.

As discussed previously, in polar coordinates $z = s + it$ and $z = e^{\rho + i\theta}$, with

$$a_r(w, \rho, \theta) = a_r(w, e^{\rho+i\theta})$$

$$f_r(w, \rho, \theta) = e^{-\rho} \cos\theta \; h_r(w, e^{\rho+i\theta}) - e^{-\rho} \sin\theta \; f_r(w, e^{\rho+i\theta})$$

$$h_r(w, \rho, \theta) = e^{-\rho} \cos\theta \; f_r(w, e^{\rho+i\theta}) + e^{-\rho} \sin\theta \; h_r(w, e^{\rho+i\theta}),$$

the curvature equation takes the form

$$\partial_\rho a_r - dh_r = *(\partial_\theta a_r - df_r + i(\bar\alpha_r \beta_r + \alpha_r \bar\beta_r))$$

$$\partial_\rho f_r - \partial_\theta h_r = e^{2\rho} * (F_{a_r} + \frac{i}{2}(|\alpha_r|^2 - |\beta_r|^2)\omega).$$

We are now interested in studying the convergence away from arbitrarily large compact sets. That is, we are interested in the convergence on regions of the form

$$\begin{array}{rl} \tilde\Omega(r) = & T^2 \times [0,r] \times [T_r, \infty) \\ & \cup T^2 \times [0,r) \times (-\infty, -T_r] \\ & \cup T^2 \times \{\rho \in [\log T_r, \infty)\ \theta \in [-\pi/2, \pi/2]\} \end{array}.$$

We choose a sequence $T(r)$ such that, as in (34), the finite energy solutions $(\mathcal{A}_r, \Psi_r)$ on $Y(r) \times \mathbf{R}$ are exponentially close to the asymptotic values $a^\pm$ on

$$T^2 \times [-r_0, r_0] \times ([T(r), \infty) \cup (-\infty, -T(r)].$$

We write $T(r) = e^{T_r}$, with $T_r \to \infty$ as $r \to \infty$.

In order to study the convergence on the cylinder $T^2 \times [-r, r] \times \mathbf{R}$, we introduce a suitable rescaling of the coordinates on $T^2 \times \mathbf{R}^2$. In polar coordinates, we introduce the translation $\rho \mapsto \rho - T_r$. In these new coordinates, the second part of the curvature equation becomes

$$\partial_\rho f_r - \partial_\theta h_r - e^{2T_r} e^{2\rho} * (F_{a_r} + \frac{i}{2}(|\alpha_r|^2 - |\beta_r|^2)\omega) = 0.$$

Consider the unit disk $D = \{\rho \le 0\}$ in the new coordinates. By the previous analysis of the convergence on compact sets we obtain smooth convergence of a subsequence to a solution of

$$\partial_\rho a - *(\partial_\theta a - df) = 0$$

$$F_a = 0.$$

Up to a gauge transformation, this is a holomorphic map

$$a : D \to \chi_0(T^2, Y) = \mathbf{C}.$$

Namely, $a(\rho, \theta)$ is the flat connection on T^2 satisfying

$$a(\rho, \theta)(w) = a(w, \rho, \theta).$$

With a slight abuse of notation we identify $a(\rho, \theta)$ with the class in $\chi_0(T^2, Y)$ obtained modulo homotopically trivial gauge transformations. The map a has the property that it maps the center of the disk, $\rho \to -\infty$, to the flat connection a_∞, which is the limit on compact sets of the original sequence $(\mathcal{A}_r, \Psi_r)$. Moreover, by the finite energy condition, the map a is holomorphic up to the boundary, and the restriction to the boundary

$$a : S^1 = \partial D = \{\rho = 0\} \to \chi_0(T^2, Y) = \mathbf{C}$$

gives a curve in $\chi_0(T^2, Y)$. By our choice of T_r, this curve will satisfy $a(0, \pi/2) = a^+$ and $a(0, -\pi/2) = a^-$.

Thus, we have obtained that the convergence on the $T^2 \times [-r, r] \times \mathbf{R}$ part of $Y(r) \times \mathbf{R}$, away from compact sets, produces a holomorphic map. Now we want to describe the corresponding non-uniform convergence, away from compact sets, on the sub-domains of $Y(r) \times \mathbf{R}$ that correspond to the knot complement and to the tubular neighborhood of the knot. We will use a similar rescaling of the coordinates.

Let V_r be the knot complement V with a cylinder of length r,

$$V_r = V \cup_{T^2 \times \{0\}} T^2 \times [0, r].$$

On the manifold $V_r \times \mathbf{R}$ consider the domain

$$\tilde{\Omega}(r) = V_r \times ([T_r, \infty) \cup (-\infty, -T_r])$$

$$\cup T^2 \times \{\rho \in [\log T_r, \infty) \ \theta \in [-\pi/2, \pi/2]\},$$

where we use the polar coordinates (ρ, θ) introduced before. On this domain consider the change of coordinates

$$\rho \mapsto \rho - T_r$$

in the $\rho \in \mathbf{R}$ direction, and the corresponding change of coordinates

$$|t| \mapsto e^{-T_r} |t|$$

in the $t \in \mathbf{R}$ direction, where the coordinates ρ and t are related by $z = s + it$ and $z = e^{\rho + i\theta}$, as before, with $(s, t) \in [0, \infty) \times \mathbf{R}$.

This produces the rescaled domain

$$V_r \times ([T_r e^{-T_r}, \infty) \cup (-\infty, -T_r e^{-T_r}])$$

$$\cup T^2 \times \{\rho \in [\log(T_r) - T_r, \infty) \ \theta \in [-\pi/2, \pi/2]\}.$$

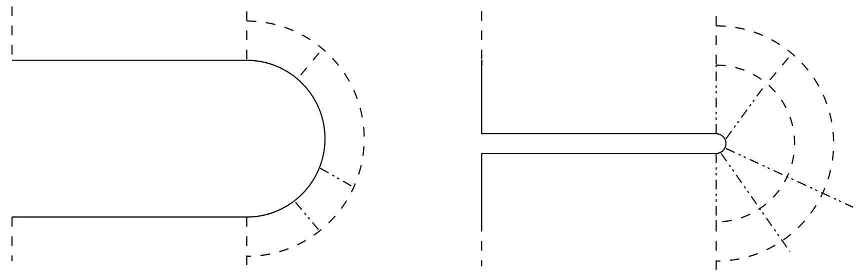

Figure 1: The domain $\tilde{\Omega}(r)$ and its rescaling

Consider the domains

$$\Omega(r) = V_r \times ([-1, -T_r e^{-T_r}] \cup [T_r e^{-T_r}, 1])$$
$$\cup T^2 \times \{\rho \in (-T_r + \log(T_r), 0]\ \theta \in [-\pi/2, \pi/2]\},$$

and

$$\Omega_1(r) = V_r \times ([-1, -T_r e^{-T_r}] \cup [T_r e^{-T_r}, 1]) \subset \Omega(r),$$
$$\Omega_2(r) = T^2 \times \{\rho \in (-T_r + \log(T_r), 0]\ \theta \in [-\pi/2, \pi/2]\} \subset \Omega(r).$$

The finite energy solutions $(\mathcal{A}_r, \Psi_r)$ of the SW equations satisfy

$$e^{-T_r} \frac{\partial}{\partial_t} A_r(t) = - * F_{A_r(t)} + \sigma(\psi_r(t), \psi_r(t))$$

$$e^{-T_r} \frac{\partial}{\partial_t} \psi_r(t) = -\partial_{A_r(t)} \psi_r(t)$$

on $\Omega_1(r)$.

The finite energy condition ensures that the left hand side converges to zero uniformly on any fixed compact set. Thus, the family $(\mathcal{A}_r, \Psi_r)$, under the present change of coordinates, converges smoothly on compact sets in to a path $(A(t), \psi(t))$ of finite energy solutions in the domain $V \times \mathbf{R}^*$, where $\mathbf{R}^* = \mathbf{R} \backslash \{0\}$, where V is endowed with an infinite cylindrical end $T^2 \times [0, \infty)$. That is, we obtain a path $(A(t), \psi(t))$ of elements satisfying

$$* F_{A(t)} + \sigma(\psi(t), \psi(t)) = 0$$

$$\partial_{A(t)} \psi(t) = 0.$$

This defines a path $[A(t), \psi(t)]$ in $\mathcal{M}_V$ for $0 < t \leq 1$. The endpoint satisfies

$$[A(1), \psi(1)] = [A'_{+\infty}, \psi'_{+\infty}],$$

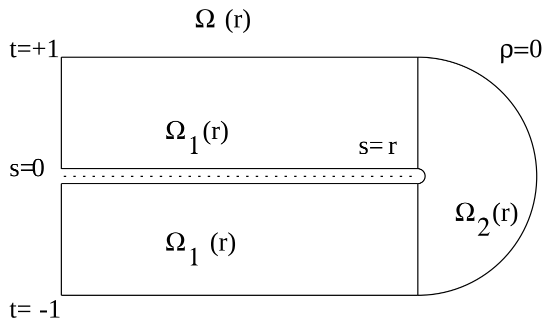

Figure 2: The domains $\Omega(r)$, $\Omega_1(r)$, and Ω_2

where $(A'_{+\infty}, \psi'_{+\infty})$ is the limit on V as $r \to \infty$ of the asymptotic values $(A_r(+\infty), \psi_r(+\infty))$ of $(\mathcal{A}_r, \Psi_r)$. Thus, we have

$$\partial_\infty(A'_{+\infty}, \psi'_{+\infty}) = a^+,$$

under the boundary value map. Moreover, we obtain an element

$$\lim_{t\to 0_+}[A(t), \psi(t)] = [A, \psi]$$

in $\mathcal{M}_V$, satisfying

$$\partial_\infty(A, \psi) = a'_\infty,$$

where a'_∞ is the asymptotic value of the limit on compact sets of the solutions $(\mathcal{A}_r, \Psi_r)$ on $V \times \mathbf{R}$, as analysed in the previous section. Similarly, we obtain a path $[A(t), \psi(t)]$ in $\mathcal{M}_V$ for $-1 \leq t < 0$, where the endpoint satisfies

$$[A(-1), \psi(-1)] = [A'_{-\infty}, \psi'_{-\infty}],$$

where $(A'_{-\infty}, \psi'_{-\infty})$ is the limit on V as $r \to \infty$ of the asymptotic values $(A_r(-\infty), \psi_r(-\infty))$ of $(\mathcal{A}_r, \Psi_r)$. Thus, we have

$$\partial_\infty(A'_{-\infty}, \psi'_{-\infty}) = a^-,$$

under the boundary value map. Similarly, the element

$$\lim_{t\to 0_-}[A(t), \psi(t)] = [\tilde{A}, \tilde{\psi}]$$

in $\mathcal{M}_V$ satisfies

$$\partial_\infty(\tilde{A}, \tilde{\psi}) = a'_\infty.$$

Notice that in general the element $[A, \psi]$ and the element $[\tilde{A}, \tilde{\psi}]$ need not be the same element in $\mathcal{M}_V$. In fact, the domain $\Omega_1(r)$ is defined by rescaling the initial domain $\tilde{\Omega}(r)$, and the values of solutions near the frontier $t = \pm T_r$ of $\tilde{\Omega}(r)$ need not approach the same limits as $r \to \infty$.

From the information on the convergence on compact sets obtained in the previous section, together with the choice of the domains, we can only derive the relation

$$\partial_\infty(A, \psi) = \partial_\infty(\tilde{A}, \tilde{\psi}).$$

Thus, both $[A, \psi]$ and $[\tilde{A}, \tilde{\psi}]$ are in the zero-dimensional fiber $\partial_\infty^{-1}(a'_\infty)$, cf. [4].

Moreover, by the previous analysis, together with the analysis of asymptotics on $V \times \mathbf{R}$ of the previous section, we obtain convergence on the $\Omega_2(r)$ to a function $a : D^+ \to \chi_0(T^2, V)$, holomorphic in a neighborhood U of the half disk

$$D^+ = \{\rho \in (-\infty, 0]\ \theta \in [-\pi/2, \pi/2]\},$$

which maps the center of the disk to a'_∞, and such that the restriction to the boundary component $\{\theta \in \{-\pi/2, \pi/2\}\} \subset \partial D^+$ gives a curve in $\partial_\infty(\mathcal{M}_V) \subset \chi_0(T^2, V)$ connecting a^+, a'_∞, and a^-. The image of the component $\{-\pi/2 \leq \theta \leq \pi/2,\ \rho = 0\}$ of ∂D^+ is mapped to another curve in $\chi_0(T^2, V)$, not necessarily contained in $\partial_\infty(\mathcal{M}_V)$, which connects the points a^+, a'_∞, and a^-. The image of ∂D^+ is a closed curve in $\chi_0(T^2, V)$, not necessarily smooth across the points $\theta = \pm\pi/2$.

3.5 The convergence theorem

We need one more preliminary Lemma, which connects the limits on compact sets, with the non-uniform limits obtained in the previous section.

