Seiberg–Witten–Floer homology of a surface times a circle for non-torsion spinC structures

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We determine the Seiberg–Witten–Floer homology groups of the 3-manifold \( \Sigma \times S^1 \), where \( \Sigma \) is a surface of genus \( g \geq 2 \), together with its ring structure, for a SpinC structure with non-vanishing first Chern class. We give applications to computing Seiberg–Witten invariants of 4-manifolds which are connected sums along surfaces and also we reprove the higher type