Lemma 3.9 *Let $(\mathcal{A}, \Psi)$ be a finite energy solution of the Seiberg–Witten equations (2) on the 4-manifold $V \times \mathbf{R}$ with the infinite end $T^2 \times [0, \infty) \times \mathbf{R}$. Let $a_\infty \in \chi_0(T^2, V)$ be the radial gauge limit of $(\mathcal{A}, \Psi)$. Let $(A(t), \psi(t))$ be gauge equivalent to the original $(\mathcal{A}, \Psi)$, in a temporal gauge in the $t \in \mathbf{R}$ direction. Then the elements $(A(t), \psi(t))$ converge in the $s \in [r_0, \infty)$ direction to a_∞, uniformly with respect to $|t| \geq T_0$, and we have limits*

$$\lim_{t \to +\infty} (A(t), \psi(t)) = (A, \psi),$$

and

$$\lim_{t \to -\infty} (A(t), \psi(t)) = (\tilde{A}, \tilde{\psi}),$$

with

$$\partial_\infty(A,\psi) = \partial_\infty(\tilde{A},\tilde{\psi}) = a_\infty.$$

Proof. In order to prove the convergence in the time direction, consider the asymptotics of solutions in radial gauge studied in the previous section. From the analysis of the system of ODE's (22), we obtained rates of decay as $\rho \to \infty$ of the form $\exp(-n\rho)$ or $\exp(n\rho - C_{nlk}e^\rho)$, or $\exp(-C_{nlk}e^\rho)$, for some positive constants $C_{nlk} > 0$.

In the last case, at a point (s,t) along the end $[r_0,\infty)\times\mathbf{R}$, the L^2-norm on T^2 is bounded by $\exp(-C(s^2+t^2)^{1/2})$. In particular, we have exponential decay in the $s \in [r_0,\infty)$ direction, uniformly in $|t| \geq T_0$, bounded by $\exp(-CT_0 s)$. Thus, if we consider a temporal gauge representative $(A'(t),\psi'(t))$ of $(\mathcal{A}',\Psi')$, for every sequence $|t_k| \to \infty$ we obtain convergence up to gauge of a subsequence to an element in $\mathcal{M}_V$, exponentially decaying in the $s \in [r_0,\infty)$ direction to the asymptotic value a'_∞, at a rate at least equal to $\exp(-|\lambda| T_0 s)$.

The case of the slow decaying solutions with asymptotics $e^{-n\rho}$ gives polynomial decay in the $s \in [r_0,\infty)$ direction for every fixed $t \in \mathbf{R}$, at a rate

$$\frac{1}{\left(1+(\frac{t}{s})^2\right)^{n/2}} \cdot \frac{1}{s^n} \leq \frac{1}{s^n},$$

uniform in $|t| \geq T_0$. Thus, in this case, the temporal gauge representative $(A'(t),\psi'(t))$ converges to solutions of (1) on V, as $t \to \pm\infty$, that decay to the asymptotic value a'_∞ at a polynomial rate $1/s^n$. The analysis of [4] on the center manifold for the equations (1) on $T^2 \times [r_0,\infty)$ (cf. [15]), imply the following. If the decay is asymptotic to $1/s^n$, with $n \geq 2$, then the actual rate of decay of the limit solution is exponential and we have

$$a'_\infty \in \partial_\infty(\mathcal{M}^*_V).$$

If the rate of decay is $1/s$, then the limit solution might also decay like $1/s$ along $s \in [r_0,\infty)$. In this case the asymptotic value is $a'_\infty = \vartheta$, the unique "bad point" in the character variety $\chi(T^2)$.
$\diamond$

We can summarize all the previous results as follows

Theorem 3.10 *On the manifold $Y(r) \times \mathbf{R}$ consider classes of finite energy solutions $[\mathcal{A}_r, \Psi_r]$ of (2), with asymptotic values*

$$[A_r(\pm\infty), \psi_r(\pm\infty)] = [A'_{\pm\infty}, \psi'_{\pm\infty}]\#^r_{a^\pm}[a^\pm, 0],$$

as $t \to \pm\infty$. Under the usual assumptions on the scalar curvature on $\nu(K)$, we obtain the following geometric limits as $r \to \infty$.

(i) a finite energy $[\mathcal{A}', \Psi']$ on $V \times \mathbf{R}$, with radial limit $a'_\infty \in \chi_0(T^2, V)$, and limits as $t \to \pm\infty$

$$[A, \psi] \text{ and } [\tilde{A}, \tilde{\psi}] \text{ in } \partial_\infty^{-1}(a'_\infty) \subset \mathcal{M}_V.$$

(ii) a solution $[a''_\infty, 0]$ on $\nu(K) \times \mathbf{R}$, with $a''_\infty \in \chi_0(T^2, \nu(K))$.

(iii) a path $[A(t), \psi(t)] \in \mathcal{M}_V$ for $t \in [-1, 0) \cup (0, 1]$ with

$$[A(\pm 1), \psi(\pm 1)] = [A'_{\pm\infty}, \psi'_{\pm\infty}]$$

and the limits as $t \to 0_\pm$ equal to $[A, \psi]$ and $[\tilde{A}, \tilde{\psi}]$.

(iv) a path $[A(t), 0]$ in $\mathcal{M}_{\nu(K)} = \chi(\nu(K))$ for $t \in [-1, 1]$ satisfying $[A(\pm 1), 0] = [a^\pm, 0]$ and $[A(0), 0] = [a''_\infty, 0]$.

(v) a holomorphic map $a : D \to \chi_0(T^2, Y)$, with $a(0) = a_\infty$, the limit on compact sets on the cylinder $T^2 \times [-r, r] \times \mathbf{R}$.

(vi) maps $a_V : D^+ \to \chi_0(T^2, V)$ and $a_\nu : D^+ \to \chi_0(T^2, \nu(K))$, holomorphic on a neighborhood of the half disk D^+ with $a_V(\rho, \pm\pi/2) = [A(t), \psi(t)]$ and $a_\nu(\rho, \pm\pi/2) = [A(t), 0]$.

Proof. Suppose given a family $(\mathcal{A}_r, \Psi_r)$ of finite energy solutions of the Seiberg-Witten equations (2) on the manifolds $Y(r) \times \mathbf{R}$, with asymptotic values $(A_r(\pm\infty), \psi_r(\pm\infty))$ as $t \to \pm\infty$ satisfying (1). Assume that, as $r \to \infty$ the asymptotic values split as

$$(A_r(\pm\infty), \psi_r(\pm\infty)) = (A'_{\pm\infty}, \psi'_{\pm\infty})\#^r_{a^\pm}\lambda(a^\pm, 0),$$

for some gauge transformation λ on $\nu(K)$, according to [4].

Then, up to gauge, there is a subsequence, which we still denote $(\mathcal{A}_r, \Psi_r)$, with the following behavior. On compact sets contained in the long cylinder $T^2 \times [-r, r] \times \mathbf{R}$ it converges smoothly to a flat connection a_∞ on $T^2 \times \mathbf{R}^2$, constant in the $\mathbf{R}^2$ directions. On compact sets in $V \times \mathbf{R}$ it converges smoothly to a finite energy solution $(\mathcal{A}', \Psi')$ of the Seiberg-Witten equations (2) on $V \times \mathbf{R}$, which is exponentially decaying in radial gauge on the end

$T^2 \times [0,\infty) \times \mathbf{R}$ to a flat connection a'_∞. This follows from the analysis of the asymptotics in the previous section.

The solution $(\mathcal{A}', \Psi')$ in a temporal gauge in the $t \in \mathbf{R}$ direction converges to solutions (A,ψ) and $(\tilde{A},\tilde{\psi})$ of the Seiberg-Witten equations (1) on V, as $t \to \pm\infty$ respectively. The elements (A,ψ) and $(\tilde{A},\tilde{\psi})$ have asymptotic value along the end $T^2 \times \mathbf{R}$ of V

$$\partial_\infty(A,\psi) = \partial_\infty(\tilde{A},\tilde{\psi}) = a'_\infty,$$

under the map ∂_∞ analyzed in [4]. This result follows from Lemma 3.9

Finally, again from the arguments presented in the previous sections, on compact sets in $\nu(K) \times \mathbf{R}$ the subsequence $(\mathcal{A}_r, \Psi_r)$ converges to a solution of the abelian ASD equation, that is, up to gauge, to a flat connection a''_∞ on T^2.

Up to the changes of coordinates $\rho \mapsto \rho - T_r$ and $|t| \mapsto e^{-T_r}|t|$, for a sufficiently fast growing $T_r \to \infty$, there is a subsequence of $(\mathcal{A}_r, \Psi_r)$ that converges smoothly on the domain Ω_1 defined above to a path $[A(t),\psi(t)]$ in $\mathcal{M}_V$ for $t \in [-1,0) \cup (0,1]$ with

$$[A(-1),\psi(-1)] = [A'_{-\infty}, \psi'_{-\infty}], \quad \text{with} \quad \partial_\infty(A'_{-\infty}, \psi'_{-\infty}) = a^-$$

$$\lim_{t\to 0_-}[A(t),\psi(t)] = [\tilde{A},\tilde{\psi}], \quad \text{with} \quad \partial_\infty(A,\psi) = a'_\infty$$

$$\lim_{t\to 0_+}[A(t),\psi(t)] = [A,\psi], \quad \text{with} \quad \partial_\infty(A,\psi) = a'_\infty$$

$$[A(1),\psi(1)] = [A'_{+\infty}, \psi'_{+\infty}] \quad \text{with} \quad \partial_\infty(A'_{+\infty}, \psi'_{+\infty}) = a^+.$$

Under the same change of coordinates there is a subsequence that converges smoothly on the domain Ω_2 described above to a function $a_V : D^+ \to \chi_0(T^2, Y) = \mathbf{C}$, holomorphic in a neighborhood U of the half disk D^+, which maps the center of the disk to a'_∞ and such that the restriction to the boundary component $\theta \in \{-\pi/2, \pi/2\}$ of D^+ gives a curve in $\chi_0(T^2,Y)$ connecting a^+, a'_∞, and a^-.

Similarly, on the manifold $\nu(K) \times \mathbf{R}$ we can consider similarly defined domains Ω_1 and Ω_2, and an analogous reparametrization. We obtain convergence on Ω_1 to a path $[A(t),0]$ in $\mathcal{M}_{\nu(K)} = \chi(\nu(K))$ for $t \in [-1,1]$ with

$$[A(-1),0] = [a^-,0],$$

$$[A(0),0] = [a''_\infty,0],$$

$$[A(1),0] = [a^+,0],$$

where a''_∞ is the limit on compact sets of the $(\mathcal{A}_r, \Psi_r)$ on $\nu(K) \times \mathbf{R}$, up to gauge. On Ω_2 we obtain convergence to a function $a_\nu : D^+ \to \chi_0(T^2, Y)$, holomorphic in a neighborhood U of the half disk D^+, which maps the center of the disk to a''_∞, and such that the restriction to the boundary component $\theta \in \{-\pi/2, \pi/2\}$ of D^+ gives a curve in $\chi_0(T^2, Y)$ connecting a^+, a''_∞, and a^-.

Finally, the reparametrization $\rho \mapsto \rho - T_r$ on the long cylinder $(s,t) \in [-r, r] \times \mathbf{R}$ gives convergence on the disk $D = \{\rho \leq 0\}$ to a holomorphic map $a : D \to \chi_0(T^2, Y)$, which maps the center of the disk to a_∞.

There are subdomains D_1^+ and D_2^+ inside the disk D, and conformal equivalences $\varphi_i : D^+ \to D_i^+$, such that we have

$$a|_{D_1^+} = a_V \circ \varphi_1^{-1}$$

$$a|_{D_2^+} = a_\nu \circ \varphi_2^{-1}.$$

In the special case where $a^+ = a^- = a_\infty$ up to gauge, the holomorphic functions are constant and the paths $[A(t), \psi(t)]$ in $\mathcal{M}_V$ and $\mathcal{M}_{\nu(K)}$ are also constant.

$\diamond$

Notice that, in general, the points a'_∞, a''_∞, and a_∞ are distinct points in $\chi_0(T^2, Y)$. The point a'_∞ is constrained to be on the 1-dimensional subspace $\partial_\infty(\mathcal{M}_V)$ and the point a''_∞ is contained in $\mathcal{M}^{red}_{\nu(K),\mu} = \chi(\nu(K))$.

We have seen that, if the three points a^+, a^-, and a_∞ coincide, $a^+ = a^- = a_\infty$ up to gauge, then the flow line $(\mathcal{A}_r, \Psi_r)$ on $Y(r) \times \mathbf{R}$ must connect critical points which break through the same asymptotic values, $a^+ = a^- = a_\infty$.

We can also observe that the following Corollary holds.

Corollary 3.11 *Assume that the points a'_∞ and a''_∞ coincide up to gauge. Assume, moreover, that they are distinct from the bad point ϑ, and from both a^+ and a^-,*

$$a^+ \neq a'_\infty \neq a^-.$$

then the original flow line $(\mathcal{A}_r, \Psi_r)$ on $Y(r) \times \mathbf{R}$ must connect critical points with relative index at least two.

Proof. Let $(\mathcal{A}', \Psi')$ be the limit on compact sets in $V \times \mathbf{R}$ of the family $(\mathcal{A}_r, \Psi_r)$ as $r \to \infty$. Then the point a'_∞ is the radial limit of $(\mathcal{A}', \Psi')$. By

Lemma 3.9 we know that as $t \to \pm\infty$ the solution $(\mathcal{A}', \Psi')$ has asymptotic values $[A, \psi]$ and $[\tilde{A}, \tilde{\psi}]$ in $\mathcal{M}^*_V$ with

$$\partial_\infty[A, \psi] = \partial_\infty[\tilde{A}, \tilde{\psi}] = a_\infty.$$

By gluing this solution with the solution $(a''_\infty, 0)$ on the $\nu(K) \times \mathbf{R}$ side, we obtain a flow line in $Y(r) \times \mathbf{R}$ which connects the critical points

$$[A, \psi]\#^r_{a_\infty}[a_\infty, 0]$$

and

$$[\tilde{A}, \tilde{\psi}]\#^r_{a_\infty}[a_\infty, 0].$$

This shows that there are two intermediate critical points between the endpoints of the flow line $(\mathcal{A}_r, \Psi_r)$.
$\diamond$

4 Gluing

In this section we describe how to produce an approximate solution to the 4-dimensional Seiberg–Witten equations on $Y(r(T)) \times \mathbf{R}$, with gluing parameter $T \geq T_0$, by pasting together the various geometric limits of Theorem 3.10.

4.1 The approximate solutions

We define the following quantities that will be used throughout this section:

$$R(T) = \frac{1}{\pi}(e^T + T),$$

$$\ell(T) = \frac{1}{\pi}(e^T + T) \sin\left(\frac{\pi T}{e^T + T}\right),$$

$$r(T) = \left(r_0 + \frac{1}{\pi}(e^T + T) \cos\left(\frac{\pi T}{e^T + T}\right)\right).$$

Moreover, we define

$$\Upsilon(t, T) = r(T) - (R(T)^2 - t^2)^{1/2},$$

for $|t| \leq R(T)$, and

$$\tilde{\Upsilon}(t, T) = r_0 + \cos\left(\frac{\pi T}{e^T + T}\right) - \min\left\{(1 - t^2)^{1/2}, \cos\left(\frac{\pi T}{e^T + T}\right)\right\},$$

for $|t| \leq 1$.

Consider the path $[A(t), \psi(t)]$ on the domain

$$V_{r(T)} \times ([-1, -\ell(T)/R(T)] \cup [\ell(T)/R(T), 1], \tag{35}$$

with T large enough, so that the elements $[A(\pm\ell(T)/R(T)), \psi(\pm\ell(T)/R(T))]$ are very close to the elements $[A, \psi]$ and $[\tilde{A}, \tilde{\psi}]$, and with r_0 large enough, so that at $s = r_0$ the elements $[A(t), \psi(t)]$ are very close to the asymptotic values $a_\infty(t)$, up to an error of the order of $e^{-\delta r_0}$. Consider the rescaled domain

$$V_{r(T)} \times ([-R(T), -\ell(T)] \cup [\ell(T), R(T)]), \tag{36}$$

under the change of coordinates $|t| \to R(T)|t|$, and write $[A(t), \psi(t)]_T$ for the corresponding rescaled element. Consider then the path $[A(t), \psi(t)]_T$ restricted over the domain

$$\mathcal{R}_1(T) = \begin{array}{l} (V \times I(T)) \cup_{T^2 \times \{s=0\} \times I(T)} \\ T^2 \times \{(s,t) \mid s \in [0, \Upsilon(t,T)], \quad t \in I(T)\}, \end{array} \tag{37}$$

with

$$I(T) = [-R(T), -\ell(T)] \cup [\ell(T), R(T)], \tag{38}$$

inside the rescaled domain (36).

By Theorem 3.10, we can consider a disk D_T of radius $R(T)$, and a map $a_T : D_T \to \chi_0(T^2, Y)$, such that the center of the circle is mapped to the limit a_∞ of Theorem 3.10, and the points along the boundary corresponding to the angles $\{-\pi, -\pi/2, 0, \pi/2\}$ are mapped to the points $\{a''_\infty, a^-, a'_\infty, a^+\}$, respectively. The function $a_T : D_T \to \chi_0(T^2, Y)$ agrees with the map $a : D \to \chi_0(T^2, Y)$ of Theorem 3.10 restricted to a subdomain of the unit disk D homeomorphic to D_T. We can guarantee, by choosing T large enough, that on the arc of length $2T$ on ∂D_T, centered at the angle $\theta = 0$, the values of the function a_T are sufficiently close to a'_∞, and on the similar arc centered at the angle $\theta = -\pi$ the values of the function a_T are sufficiently close to a''_∞.

Consider then the element $(\mathcal{A}', \Psi')$ restricted over the domain

$$\mathcal{R}_3(T) = V_{r_0} \times [-\ell(T), \ell(T)], \tag{39}$$

For $T \geq T_0$ large enough, we have that, near $t = \pm\ell(T)$, $(\mathcal{A}', \Psi')$ is sufficiently close to the elements $[A, \psi]$ and $[\tilde{A}, \tilde{\psi}]$, and, near $s = r_0$, we have that $(\mathcal{A}', \Psi')$ is sufficiently close to the asymptotic value a_∞.

Moreover, consider the rescaled path $(a''(t),0)_T$ on the domain

$$\mathcal{R}_4(T) = \begin{array}{l} (\nu(K) \times I(T)) \cup_{T^2 \times \{s=0\} \times I(T)} \\ T^2 \times \{(s,t) \mid s \in [0, \Upsilon(t,T)], \quad t \in I(T)\}, \end{array} \tag{40}$$

with $I(T)$ as in (38).

Consider then the element $(a''_\infty, 0)$ on the domain

$$\mathcal{R}_5(T) = \nu(K)_{r_0} \times [-\ell(T), \ell(T)], \tag{41}$$

Now consider the manifold $Y(r(T)) \times \mathbf{R}$, with the long cylinder $T^2 \times [-r(T), r(T)] \times \mathbf{R}$. Consider the product region

$$T^2 \times \{s \in [-r(T), r(T)], \quad t \in [-R(T), R(T)]\} \tag{42}$$

in $Y(r(T)) \times \mathbf{R}$ as illustrated in part *(a)* of Figure 3.

Consider inside (42) the region $\mathcal{R}_2(T)$ given by $T^2 \times D_T$, with the disk D_T centered at $s=0$, $t=0$, as in Figure 3. We identify the regions $\mathcal{R}_i(T)$ of (37), (39), (40), (41), with the corresponding regions as in Figure 3. The smaller strip of regions $\mathcal{R}_3(T)$ and $\mathcal{R}_5(T)$ has height

$$2\ell(T) = \frac{2}{\pi}(e^T + T) \sin\left(\frac{\pi T}{e^T + T}\right),$$

and the larger horizontal strip of the regions $\mathcal{R}_1(T)$ and $\mathcal{R}_4(T)$ has height $2R(T) = \frac{2}{\pi}(e^T + T)$. The arc of ∂D_T cut out by the smaller strip has length $2T$.

We construct an approximate solution on $Y(r(T)) \times \mathbf{R}$ supported on the strip $|t| \leq R(T)$.

We consider smooth functions $\eta_{T,1}(s,t)$, $\eta_{T,2}(s,t)$, $\eta_{T,3}(s,t)$, $\eta_{T,4}(s,t)$, and $\eta_{T,5}(s,t)$ with values in $[0,1]$, supported in the corresponding shaded regions in Figure 3, respectively, such that

$$\eta_{T,1} + \eta_{T,2} + \eta_{T,3} + \eta_{T,4} + \eta_{T,5} \equiv 1$$

is satisfied everywhere in (42). We extend by 1 the functions

$$\eta_{T,1}(s,t), \quad \eta_{T,3}(s,t), \quad \eta_{T,4}(s,t), \quad \text{and} \quad \eta_{T,5}(s,t)$$

on the sides V and $\nu(K)$ of the corresponding domains $\mathcal{R}_i(T)$ which are not represented in the figure.

We finally choose a smooth cutoff function $\chi(t)$ with values in $[0,1]$, supported in $[-R(T), R(T)]$ and satisfying $\chi(t) \equiv 1$ for $t \in [-R(T)+1, R(T)-1]$. Consider also cutoff functions $\chi_{\pm}(t)$ supported in $[R(T)-1, \infty)$ and $(-\infty, -R(T)+1]$, and with $\chi_{\pm}(t) \equiv 1$ for $|t| \geq R(T)$. We define the approximate solution as

$$\begin{aligned}(\mathcal{A}, \Psi)_T = \quad & \chi \cdot (\eta_{T,1}(A(t), \psi(t))_T + \eta_{T,2} a_T \\ & + \eta_{T,3}(\mathcal{A}', \Psi') + \eta_{T,4}(a''(t), 0)_T + \eta_{T,5}(a''_\infty, 0)) \\ & + \chi_-(A^-, \psi^-)_T + \chi_+(A^+, \psi^+)_T.\end{aligned} \tag{43}$$

Here we use the notation $(A(t), \psi(t))_T = (A(R(T)\tau), \psi(R(T)\tau))$, where $(A(\tau), \psi(\tau))$ are the paths in $\mathcal{M}^*_V$, that appear in the geometric limits, for $\tau \in [\ell(T)/R(T), 1]$, and $t = R(T)\tau$. We also denote by $(A^{\pm}, \psi^{\pm})_T$ representatives of the elements in $\mathcal{M}_{Y(r(T))}$ given by the gluing

$$(A^{\pm}, \psi^{\pm})_T = (A^{\pm}, \psi^{\pm}) \#_{r(T)} (a^{\pm}, 0),$$

with

$$(A^{\pm}, \psi^{\pm}) = (A(\pm 1), \psi(\pm 1)) \in \mathcal{M}^*_V.$$

4.2 Linearizations

Let $L_{A(t),\psi(t)}$ be the linearization of the 3-dim Seiberg-Witten equations (1) at the solutions $(A(t), \psi(t))$. Consider the operator $L_{A(t),\psi(t)}$ acting on pairs (α, ϕ) of L^2_1 1-forms and spinors.

First, let us recall from [4] that, given an element $[A, \psi]$ in $\mathcal{M}^*_V$, the operator $L_{A,\psi}$ is a Fredholm operator on the weighted Sobolev spaces of 1-forms and spinors on V,

$$L_{A,\psi} : L^2_{1,\delta}(V) \to L^2_\delta(V), \tag{44}$$

with

$$\delta = \frac{1}{2} \min\{|\lambda|, \quad \lambda \in spec(Q_{a'_\infty})\},$$

and $a'_\infty = \partial_\infty([A, \psi])$. By Theorem 6.2 and 7.4 of [10], we can consider the Fredholm operator $L_{A,\psi}$ acting on L^2_1 1-forms and spinors. For a generic choice of the perturbation of the equations on V, as in [4], the operator

$$L_{A,\psi} : L^2_1(V) \to L^2(V) \tag{45}$$

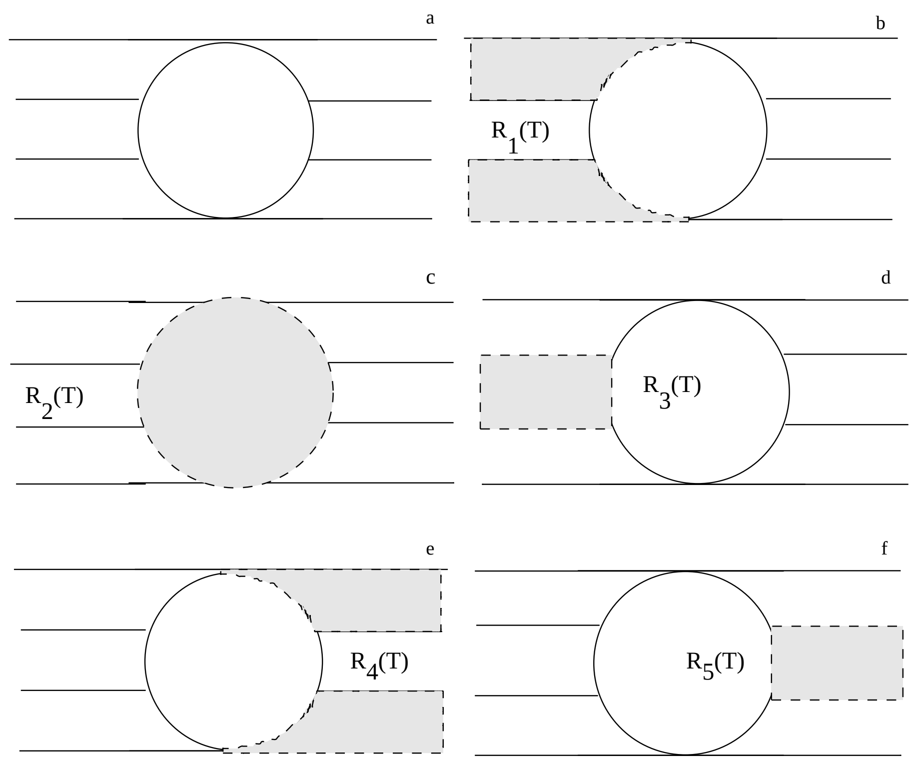

Figure 3: The construction of the approximate solution

is injective and surjective. In the following section we shall discuss the effect of adding the perturbation. For the purpose of this section, let us just assume that injectivity and surjectivity are achieved. On the manifold V_r, with $r \geq r_0$, we can then consider the operator $L_{A,\psi}$ acting between

$$L_{A,\psi} : L^2_1(V_r, P_+ \oplus \ell) \to L^2(V_r), \tag{46}$$

that is, on the space of 1-forms and spinors on V_r, with boundary conditions defined by the APS condition [1]

$$P_+ = L^2\text{-closure of } \{\phi_k | \mu_k > 0\},$$

with ϕ_k eigenfunctions of the operator Q_{a_∞} on T^2 with eigenvalue μ_k, and by the Lagrangian subspace ℓ in $H^1(T^2, i\mathbf{R})$ defined by the asymptotic values of the extended L^2-solutions of $L_{A,\psi}(\alpha, \phi) = 0$ on V. By Proposition 2.4 of [3], we have that (46) is a self-adjoint Fredholm operator with

$$Ker(L_{A,\psi} : L^2_1(V) \to L^2(V)) \cong Ker(L_{A,\psi} : L^2_1(V_r, P_+ \oplus \ell) \to L^2(V_r)).$$

Lemma 4.1 *Consider the operators* $L_{A(t),\psi(t)}$, $L_{A,\psi}$, *and* $L_{\tilde{A},\tilde{\psi}}$ *acting on* L^2_1 *1-forms and spinors on the manifold* V *with the infinite cylindrical end* $T^2 \times [0,\infty)$. *Assume that these operators* $L_{A(t),\psi(t)}$ *are surjective for all* $t \in [-1,0) \cup (0,1]$, *and that the operators* $L_{A,\psi}$ *and* $L_{\tilde{A},\tilde{\psi}}$ *are also surjective. Moreover, assume also that the path* $[A(t),\psi(t)]$ *and the elements* $[A,\psi]$ *and* $[\tilde{A},\tilde{\psi}]$ *in* $\mathcal{M}_V$ *are contained in the complement of* $\partial_\infty^{-1}(U_\vartheta)$, *where* ∂_∞ *is the asymptotic value map, and* U_ϑ *is a small neighborhood of the bad point* ϑ *in the character variety of* T^2. *Then, for all* $T > T_0$, *the operator*

$$\mathcal{T}_T = R(T)^{-1}\frac{\partial}{\partial t} + L_{A(t),\psi(t)}$$

acting on pairs (α,ϕ) *in the* L^2_1 *completion of compactly supported 1-forms and spinors in the domain*

$$\begin{aligned} \mathcal{S}_1(T) = (V \times [-1,1]) \cup_{T^2\times\{s=0\}\times[-1,1]} \\ T^2 \times \{(s,t) \mid s \in [0,\tilde{\Upsilon}(t,T)], \quad t \in [-1,1]\} \end{aligned} \tag{47}$$

is also surjective.

Proof. From the previous discussion we can conclude that, assuming the operators $L_{A(t),\psi(t)}$, $L_{A,\psi}$ and $L_{\tilde{A},\tilde{\psi}}$, with domain as in (45) are surjective, we obtain that the operator $L_{A(t),\psi(t)}$, with domain as in (46), with $r = \tilde{\Upsilon}(t,T)$, has trivial kernel and is surjective. Similarly, the operators $L_{A,\psi}$ and $L_{\tilde{A},\tilde{\psi}}$, with domain as in (46), with $r = r_0$, have trivial kernel and are surjective.

For simplicity of notation, we just write L for the surjective operators we are considering. Given any h, we want to find a function f in the domain satisfying

$$h = (\frac{1}{R(T)}\frac{\partial}{\partial t} + L)f.$$

We know that we have $h = Lg$ for some g. Moreover, we have an estimate

$$\|g(t)\|_{L^2(V_{\tilde{\Upsilon}(t,T)}\times\{t\})} \le C_t \|h(t)\|_{L^2(V_{\tilde{\Upsilon}(t,T)}\times\{t\})},$$

which follows from Parseval's formula (cf. [1] pg.50), where the constant C_t is determined by the smallest absolute value of the non-zero eigenvalues of the asymptotic operator $Q_{a_\infty(t)}$,

$$C_t = \frac{c}{\lambda_\infty(t)},$$

with

$$\lambda_\infty(t) = \min\{|\lambda| \mid \lambda \in spec(Q_{a_\infty(t)})\}.$$

In the character variety $\chi(T^2)$ consider a small open neighborhood U_ϑ of the "bad point" ϑ (see [4]). We have the asymptotic value map

$$\partial_\infty : \mathcal{M}_V \to \chi_0(T^2, V)$$

defined as in [4], with

$$\partial_\infty(A(t), \psi(t)) = a_\infty(t).$$

Notice that, if we assume that the path $(A(t), \psi(t))$, for $0 < t \leq 1$ is in the complement of $\partial_\infty^{-1}(U_\vartheta)$, then the constant C_t can be replaced with the uniform constant

$$C = c \cdot (\min_{0 \leq t \leq 1} \lambda_\infty(t))^{-1},$$

independent of t and of the gluing parameter T.

So we can write

$$h = (\frac{1}{R(T)} \frac{\partial}{\partial t} + L)g - \frac{1}{R(T)} \frac{\partial}{\partial t} g.$$

Again by the surjectivity of L we can write

$$-\frac{\partial}{\partial t} g = L g_1.$$

Again, we have an estimate $\|g_1\| \leq C\|g\|$, hence iterating the process we get

$$h = \left(\frac{1}{R(T)} \frac{\partial}{\partial t} + L \right) \left(\sum_{k=0}^{n} (-1)^k \left(\frac{1}{R(T)} \right)^k g_k \right) + (-1)^{n+1} \left(\frac{1}{R(T)} \right)^{n+1} \frac{\partial}{\partial t} g_n.$$

For all $T \geq T_0$ we obtain

$$\sum_{k=0}^{\infty} \left(\frac{1}{R(T)} \right)^k C^k < \infty,$$

since $R(T) \sim e^T$ for large $T \geq T_0$. Thus, as $n \to \infty$, we have uniform convergence of the series

$$\sum_{k=0}^{n} (-1)^k \left(\frac{1}{R(T)} \right)^k g_k \to f,$$

and of the sequence

$$(-1)^{n+1}\left(\frac{1}{R(T)}\right)^{n+1}\frac{\partial}{\partial t}g_n \to 0.$$

$\diamond$

Over the unit disk D, consider the operator

$$D_1 = \begin{pmatrix} \partial_\rho - *\partial_\theta & R(T)^{-2} * d \\ -e^{2\rho} * d & \partial_\rho \end{pmatrix}$$

acting on pairs $(\tilde{a}, \tilde{f})$ where $\tilde{a}(w,\rho,\theta)$ is 1-form and $\tilde{f}(w,\rho,\theta)$ is a function. Consider also the operator

$$D_2 = \begin{pmatrix} \frac{1}{R(T)}(\partial_\rho - i\partial_\theta) & -e^{\rho+i\theta} i\bar{\partial}_a^* \\ e^{\rho-i\theta} i\bar{\partial}_a & \frac{1}{R(T)}(\partial_\rho + i\partial_\theta) \end{pmatrix},$$

acting on a spinor $(\tilde{\alpha}, \tilde{\beta})$. Here $a(\rho,\theta)$ is the image under the map $a : D \to \chi_0(T^2, Y)$ of Theorem 3.10, restricted to a subdomain of D homeomorphic to the disk. That is, for each (ρ,θ), $a(\rho,\theta)$ is a flat connection on T^2.

Lemma 4.2 *Without loss of generality, we can assume that the image of the map $a : D \to \chi_0(T^2, Y)$ avoids the lattice of bad points ϑ in $\chi_0(T^2, Y)$*

Proof. We have unique bad point ϑ in the character variety of T^2, [4], which corresponds to a lattice of bad points in the universal cover $\chi_0(T^2, Y)$.

First notice that, according to the proof of Lemma 3.9, the condition that the endpoint a'_∞ is the bad point ϑ is non-generic: by direct inspection of the characteristic polynomial of the system (29) this case corresponds to the vanishing of the coefficients C_{nlk} that give the faster decay $e^{\pm n\rho - C_{nlk}e^{\rho}}$. Thus, under generic choice of the perturbation, as will be discussed in the next section, we can assume that $a'_\infty \neq \vartheta$.

This implies that, under the maps a_V and a_ν of Theorem 3.10, the image of the side $\theta \in \{-\pi/2, \pi/2\}$ of the boundary of the half disk D^+ is a smooth path in $\partial_\infty \mathcal{M}_V$ or in $\partial_\infty \mathcal{M}_{\nu(K)}$, connecting the points $a^\pm$ which avoids the lattice of bad points. We can always replace the domain D^+ with some smaller domain containing the same component $\theta \in \{-\pi/2, \pi/2\}$ and $\rho \in (-\infty, 0]$ of the boundary, in such a way that the image of the rest of the domain also avoids the lattice of bad points. That still determines

uniquely the holomorphic function. Thus, we can assume for simplicity that the image of the subdomain of D under the map $a : D \to \chi_0(T^2, Y)$ also contains no bad point ϑ in $\chi_0(T^2, Y)$.
$\diamond$

We want to consider $D_1 \oplus D_2$ acting on elements $(\tilde{a}, \tilde{f}, \tilde{\alpha}, \tilde{\beta})$ that represent deformations of $(a, 0, 0, 0)$ in the L^2_1 completion of compactly supported 1-forms and spinors on D.

Lemma 4.3 *The operator $D_1 \oplus D_2$ acting on the L^2_1 completion of compactly supported 1-forms and spinors on D, has trivial Cokernel.*

Proof. We know [4] that the operator

$$\begin{pmatrix} 0 & -e^{\rho+i\theta}\bar{\partial}^*_a \\ e^{\rho-i\theta}\bar{\partial}_a & 0 \end{pmatrix}$$

is surjective if the flat connection $a(\rho, \theta)$ avoids the lattice of bad points ϑ in $\chi_0(T^2, Y)$. Then, by the same argument of Lemma 4.1, we obtain that for sufficiently large $T \geq T_0$, the operator D_2 is surjective.

Suppose given a pair (β, h) of a 1-form and a function, such that (β, h) is orthogonal to the range of D_1. Then, the pair (β, h) satisfies the equations

$$\partial_\rho h = R(T)^{-2} d^* \beta$$

$$\partial_\rho \beta = *\partial_\theta \beta + *d(e^{-2\rho} h).$$

The Fourier coefficients satisfy the ODE

$$\begin{aligned} h'_{nlk} &= (R(T))^{-2}(-ilv_{nlk} + inu_{nlk}) \\ u'_{nlk} &= inv_{nlk} - e^{-2\rho} ikh_{nlk} \\ v'_{nlk} &= -inu_{nlk} + e^{-2\rho} ilh_{nlk}. \end{aligned} \tag{48}$$

We can show that solutions of (48) diverge sufficiently fast at $\rho \to -\infty$, so that they fail to be in L^2. Since we are interested in the $\rho \to -\infty$ behavior, we can isolate in the system (48) a leading term given by the system

$$\begin{aligned} h'_{nlk} &= R(T)^{-2}(-ilv_{nlk} + inu_{nlk}) \\ u'_{nlk} &= -e^{-2\rho} ikh_{nlk} \\ v'_{nlk} &= e^{-2\rho} ilh_{nlk}, \end{aligned} \tag{49}$$

and treat the remaining terms as a perturbation.

Solutions of the system (49) are solutions of

$$h''_{nlk} = e^{-2\rho}(R(T))^{-2}(k^2 + l^2)h_{nlk}.$$

These are combinations of Bessel functions of type $I(0,z)$ and $K(0,z)$, with the variable $z = c_{nlk}\sqrt{(R(T))^{-2}e^{-2\rho}}$, for some constants $c_{nlk} > 0$. None of these functions is in $L^2(D)$.

◇

Consider the operator $\mathcal{T}_{a''(t)}$ of the form

$$\frac{1}{R(T)}\frac{\partial}{\partial t} + \begin{pmatrix} *d & -d & 0 \\ -d^* & 0 & 0 \\ 0 & 0 & \partial_{a''(t)} \end{pmatrix}, \tag{50}$$

where $\partial_{a''(t)}$ is the 3-dimensional Dirac operator, twisted with the flat connection

$$a''(w,s,t) = a''(t)(w,s).$$

Lemma 4.4 *If the path $a''(t)$ of connections in $\chi_0(T^2,Y)$ avoids the lattice of bad points, then the operator $\mathcal{T}_{a''(t)}$, acting on the L^2_1 completion of the space of compactly supported 1-forms and spinors on the domain*

$$\begin{aligned} \mathcal{S}_4(T) = (\nu(K) \times [-1,1]) \cup_{T^2\times\{s=0\}\times[-1,1]} \\ T^2 \times \{(s,t) \mid s \in [0, \tilde{\Upsilon}(t,T)], \quad t \in [-1,1]\} \end{aligned} \tag{51}$$

is surjective for sufficiently large $T \geq T_0$.

Proof. The operator

$$\frac{1}{R(T)}\frac{\partial}{\partial t} + \partial_{a''(t)}$$

is surjective, for sufficiently large $T \geq T_0$, by the argument of Lemma 4.1. Suppose given an element (β, h) orthogonal to the range of

$$\frac{1}{R(T)}\frac{\partial}{\partial t} + \begin{pmatrix} *d & -d \\ -d^* & 0 \end{pmatrix}.$$

In the large $T >> 0$ limit, the element (β, h) satisfies

$$d^*\beta = 0 \quad \text{and} \quad *d\beta - dh = 0.$$

This implies that h is harmonic, in the L^2_1 completion of the space of compactly supported function, hence it is vanishing by the maximum principle. On the other hand β satisfies $(d+d^*)\beta = 0$, which also implies $\beta \equiv 0$ in the L^2_1 completion of the space of compactly supported forms on (51).
$\diamond$

4.3 The solutions

In this section we prove that every approximate solution can be deformed to an actual solution of the Seiberg–Witten equations. We first prove that the linearization $\mathcal{D}_T$ at the approximate solution is surjective. Here the operator

$$\mathcal{D}_T = \mathcal{D}_{\mathcal{A},\Psi}$$

is the linearization of the 4-dimensional Seiberg–Witten equations (2) on $Y(r(T)) \times \mathbf{R}$, and $(\mathcal{A}, \Psi)$ is the approximate solution (43). Then, the gluing theorem follows as a fixed point argument, , as we are going to discuss, cf. similar arguments in [4], [5], [7], [11], [17]. Our case here is similar to the construction of [5], since we want to prove a gluing theorem where the underlying geometry also changes along with the gluing parameter T. In particular, we will obtain the actual solution close to the approximate solution by a fixed point arguments with constants depending on T.

In particular, we have seen from the convergence result of Theorem 3.10, that the non-uniform geometric limits are obtained from solutions on $Y(r) \times \mathbf{R}$ after suitable rescaling of the coordinates. Thus, in order to prove the gluing theorem, we will introduce a norm on the domain $Y(r(T)) \times \mathbf{R}$ which takes into account the necessary rescaling. That will allow us to compare the operator $\mathcal{D}_T$ with the operators $\mathcal{D}^T_i$ defined in the previous subsection on the corresponding regions $\mathcal{S}_i(T)$,

$$\mathcal{S}_1(T) = (V \times [-1,1]) \cup_{T^2 \times \{s=0\} \times [-1,1]}$$
$$T^2 \times \{(s,t) \mid s \in [0, \tilde{\Upsilon}(t,T)], \quad t \in [-1,1]\}$$

$\mathcal{S}_2(T) = D$, the unit disk. $\mathcal{S}_i(T) = \mathcal{R}_i(T)$ for $i = 3, 5$, and

$$\mathcal{S}_4(T) = (\nu(K) \times [-1,1]) \cup_{T^2 \times \{s=0\} \times [-1,1]}$$
$$T^2 \times \{(s,t) \mid s \in [0, \tilde{\Upsilon}(t,T)], \quad t \in [-1,1]\}.$$

We introduce a T-dependent Banach norm, which takes into account the effect of rescaling. We need to define rescaled L^2 and L^2_1 norms on the spaces

$\Lambda^1 \oplus \Gamma(W^+)$ and $\Lambda^{2+} \oplus \Gamma(W^-)$ on $\cup_i \mathcal{R}_i(T)$. We introduce the notation

$$\epsilon(T) = \frac{1}{R(T)}.$$

First notice the following rescaling of the volume elements and norms. Let us denote by

$$\varphi : \mathcal{S}_i(T) \to \mathcal{R}_i(T)$$

the diffeomorphism given by the rescaling. In particular, we analyze the two distinct cases

$$\varphi : \mathcal{S}_1(T) \to \mathcal{R}_1(T)$$
$$\varphi(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = (x = \tilde{x}, y = \tilde{y}, z = \tilde{z}, t = R(T)\tilde{t}),$$

and

$$\varphi : \mathcal{S}_2(T) \to \mathcal{R}_2(T)$$
$$\varphi(\tilde{u}, \tilde{v}, \tilde{s}, \tilde{t}) = (u = \tilde{u}, v = \tilde{v}, s = R(T)\tilde{s}, t = R(T)\tilde{t}).$$

We have

$$\int_{\mathcal{R}_i(T)} \omega = \int_{\mathcal{S}_i(T)} \varphi^*(\omega),$$

for any 4-form ω. Moreover, the Hodge $*$-operator satisfies

$$\tilde{*}_{\mathcal{S}_1(T)} e^i = e^j \wedge e^k \wedge e^t$$

$$\tilde{*}_{\mathcal{S}_1(T)} e^t = e^i \wedge e^j \wedge e^k$$

where the orthonormal basis of 1-form in the rescaled metric on $\mathcal{S}_1(T)$ is given by $e^i = dx^i$ and $e^t = R(T)d\tilde{t}$. Similarly, the Hodge $*$-operator satisfies the same relations for the orthonormal basis of 1-forms du, dv, $R(T)d\tilde{s}$, $R(T)d\tilde{t}$ on $\mathcal{S}_2(T)$.

Consider first the space $\Lambda^1 \oplus \Gamma(W^+)$. We write a 1-form as $a_V + f dt$ and a spinor ψ on $\mathcal{R}_1(T)$. We denote by $\tilde{a}_V$, $\tilde{f}$, $\tilde{\psi}$, etc. for the composite $f \circ \varphi$ with the change of coordinates. We obtain

$$\int_{\mathcal{R}_1(T)} a_V \wedge *_4 a_V = \int_{\mathcal{S}_1(T)} \varphi^*(a_V \wedge *_4 a_V) = \int_{\mathcal{S}_1(T)} \varphi^*(a_V) \wedge \tilde{*}_4 \varphi^*(a_V),$$

$$\int_{\mathcal{R}_1(T)} f\, dt \wedge *_4(f\, dt) = \int_{\mathcal{S}_1(T)} \varphi^*(f\, dt \wedge *_4(f\, dt)) =$$

$$\int_{\mathcal{S}_1(T)} (R(T)\tilde{f} d\tilde{t}) \wedge \tilde{*}_4(R(T)\tilde{f} d\tilde{t}) = \int_{\mathcal{S}_1(T)} \varphi^*(f\, dt) \wedge \tilde{*}_4 \varphi^*(f\, dt).$$

$$\int_{\mathcal{R}_1(T)} \|\psi\|^2 dt = R(T) \int_{\mathcal{S}_1(T)} \|\tilde{\psi}\|^2 d\tilde{t}.$$

Thus, over this region, the elements $\xi_T = (a_v, R(T) f\, dt, \epsilon(T)^{1/2}\psi)$ satisfy

$$\|\xi_T\|_{L^2(\mathcal{R}_1(T))} = \|\varphi^*(\xi)\|_{L^2(\mathcal{S}_1(T))},$$

for $\xi = (a_v, f\, dt, \psi)$.

Now consider the case of $\Lambda^{2+} \oplus \Gamma(W^-)$, again over the region $\mathcal{R}_1(T)$. We have 2-forms $\Omega = \omega_V + \eta_V \wedge dt$ and a spinors ψ satisfying

$$\int_{\mathcal{R}_1(T)} \omega_V \wedge *_4\omega_V = \int_{\mathcal{S}_1(T)} \varphi^*(\omega_V \wedge *_4\omega_V) = \int_{\mathcal{S}_1(T)} \varphi^*(\omega_V) \wedge \tilde{*}_4\varphi^*(\omega_V),$$

$$\int_{\mathcal{R}_1(T)} (\eta_V \wedge dt) \wedge *_4(\eta_V \wedge dt) = \int_{\mathcal{S}_1(T)} \varphi^*((\eta_V \wedge dt) \wedge *_4(\eta_V \wedge dt)) =$$

$$\int_{\mathcal{S}_1(T)} \varphi^*(\eta_V \wedge dt) \wedge \tilde{*}_4\varphi^*(\eta_V \wedge dt),$$

$$\int_{\mathcal{R}_1(T)} \|\psi\|^2 dt = R(T) \int_{\mathcal{S}_1(T)} \|\tilde{\psi}\|^2 d\tilde{t}.$$

Thus, in this case again, the elements $\xi_T = (\omega_V, \eta_V \wedge dt, \epsilon(T)^{1/2}\psi)$ satisfy

$$\|\xi_T\|_{L^2(\mathcal{R}_1(T))} = \|\varphi^*(\xi)\|_{L^2(\mathcal{S}_1(T))},$$

for $\xi = (\omega_V, \eta_V \wedge dt, \psi)$.

Now consider the regions $\mathcal{R}_2(T)$ and $\mathcal{S}_2(T)$. In the case of $\Lambda^1 \oplus \Gamma(W^+)$, we have 1-forms $a_{T^2} + f\, ds + h\, dt$ and spinors $\psi = (\alpha, \beta)$, satisfying

$$\int_{\mathcal{R}_2(T)} a_{T^2} \wedge *_4 a_{T^2} = \int_{\mathcal{S}_2(T)} \varphi^*(a_{T^2} \wedge *_4 a_{T^2}) = \int_{\mathcal{S}_2(T)} \varphi^*(a_{T^2}) \wedge \tilde{*}_4\varphi^*(a_{T^2}),$$

$$\int_{\mathcal{R}_2(T)} f\, ds \wedge *_4(f\, ds) = \int_{\mathcal{S}_2(T)} \varphi^*(f\, ds \wedge *_4(f\, ds)) =$$

$$\int_{\mathcal{S}_2(T)} \varphi^*(f\, ds) \wedge \tilde{*}_4\varphi^*(f\, ds),$$

$$\int_{\mathcal{R}_2(T)} h\, dt \wedge *_4(h\, dt) = \int_{\mathcal{S}_2(T)} \varphi^*(h\, dt \wedge *_4(h\, dt)) =$$

$$\int_{\mathcal{S}_2(T)} \varphi^*(h\, dt) \wedge \tilde{*}_4\varphi^*(h\, dt),$$

$$\int_{\mathcal{R}_2(T)} \|\psi\|^2 ds\, dt = R(T)^2 \int_{\mathcal{S}_2(T)} \|\tilde{\psi}\|^2 d\tilde{s}\, d\tilde{t}.$$

Thus, the elements $\xi_T = (a_{T^2}, f\, ds, h\, dt, \epsilon(T)\psi)$ satisfy

$$\|\xi_T\|_{L^2(\mathcal{R}_2(T))} = \|\varphi^*(\xi)\|_{L^2(\mathcal{S}_2(T))}$$

for $\xi = (a_{T^2}, f\, ds, h\, dt, \psi)$

Similarly, for $\Lambda^{2+} \oplus \Gamma(W^-)$ we have 2-forms $\omega_{T^2} + \eta \wedge ds + \gamma \wedge dt + h\, ds \wedge dt$, and spinors ψ satisfying

$$\int_{\mathcal{R}_2(T)} \omega_{T^2} \wedge *_4 \omega_{T^2} = \int_{\mathcal{S}_2(T)} \varphi^*(\omega_{T^2} \wedge *_4 \omega_{T^2}) = \int_{\mathcal{S}_2(T)} \varphi^*(\omega_{T^2}) \wedge \tilde{*}_4 \varphi^*(\omega_{T^2}),$$

$$\int_{\mathcal{R}_2(T)} \eta \wedge ds \wedge *_4(\eta \wedge ds) = \int_{\mathcal{S}_2(T)} \varphi^*(\eta \wedge ds) \wedge \tilde{*}_4 \varphi^*(\eta \wedge ds),$$

$$\int_{\mathcal{R}_2(T)} \gamma \wedge dt \wedge *_4(\gamma \wedge dt) = \int_{\mathcal{S}_2(T)} \varphi^*(\gamma \wedge dt) \wedge \tilde{*}_4 \varphi^*(\gamma \wedge dt),$$

$$\int_{\mathcal{R}_2(T)} h\, ds \wedge dt \wedge *_4(h\, ds \wedge dt) = \int_{\mathcal{S}_2(T)} \varphi^*(h\, ds \wedge dt) \wedge \tilde{*}_4 \varphi^*(h\, ds \wedge dt),$$

$$\int_{\mathcal{R}_2(T)} \|\psi\|^2 ds\, dt = R(T)^2 \int_{\mathcal{S}_2(T)} \|\tilde{\psi}\|^2 d\tilde{s}\, d\tilde{t}.$$

Thus, the elements $\xi_T = (\omega_{T^2}, \eta \wedge ds, \gamma \wedge dt, h, \epsilon(T)\psi)$ satisfies

$$\|\xi_T\|_{L^2(\mathcal{R}_2(T))} = \|\varphi^*(\xi)\|_{L^2(\mathcal{S}_2(T))}$$

for $\xi = (\omega_{T^2}, \eta \wedge ds, \gamma \wedge dt, h, \psi)$.

Now consider the following T-rescaled L^2-norm for an element ξ in $\Lambda^1 \oplus \Gamma(W^+)$ over $\cup_i \mathcal{R}_i(T)$:

$$\begin{aligned} \|\xi\|^2_{0,T} := \quad & \|(\eta_{T,1} + \eta_{T,4})(a_V, f\, dt, \epsilon(T)^{1/2}\psi)\|^2_{L^2(\mathcal{R}_1(T) \cup \mathcal{R}_4(T))} + \\ & \|(\eta_{T,3} + \eta_{T,5})(a_V, f\, dt, \psi)\|^2_{L^2(\mathcal{R}_3(T) \cup \mathcal{R}_5(T))} + \\ & \|\eta_{T,2}(a_{T^2}, f\, ds, h\, dt, \epsilon(T)(\alpha, \beta))\|^2_{L^2(\mathcal{R}_2(T))}. \end{aligned} \tag{52}$$

Similarly, we define the T-rescaled L^2-norm for an element ξ in $\Lambda^{2+} \oplus \Gamma(W^-)$ over $\cup_i \mathcal{R}_i(T)$:

$$\begin{aligned} \|\xi\|^2_{0,T} := \quad & \|(\eta_{T,1} + \eta_{T,4})(\omega_V, \eta_V \wedge dt, \epsilon(T)^{1/2}\psi)\|^2_{L^2(\mathcal{R}_1(T) \cup \mathcal{R}_4(T))} + \\ & \|(\eta_{T,3} + \eta_{T,5})(\omega_V, \eta_V \wedge dt, \psi)\|^2_{L^2(\mathcal{R}_3(T) \cup \mathcal{R}_5(T))} + \\ & \|\eta_{T,2}(\omega_{T^2}, \eta \wedge ds, \gamma \wedge dt, h, \epsilon(T)(\alpha, \beta))\|^2_{L^2(\mathcal{R}_2(T))}. \end{aligned} \tag{53}$$

With this choice of norms we have

$$\|\xi\|_{0,T\,\mathcal{R}_i(T)} = \|\varphi^*(\xi)\|_{L^2(\mathcal{S}_i(T))}.$$

Moreover, we define a T-rescaled L^2_1 norm on $\Lambda^1 \oplus \Gamma(W^+)$ on $\cup_i \mathcal{R}_i(T)$ by setting

$$\begin{aligned}
\|\xi\|^2_{1,T} := \; & \|(\eta_{T,1}+\eta_{T,4})(a_V, f\,dt, \epsilon(T)^{1/2}\psi)\|^2_{L^2(\mathcal{R}_1(T)\cup\mathcal{R}_4(T))}+ \\
& \textstyle\sum_{i=1}^3 \|(\eta_{T,1}+\eta_{T,4})\nabla_i(a_V, f\,dt, \epsilon(T)^{1/2}\psi)\|^2_{L^2(\mathcal{R}_1(T)\cup\mathcal{R}_4(T))}+ \\
& \|(\eta_{T,1}+\eta_{T,4})R(T)\tfrac{\partial}{\partial t}(a_V, f\,dt, \epsilon(T)^{1/2}\psi)\|^2_{L^2(\mathcal{R}_1(T)\cup\mathcal{R}_4(T))}+ \\
& \|(\eta_{T,3}+\eta_{T,5})(a_V, f\,dt, \psi)\|^2_{L^2(\mathcal{R}_3(T)\cup\mathcal{R}_5(T))}+ \\
& \textstyle\sum_{i=1}^3 \|(\eta_{T,3}+\eta_{T,5})(a_V, f\,dt, \psi)\|^2_{L^2(\mathcal{R}_3(T)\cup\mathcal{R}_5(T))}+ \\
& \|(\eta_{T,3}+\eta_{T,5})R(T)\tfrac{\partial}{\partial t}(a_V, f\,dt, \psi)\|^2_{L^2(\mathcal{R}_3(T)\cup\mathcal{R}_5(T))}+ \\
& \|\eta_{T,2}(\omega_{T^2}, \eta\wedge ds, \gamma\wedge dt, h, \epsilon(T)(\alpha,\beta))\|^2_{L^2(\mathcal{R}_2(T))}+ \\
& \textstyle\sum_{i=1}^3 \|\eta_{T,2}\nabla_i(\omega_{T^2}, \eta\wedge ds, \gamma\wedge dt, h, \epsilon(T)(\alpha,\beta))\|^2_{L^2(\mathcal{R}_2(T))}+ \\
& \|\eta_{T,2}R(T)\tfrac{\partial}{\partial t}(\omega_{T^2}, \eta\wedge ds, \gamma\wedge dt, h, \epsilon(T)(\alpha,\beta))\|^2_{L^2(\mathcal{R}_2(T))}
\end{aligned} \tag{54}$$

On $\Lambda^1 \oplus \Gamma(W^+)$ on the $\mathcal{S}_i(T)$ we consider the L^2_1 norm defined as

$$\begin{aligned}
\|\xi\|^2_{1,T\,\mathcal{S}_1(T)\cup\mathcal{S}_3(T)} := \; & \|\xi\|^2_{L^2(\mathcal{S}_1(T)\cup\mathcal{S}_3(T))}+ \\
& \textstyle\sum_{i=1}^3 \|\nabla_i\xi\|^2_{L^2(\mathcal{S}_1(T)\cup\mathcal{S}_3(T))} + \|\frac{\partial}{\partial \tilde{t}}\xi\|^2_{L^2(\mathcal{S}_1(T)\cup\mathcal{S}_3(T))}, \\
\|\xi\|^2_{1,T\,\mathcal{S}_3(T)\cup\mathcal{S}_5(T)} := \; & \|\xi\|^2_{L^2(\mathcal{S}_1(T)\cup\mathcal{S}_3(T))}+ \\
& \textstyle\sum_{i=1}^3 \|\nabla_i\xi\|^2_{L^2(\mathcal{S}_1(T)\cup\mathcal{S}_3(T))} + \|\frac{\partial}{\partial \tilde{t}}\xi\|^2_{L^2(\mathcal{S}_1(T)\cup\mathcal{S}_3(T))}, \\
\|\xi\|^2_{1,T\,\mathcal{S}_2(T)} := \; & \|\xi\|^2_{L^2(\mathcal{S}_2(T))}+ \\
& \|\partial_u\xi\|^2_{L^2(\mathcal{S}_2(T))} + \|\partial_v\xi\|^2_{L^2(\mathcal{S}_2(T))}+ \\
& \|\partial_{\tilde{t}}\xi\|^2_{L^2(\mathcal{S}_2(T))} + \|\partial_{\tilde{s}}\xi\|^2_{L^2(\mathcal{S}_2(T))}.
\end{aligned}$$

With this choice of norms, we have

$$\|\xi\|_{1,T\,\mathcal{R}_i(T)} = \|\varphi^*(\xi)\|_{L^2_1\mathcal{S}_i(T)}.$$

We can now prove the main Lemma for the gluing theorem: this can be regarded as an analogue, in our context, of Lemma 4.4 and 4.5 of [5]. We denote in the following by $\tilde{\mathcal{D}}_i^T$ the operators $\mathcal{D}_i^T$ introduced before, rescaled under $\mathcal{S}_i(T) \to \mathcal{R}_i(T)$.

Lemma 4.5 *Suppose given a sequence ξ_k of connections and spinors, which we can write as*

$$\xi_k = (A_k, f_k, \Psi_k)$$

over the domains $\cup_{i\neq 2}\mathcal{R}_i(T_k)$, and

$$\xi_k = (a_k, f_k, h_k, \alpha_k, \beta_k)$$

over $\mathcal{R}_2(T_k)$, for a sequence $T_k \to \infty$. Let $\xi_{k,i} = \eta_{T_k,i}\xi_k$. We assume that the elements $\xi_{k,i}$ are in the orthogonal complement of the Kernels of the operators $\tilde{\mathcal{D}}_i^{T_k}$,

$$\xi_{k,i} \in Ker(\tilde{\mathcal{D}}_i^{T_k})^{\perp},$$

with respect to the L^2_{1,T_k}-norms. Moreover, we assume that the operators $\tilde{\mathcal{D}}_i^{T_k}$ have trivial cokernels in the space $L^2_{0,T_k}(\mathcal{R}_i(T_k))$. Then, under this hypothesis, the convergence

$$\|\mathcal{D}_{T_k}\xi_k\|_{0,T_k} \to 0,$$

where $\mathcal{D}_{T_k}$ is the linearization at the approximate solutions, implies

$$\|\xi_k\|_{1,T_k} \to 0.$$

Proof. We first observe that, in the operator norm over each $\mathcal{R}_i(T_k)$, we have

$$\|\mathcal{D}_{T_k,i} - \tilde{\mathcal{D}}_i^{T_k}\| \to 0$$

as $T_k \to \infty$. We also assume that the cutoff functions satisfy

$$sup_{\mathcal{R}_i(T_k)}|\nabla \eta_{T_k,i}(s,t)| \leq q(T_k).$$

We assume that $q(T_k) \to 0$, where a bound on the rate of decay to zero of $q(T)$ as $T \to \infty$ will be specified in the proof of Lemma 4.8 below.

The argument then is similar to [17] §2.5. Suppose given a sequence ξ_k, and parameters $T_k \to \infty$, satisfying

$$\|\mathcal{D}_{T_k}\xi_k\|_{0,T_k} \to 0.$$

Moreover, assume that, for all k, we have

$$\|\xi_k\|_{1,T_k} = 1.$$

We have an estimate

$$\|\tilde{\mathcal{D}}_i^{T_k}\xi_{k,i}\|_{0,T_k} \leq Cq(T_k)\|\xi_k\|_{1,T_k} + \|\eta_{T_k,i}\tilde{\mathcal{D}}_i^{T_k}\xi_k\|_{0,T_k}$$

$$\leq Cq(T_k) + \|\mathcal{D}_{T_k} - \tilde{\mathcal{D}}_i^{T_k}\| \, \|\xi_k\|_{1,T_k} + \|\mathcal{D}_{T_k}\xi_k\|_{0,T_k}.$$

The terms in the right hand side decay to zero, thus we obtain that the elements $\xi_{k,i}$ are in the span of the low modes of $\tilde{\mathcal{D}}_i^{T_k}$. Let us denote by $\mathcal{V}_{k,i}$ the space of eigenvectors of $(\tilde{\mathcal{D}}_i^{T_k})^*\tilde{\mathcal{D}}_i^{T_k}$ acting on completion of the space of compactly supported 1-forms and spinors over $\mathcal{R}_i(T_k)$ in the L^2_{1,T_k} norm, with eigenvalues $\lambda_{T_k} \to 0$ as $T_k \to \infty$. This is the space of low modes of $\tilde{\mathcal{D}}_i^{T_k}$. We have obtained $\xi_{k,i} \in \mathcal{V}_{k,i}$, from the previous estimate. We use the notation $\mathcal{V}^{\#}_{k,i}$ for the span of low modes of the L^2_{1,T_k} adjoint.

Now we have

$$\dim \mathcal{V}_{k,i} \geq \dim Ker(\tilde{\mathcal{D}}_i^{T_k}).$$

Moreover, for $T_k \to \infty$, we have

$$\dim \mathcal{V}_{k,i} - \dim \mathcal{V}^{\#}_{k,i} = \dim Ker(\tilde{\mathcal{D}}_i^{T_k}) - \dim Coker(\tilde{\mathcal{D}}_i^{T_k}).$$

Under the assumption that $Coker(\tilde{\mathcal{D}}_i^{T_k}) = 0$, we obtain the reverse estimate

$$\dim \mathcal{V}_{k,i} \leq \dim Ker(\tilde{\mathcal{D}}_i^{T_k}).$$

Thus, we can identify $\mathcal{V}_{k,i} \cong Ker(\tilde{\mathcal{D}}_i^{T_k})$. Thus, under these hypotheses we would have $\xi_{k,i} \in Ker(\tilde{\mathcal{D}}_i^{T_k})$. This contradicts the initial assumption $\xi_{k,i} \in Ker(\tilde{\mathcal{D}}_i^{T_k})^{\perp}$.
$\diamond$

The assumption $Coker(\tilde{\mathcal{D}}_i^{T_k}) = 0$ has been discussed in the previous subsection, and it follows from the results of the last Section, about perturbed operators.

We can derive from Lemma 4.5 the following Corollary.

Corollary 4.6 *There are constants T_0 and $c_1 > 0$, independent of T, such that we have an estimate*

$$\|\xi\|_{1,T} \leq c_1\|\mathcal{D}_T\xi\|_{0,T},$$

for all ξ satisfying

$$\eta_{T,i}\xi \in Ker(\tilde{\mathcal{D}}_i^{T})^{\perp}.$$

Moreover, we have a similar estimate

$$\|\xi\|_{0,T} \leq C\|\mathcal{D}_T^*\xi\|_{1,T},$$

for a constant $C > 0$ independent of $T \geq T_0$, under the assumption that the operators $\tilde{\mathcal{D}}_i^T$ have trivial Cokernels. Here $\mathcal{D}_T^*$ is the adjoint with respect to the T-dependent norm.

Proof. By the choice of the T-dependent norms, c_1 is independent of T. The rest of the statement follows from Lemma 4.5. The second estimate follows by a similar argument.
$\diamond$

Now we can state the second main result of this part of the work, namely the gluing theorem.

Theorem 4.7 *Given any approximate solution $\Xi_0 = (\mathcal{A}, \Psi)$, constructed as in (43), for all sufficiently large $T \geq T_0$, there exists a solution Ξ of the equations (2) on $Y(r(T)) \times \mathbf{R}$, satisfying $\|\Xi - \Xi_0\|_{1,T} \leq c\epsilon(T)^{1/2}$, for a constant $c > 0$ independent of $T \geq T_0$.*

Proof. We divide the proof in several steps. On the domain $Y(r(T)) \times \mathbf{R}$, consider the map

$$\sigma_T(\mathcal{A}, \Psi) = \begin{cases} F_{\mathcal{A}}^+ - \tau(\Psi, \Psi) \\ D_{\mathcal{A}}\Psi \end{cases}$$

Lemma 4.8 *Let Ξ_0 be the approximate solution as in (43). The estimate*

$$\|\sigma_T(\Xi_0)\|_{0,T} \leq c_0\epsilon(T)^{1/2}$$

is satisfied, with $c_0 > 0$ independent of $T \geq T_0$.

Proof. Consider first the region $\mathcal{R}_2(T) \cup \mathcal{R}_3(T) \cup \mathcal{R}_5(T)$. Over this region we have $\sigma_T(\Xi_0)(w, s, t) = 0$ except on the overlap between the supports of the functions $\eta_{T,i}$ used in (43). On $\mathcal{R}_i(T) \cap supp(\eta_{T,j})$ in this region, we can estimate

$$\|\sigma_T(\Xi_0)\|_{0,T\,(\mathcal{R}_i(T)\cap supp(\eta_{T,j}))} \leq \sup_{\mathcal{R}_j(T)} |\nabla\eta_{T,j}|\ \|(\mathcal{A}, \Psi)\|_{0,T\,(\mathcal{R}_i(T))}.$$

Recall that we have assumed $\sup_{\mathcal{R}_j(T)} |\nabla\eta_{T,j}| = q(T)$. It is sufficient to choose the cutoff functions as in Lemma 4.5, with the hypothesis that $q(T) \sim \epsilon(T)^{1/2}$, and we get the desired estimate. Then consider the case of the region $\mathcal{R}_1(T)$. Here the condition $\sigma_T(\Xi_0)(w,s,t) = 0$ is not satisfied, but we can estimate the error term by

$$\|\sigma_T(\Xi_0)\|^2_{0,T\,(\mathcal{R}_1(T))} \leq \int_{I(T)} (\|\frac{\partial}{\partial t}A(t)\|^2_{L^2(V)} + \epsilon(T)\|\frac{\partial}{\partial t}\psi(t)\|^2_{L^2(V)})dt$$

$$\leq \epsilon(T)\int_{-1}^{1} \|\frac{\partial}{\partial \tilde{t}}(A(\tilde{t}),\psi(\tilde{t}))\|^2_{L^2(V)} d\tilde{t} \leq c^2\epsilon(T).$$

The remaining case of the region $\mathcal{R}_4(T)$ is analogous, with a similar resulting estimate with the constant

$$c \geq \left(\int_{-1}^{1} \|\partial_{\tilde{t}} a''(\tilde{t})\|^2_{L^2(T^2)} d\tilde{t}\right)^{1/2},$$

where $a''(\tilde{t})$ is the path in $\mathcal{M}_{\nu(K)}$ obtained in the geometric limits.
◇

Now define elements as follows

$$\Xi_1 = \Xi_0 + \xi_0 \quad \xi_0 = \mathcal{D}_T^*\eta_0 \quad \mathcal{D}_T\mathcal{D}_T^*\eta_0 = -\sigma_T(\Xi_0).$$

The elements are well defined because of the following.

Lemma 4.9 *The equation* $\mathcal{D}_T\mathcal{D}_T^*\eta_0 = -\sigma_T(\Xi_0)$ *admits a unique solution* η_0 *with* $\|\eta\|_{0,T} \leq C$. *Moreover, we have*

$$\|\xi_0\|_{1,T} \leq c_1\|\sigma_T(\Xi_0)\|_{0,T} \leq c_0c_1\epsilon(T)^{1/2}.$$

with $c_1 > 0$ *independent of* $T \geq T_0$.

Proof. The operator $\mathcal{D}_T\mathcal{D}_T^*$ is invertible, under the assumptions of Lemma 4.5. The estimate then follows from the previous Lemma 4.5, Corollary 4.6 and Lemma 4.8.
◇

The linearization of the map σ_T at $\Xi_0 = (\mathcal{A}, \Psi)$ is given by

$$d\sigma_{T,\Xi_0}(\eta,\Phi) = \begin{cases} d^+\eta - \frac{1}{2}Im(\Psi,\Phi) \\ D_{\mathcal{A}}\Phi + \eta.\Psi. \end{cases}$$

By construction we have

$$d\sigma_{T,\Xi_0}(\xi) = \mathcal{D}_T(\xi).$$

Consider the non-linear part of the map σ_T, that is the expression

$$\mathcal{N}\sigma_T(\xi) = \sigma_T(\Xi_0 + \xi) - \sigma_T(\Xi_0) - d\sigma_{T,\Xi_0}(\xi).$$

We have

$$\sigma_T(\Xi_1) = \sigma_T(\Xi_0 + \xi_0) - \sigma_T(\Xi_0) - d\sigma_{T,\Xi_0}(\xi_0) = \mathcal{N}\sigma_T(\xi_0).$$

Lemma 4.10 *We have an estimate*

$$\|\sigma_T(\Xi_1)\|_{0,T} \leq c_2\|\xi_0\|_{0,T}^2 \leq c_0c_1c_2\epsilon(T)^{1/2}\|\xi_0\|_{0,T},$$

Proof. The non-linear part is given by

$$\mathcal{N}\sigma_T(\Omega,\Phi) = \begin{cases} \tau(\Phi,\Phi) \\ \Omega.\Phi, \end{cases}$$

for (Ω,Φ) in $\Lambda^1 \oplus \Gamma(W^+)$. Thus, in the L^2-norms we have

$$\|\mathcal{N}\sigma_T(\xi_0)\|_{L^2} \leq c_2\|\xi_0\|_{L^2}^2.$$

This is sufficient to obtain the desired estimate on the regions $\mathcal{R}_3(T) \cup \mathcal{R}_5(T)$. If we write $\Omega = \omega + f dt$, we can write the above estimate more precisely as

$$\|\omega.\Phi\|_{L^2} + \|f dt.\Phi\|_{L^2} \leq c(\|\omega\|_{L^2}\|\Phi\|_{L^2} + \|f\|_{L^2}\|\Phi\|_{L^2}),$$

and similarly, for the term $\tau(\Phi,\Phi)$, we have $\|\tau(\Phi,\Phi)\|_{L^2} \leq c\|\Phi\|_{L^2}^2$. We have

$$\|\tau(\Phi,\Phi)\|_{0,T} = \|\varphi^*(\tau(\Phi,\Phi))\|_{L^2(\mathcal{S}_i(T))},$$

and

$$\|\Omega.\Phi\|_{0,T} = \|\varphi^*(\Omega.\Phi)\|_{L^2(\mathcal{S}_i(T))},$$

by our choice of weights on $\Lambda^{2+} \oplus \Gamma(W^-)$. Moreover, we have

$$\varphi^*(\tau(\Phi,\Phi)) = \tilde{\tau}(\tilde{\Phi},\tilde{\Phi}) \quad \text{and} \quad \varphi^*(\Omega.\Phi) = \varphi^*(\Omega)\tilde{.}\tilde{\Phi}.$$

Thus, we obtain

$$\|\tau(\Phi,\Phi)\|_{0,T} \leq c_2 \|\tilde{\Phi}\|^2_{L^2(\mathcal{S}_i(T))}$$
$$\leq c_2 \|\Phi\|^2_{0,T\,(\mathcal{R}_i(T))},$$

where the last inequality comes form the rescaling of the norm on $\Lambda^1 \oplus \Gamma(W^+)$. Similarly, we have

$$\|\Omega.\Phi\|_{0,T} \leq c_2 \|\varphi^*(\Omega)\|_{L^2(\mathcal{S}_i(T))} \cdot \|\tilde{\Phi}\|_{L^2(\mathcal{S}_i(T))}$$
$$\leq c_2 \|\Omega\|_{0,T\,(\mathcal{R}_i(T))} \cdot \|\Phi\|_{0,T\,(\mathcal{R}_i(T))}.$$

Thus, we have obtained an estimate

$$\|\sigma_T(\Xi_1)\|_{0,T} \leq c_2 \|\xi_0\|^2_{0,T}.$$

We also have an estimate

$$\|\xi_0\|_{1,T} \leq c_0 c_1 \epsilon(T)^{1/2},$$

from the previous Lemmas, which gives

$$\|\sigma_T(\Xi_1)\|_{0,T} \leq c_0 c_1 c_2 \epsilon(T)^{1/2} \|\xi_0\|_{0,T}.$$

This proves the claim.
$\diamond$

We then proceed inductively, as in [5]. We set

$$\Xi_{\nu+1} = \Xi_\nu + \xi_\nu \quad \xi_\nu = \mathcal{D}_T^* \eta_\nu \quad \mathcal{D}_T \mathcal{D}_T^* \eta_\nu = -\sigma_T(\Xi_\nu).$$

Lemma 4.11 *The following estimates hold:*

$$\|\xi_\nu\|_{1,T} \leq C(\nu)\epsilon(T)^{1/2}$$
$$\|\sigma_T(\Xi_{\nu+1})\|_{0,T} \leq \hat{C}(\nu) \|\xi_\nu\|_{0,T}.$$

Moreover, we can always assume that an estimate

$$\|\sigma_T(\Xi_1)\|_{0,T} \leq c_0 c_1 c_2 \|\xi_0\|_{0,T},$$

with $c = c_0 c_1^2 c_2 < 1$, is satisfied. This implies that the iteration process converges to a solution Ξ of the SW equations on $Y(r(T)) \times \mathbf{R}$, satisfying $\|\Xi - \Xi_0\|_{1,T} \leq C\epsilon(T)^{1/2}$.

Proof. First of all notice that we can always include a factor $\epsilon(T_0)^{1/2}$ in the constant $c_0c_1c_2$ in the estimate of Lemma 4.10, for sufficiently large T_0, so that the condition $c_0c_1^2c_2 < 1$ is satisfied. Then define recursively $C(0) = c_0c_1$ and $\hat{C}(0) = c_0c_1c_2$ and $C(\nu+1) = c_0c_1^2c_2C(\nu)$ and $\hat{C}(\nu+1) = c_0c_1^2c_2\hat{C}(\nu)$. The result then follows inductively using the estimates

$$\|\xi_\nu\|_{1,T} \leq c_1\|\sigma_T(\Xi_\nu)\|_{0,T},$$

as in Lemma 4.5 and

$$\|\sigma_T(\Xi_{\nu+1})\|_{0,T} \leq c_2\|\xi_\nu\|_{1,T}^2,$$

as in Lemma 4.10. The estimate for the solution Ξ is obtained from

$$\|\Xi_\nu - \Xi_0\|_{1,T} \leq \sum_{j=0}^{\nu-1} \|\xi_j\|_{1,T}.$$

$\diamond$

This completes the proof of the gluing Theorem 4.7.

$\diamond$

5 Perturbation

The gluing theorem stated in the previous section relies on the assumption that the linearization $L_{A(t),\psi(t)}$ is surjective, for all the elements $[A(t),\psi(t)]$ in $\mathcal{M}_V$, as in Lemma 4.1. We know that, in general, these conditions are satisfied only after introducing a suitable perturbation.

In [4] we defined a class of perturbations $\mathcal{P}$ of the Chern–Simons–Dirac functional, such that, for a generic element $P = (U,V) \in \mathcal{P}$, the operators $L_{A,\psi}$ and $\mathcal{D}_{\mathcal{A},\Psi}$ are both surjective. These operators, in the perturbed case, are the linearizations of the corresponding perturbed critical point equation

$$\begin{cases} *F_A = \sigma(\psi,\psi) + \sum_{j=1}^N \frac{\partial U}{\partial \tau_j}\mu_j \\ \partial_A(\psi) + \sum_{j=1}^K \frac{\partial V}{\partial \zeta_j}\nu_j.\psi = 0. \end{cases}$$

and of the perturbed flow line equation

$$\begin{cases} \frac{\partial A}{\partial t} & = & -*F_A + \sigma(\psi,\psi) + \sum_{j=1}^N \frac{\partial U}{\partial \tau_j}\mu_j \\ \frac{\partial \psi}{\partial t} & = & -\partial_A\psi - \sum_{j=1}^K \frac{\partial V}{\partial \zeta_j}\nu_j.\psi, \end{cases}$$

respectively.

For simplicity, we use the notation

$$P_U = \sum_{j=1}^{N} \frac{\partial U}{\partial \tau_j} \mu_j,$$

$$P_V = \sum_{j=1}^{K} \frac{\partial V}{\partial \zeta_j} \nu_j.$$

Thus, we have

$$\begin{cases} *F_A = \sigma(\psi, \psi) + P_U \\ \partial_A(\psi) + P_V.\psi \end{cases} \tag{55}$$

and

$$\begin{cases} \frac{\partial A}{\partial t} = - * F_A + \sigma(\psi, \psi) + P_U \\ \frac{\partial \psi}{\partial t} = -\partial_A \psi - P_V.\psi. \end{cases} \tag{56}$$

On the manifold $Y(r)$ with a long cylinder, and on the non-compact manifold V with an infinite cylinder, we refine the definition of the class of perturbations, as in [4], by including the requirement that the perturbation is exponentially small in the region $T^2 \times [-r, r] \cup \nu(K)$, and on the end $T^2 \times [0, \infty)$. We denote the corresponding class with $\mathcal{P}_\delta$, where δ is the rate of decay, depending on the smallest absolute value of the non-trivial eigenvalues of the asymptotic operator Q_{a_∞} on T^2, cf. [4].

We need to check that the main results in the previous sections can be extended to the case where the perturbation $P \in \mathcal{P}_\delta$ is introduced.

5.1 Estimates in the perturbed case

In order to extend the results of Section 2 to the case of the equations (56), we need the analogue of the uniform estimate on the energy, and then of Lemma 2.3 and Lemma 2.5. The results of Section 2 then extend to this case without significant changes.

Lemma 5.1 *Let $(\mathcal{A}_r, \Psi_r)$ be a finite energy solution of (56) on $Y(r) \times \mathbf{R}$. Then, for $r \geq r_0$, and for any interval $[t_0, t_1]$ of length $\ell = t_1 - t_0$, the estimates of Lemma 2.3 and Lemma 2.5 hold, with*

$$s_0 = \max_{Y(r_0)} \{-s(x) + C(P), 0\},$$

where $C(P)$ is a positive constant depending only on the perturbation $P \in \mathcal{P}_\delta$.

Proof. Recall that the class of perturbations $\mathcal{P}_\delta$ is defined [4] by considering complete L^2 bases $\{\nu_i(r)\}_{i=1}^\infty$ and $\{\mu_j(r)\}_{j=1}^\infty$, satisfying

$$\sup_{T^2\times[-r,r]} |\nu_i(r)| \leq \sup_{T^2\times[-r_0,r_0]} |\nu_i(r_0)|$$

$$\sup_{T^2\times[-r,r]} |\mu_j(r)| \leq \sup_{T^2\times[-r_0,r_0]} |\mu_j(r_0)|,$$

and rescaling them with a function

$$f_r(s) = e^{-\delta(s+r)},$$

for $-r+\epsilon \leq s \leq r-\epsilon$, on the cylinder $T^2 \times [-r,r]$, with the weight δ satisfying

$$\delta \geq \frac{1}{2} min\{\lambda_{a_\infty} | a_\infty \in \chi(T^2)\backslash U_\vartheta\}.$$

The elements $\{f_r\nu_i(r)\}_{i=1}^\infty$ and $\{f_r\mu_j(r)\}_{j=1}^\infty$ still give complete bases, which we use to define the perturbation P_r in the class $\mathcal{P}_\delta$ on $Y(r)$.

Thus, the pointwise estimate obtained from the Weitzenböck formula gives

$$\begin{aligned} 0 \geq \tfrac{s}{2}|\psi|^2 - \langle *F_A.\psi,\psi\rangle + \langle \tfrac{\partial U}{\partial \zeta_i} f_r\nu_i(r).\psi, \tfrac{\partial U}{\partial \zeta_j} f_r\nu_i(r).\psi\rangle \\ -\langle \tfrac{\partial U}{\partial \zeta_i} d(f_r\nu_i(r)).\psi,\psi\rangle \geq \tfrac{s}{2}|\psi|^2 + \tfrac{1}{2}|\psi|^4 - C(P_r)|\psi|^2. \end{aligned} \tag{57}$$

The constant satisfies $C(P_r) \leq C(P_0)$.

We can prove a uniform bound on the energy. In fact, the energy is now defined as the variation of the perturbed Chern–Simons–Dirac functional

$$CSD_P(A,\psi) = CSD(A,\psi) + U(\tau_1(A,\psi),\cdots,\tau_N(A,\psi))$$

$$+V(\zeta_1(A,\psi),\cdots,\zeta_K(A,\psi)),$$

as in [4].

If $(\mathcal{A}_r,\Psi_r)$ is a family of finite energy solutions on $Y(r)\times\mathbf{R}$, with asymptotic values $(A_r(\pm\infty),\psi_r(\pm\infty))$ as $t\to\pm\infty$ satisfying the perturbed equations (55), the energy

$$\mathcal{E}_r = CSD_P(A_r(-\infty),\psi_r(-\infty)) - CSD_P(A_r(+\infty),\psi_r(+\infty))$$

is given by

$$\frac{1}{2}\int_{Y(r)} F_{\mathcal{A}_r}\wedge F_{\mathcal{A}_r} + \int_{Y(r)} \langle \psi_r(-\infty), f_r \sum \frac{\partial V_r}{\partial \zeta_j}\nu_j(r).\psi_r(-\infty)\rangle$$

$$- \int_{Y(r)} \langle \psi_r(+\infty), f_r \sum \frac{\partial V_r}{\partial \zeta_j}\nu_j(r).\psi_r(+\infty)\rangle$$

$$+U(\tau_1(A_r(-\infty),\psi_r(-\infty)),\cdots,\tau_N(A_r(-\infty),\psi_r(-\infty)))$$
$$-U(\tau_1(A_r(+\infty),\psi_r(+\infty)),\cdots,\tau_N(A_r(+\infty),\psi_r(+\infty)))$$
$$+V(\zeta_1(A_r(-\infty),\psi_r(-\infty)),\cdots,\zeta_K(A_r(-\infty),\psi_r(-\infty)))$$
$$-V(\zeta_1(A_r(+\infty),\psi_r(+\infty)),\cdots,\zeta_K(A_r(+\infty),\psi_r(+\infty))).$$

We know the first term is uniformly bounded, from the analysis of the unperturbed case. Using the notation

$$P_V(r) = f_r \sum \frac{\partial V_r}{\partial \zeta_j}\nu_j(r),$$

the second and third term can be estimated by writing

$$\begin{array}{c} |\int_{Y(r)}\langle\psi_r, P_V(r).\psi_r\rangle dv| \leq \int_{Y(r_0)} |\langle\psi, P_V(r).\psi\rangle| dv \\ + \int_{T^2\times([-r,-r_0]\cup[r_0,r])} |\langle\psi_r, P_V(r).\psi_r\rangle| dv. \end{array} \tag{58}$$

The first term in the right hand side is bounded by

$$\|\sigma(\psi_r,\psi_r)\|_{L^2(Y_{(r_0)})}\cdot\|P_V(r)\|_{L^2(Y(r_0))} \leq C\ C(P_0)^2\|P_V(r_0)\|_{L^2(Y(r_0))},$$

where the terms on the right are obtained using the uniform pointwise estimate (57). The second term in the right hand side of (58) can be estimated similarly by

$$C(r-r_0)^2\ C(P_0)^2 e^{-\delta(r-r_0)}\sup|\nu_i(r_0)|,$$

where the factor $(r-r_0)^2$ comes from factoring out the volume of the cylinder in the estimate of both $\sigma(\psi_r,\psi_r)$ and $P_V(r)$. Again we have used the pointwise estimate (57).

The remaining terms in the variation of CSD_P can be bounded as follows. We have

$$|U(\tau_j(A_r(-\infty),\psi_r(-\infty))) - U(\tau_j(A_r(+\infty),\psi_r(+\infty)))|$$

$$\leq |\tau_j(A_r(-\infty),\psi_r(-\infty)) - \tau_j(A_r(+\infty),\psi_r(+\infty))|\cdot$$

$$\|\frac{\partial U(r)}{\partial \tau_j}\mu_j(r)\|_{L^2(Y_r)}.$$

The last term is bounded uniformly, by our assumptions on the perturbation. We have

$$\tau_j(A_r(-\infty), \psi_r(-\infty)) - \tau_j(A_r(+\infty), \psi_r(+\infty))$$
$$= \int_{Y(r)} f_r(A_r(-\infty) - A_r(+\infty)) \wedge *\mu_j(r).$$

As in Lemma 4.6 of [4], up to changing the connections within the same gauge class, we have an estimate

$$\|A_r(-\infty) - A_r(+\infty)\|_{L^2(Y(r))} \leq C(P_0)\ Cr,$$

where the right hand side grows linearly in r like the volume $Vol(Y(r))$. This estimate follows from the uniform pointwise bound on the spinor, and the corresponding bound on the curvature. Thus, for all $\epsilon > 0$ we can choose $r \geq r_0$ large enough so that we have a bound

$$|\int_{Y(r)} f_r(A_r(-\infty) - A_r(+\infty)) \wedge *\mu_j(r)|$$
$$\leq CC(P_0)Vol(Y(r_0)) + \epsilon.$$

The estimate of the remaining term in the variation of CSD_P is analogous. Combining these estimates, we get a uniform bound on the energy for large enough $r \geq r_0$, hence the estimates on the finite energy solutions $(\mathcal{A}_r, \Psi_r)$ follow as in the Lemmata 2.3 and 2.5.
◇

5.2 Asymptotics

In order to adapt the results of Section 3 we need to study the asymptotics of finite energy solutions of the equations (56) on the manifold $V \times \mathbf{R}$, with the non-compact end $T^2 \times [0, \infty) \times \mathbf{R}$.

As in Section 3.2, we consider the ODE associated to the perturbed system (56) on $V \times \mathbf{R}$. We write (56) as

$$\partial_t a - dh + *(\partial_s a - df) = *i(\bar{\alpha}\beta + \alpha\bar{\beta}) + \sum_{j=1}^{N} \frac{\partial U}{\partial \tau_j} * q_j$$

$$\partial_t f - \partial_s h + *F_a = \frac{i}{2}(|\alpha|^2 - |\beta|^2) + \sum_{j=1}^{N} \frac{\partial U}{\partial \tau_j} p_j$$

$$\partial_t \alpha + h\alpha + i\partial_s \alpha + if\alpha + \bar{\partial}_a^* \beta - i\sum_{j=1}^{N} \frac{\partial V}{\partial \zeta_j}((-\nu_j^1 + i\nu_j^2)\beta + i\nu_j^0 \alpha) = 0$$

$$\partial_t \beta + h\beta - i\partial_s \beta - if\beta + \bar{\partial}_a \alpha + i\sum_{j=1}^{N} \frac{\partial V}{\partial \zeta_j}((\nu_j^1 + i\nu_j^2)\alpha - i\nu_j^0 \beta) = 0,$$

where we use the notation introduced in [4],

$$*_3 P_U = \sum_{j=1}^{N} \frac{\partial U}{\partial \tau_j}(p_j + q_j \wedge ds)$$

and

$$P_V = \sum_{i=1}^{K} \frac{\partial V}{\partial \zeta_i}(\nu_i^1 dx + \nu_i^2 dy + \nu_i^0 ds),$$

with $*\mu_j = p_j + q_j \wedge ds$, and $\nu_i = \nu_i^1 dx + \nu_i^2 dy + \nu_i^0 ds$. In radial gauge we obtain

$$\begin{aligned} \partial_\rho f &= e^{2\rho} * (F_a + \tfrac{i}{2}(|\alpha|^2 - |\beta|^2)\omega + P_0) \\ \partial_\rho a &= *(\partial_\theta a - df + i(\bar{\alpha}\beta + \alpha\bar{\beta}) + P_1) \\ \partial_\rho \alpha &= i(\partial_\theta \alpha + f\alpha + e^{\rho + i\theta}(\bar{\partial}_a^* \beta + P_{11}\alpha + P_{12}\beta)) \\ \partial_\rho \beta &= -i(\partial_\theta \beta + f\beta + e^{\rho - i\theta}(\bar{\partial}_a \alpha + P_{21}\alpha + P_{22}\beta), \end{aligned} \tag{59}$$

where we use the notation

$$\begin{aligned} P_0(s) &= \textstyle\sum_{j=1}^{N} \frac{\partial U}{\partial \tau_j} p_j(s) \\ P_1^{0,1}(s) &= \textstyle\sum_{j=1}^{N} \frac{\partial U}{\partial \tau_j} q_j^{0,1}(s) \\ P(s) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} &= \textstyle\sum_{j=1}^{N} \frac{\partial V}{\partial \zeta_j} \begin{pmatrix} i\nu_j^0 & -\nu_j^1 + i\nu_j^2 \\ \nu_j^1 + i\nu_j^2 & -i\nu_j^0 \end{pmatrix} \end{aligned} \tag{60}$$

as in [4]

The perturbation terms $P_0(s)$, $P_1(s)$ and $P(s)$ depend on (a, f, α, β) through the variables

$$\tau_j = \int (A - A_0) \wedge *\mu_j$$

and

$$\zeta_j = \int \langle \nu_j.\psi, \psi \rangle dv$$

(cf. [4]). By the condition that the perturbation is chosen in the class $\mathcal{P}_\delta$ we obtain the estimates

$$\begin{aligned} \|P_0(\rho)\|_{L^2(T^2\times\{\theta\})} \leq & \quad C(U,V)\|(a,f,h)\|_{L^2(T^2\times\{\theta\})} \\ & \cdot \textstyle\int_{-\infty}^{\infty} \exp(-\delta e^\rho \cos\theta) d\rho \end{aligned}$$

$$\begin{aligned} \|P_1(\rho)\|_{L^2(T^2\times\{\theta\})} \leq & \quad C(U,V)\|(a,f,h)\|_{L^2(T^2\times\{\theta\})} \\ & \cdot \textstyle\int_{-\infty}^{\infty} \exp(-\delta e^\rho \cos\theta) d\rho \end{aligned}$$

$$\begin{aligned} \|P(\rho)\cdot(\alpha,\beta)\|_{L^2(T^2\times\{\theta\})} \leq & \quad C(U,V)\|(\alpha,\beta)\|_{L^2(T^2\times\{\theta\})} \\ & \cdot \textstyle\int_{-\infty}^{\infty} \exp(-\delta e^\rho \cos\theta) d\rho \end{aligned} \tag{61}$$

after the change of coordinates $s + it = e^{\rho+i\theta}$.

In order to study the asymptotics of the system (59) we proceed as in Section 3. We consider the linear system given by the uncoupled systems (10) and (26) for the linear ASD and Dirac equations. We add the perturbation terms coming from the terms (60) in polar coordinates (ρ,θ). We first study the asymptotic of this perturbed system and show that the finite energy solutions are still exponentially decaying in the radial direction, as the solutions of the original systems (10) and (26). We then proceed as in the remaining of Section 3, to prove that the full system (59) has finite energy solutions that are exponentially decaying in the radial direction.

The system given by (10) and (26) together with the perturbation terms (60) is also uncoupled in the curvature and Dirac part. We discuss the behavior of the Dirac part: the curvature part is completely analogous.

Following [9], §X, Section 8, we consider all the systems of ODE's of the form (26), for all (n,l,k), with the additional terms coming from the Fourier transform of the term $P(\rho)\cdot(\alpha,\beta)$. These may no longer be uncoupled as the original (26). We can write this perturbed system in the form

$$\begin{aligned} X' &= M^- X + P^-(X,Y) \\ Y' &= M^+ Y + P^+(X,Y), \end{aligned} \tag{62}$$

where the variables X correspond to the eigenvectors of the systems (26) with eigenvalues $\lambda = \pm\lambda_i^{nlk}$, as in (27) with $Re(\lambda) < 0$, and Y corresponds to the eigenvectors with $Re(\lambda) > 0$.

Let λ_0 be the smallest absolute value of the eigenvalues with $Re(\lambda) < 0$. Consider a fixed μ with $-\lambda_0 < \mu < 0$. According to Theorem 8.1 and 8.3 of [9], §X, the finite energy solutions of (62) will be of the form

$$X(\rho) = e^{M^-(\rho-\rho_0)} X_0 + \int_{\rho_0}^{\rho} e^{M^-(\rho-\tau)} P^-(X(\tau), Y_0) d\tau$$

(cf. (8.15) of [9], §X).

Thus, the asymptotic decay of solutions is governed by the decay of solutions (28) of (26). The asymptotics of the original system (59) are then obtained by successive approximation as in Section 3.

With these results in place, the remaining of Section 3 and Section 4 extend with minor changes. The hypotheses of Lemma 4.1 are now satisfied for a generic choice of the perturbation $P \in \mathcal{P}_\delta$. Similarly, we have guaranteed that the linearization $\mathcal{D}_{\mathcal{A}',\Psi'}$ is surjective.

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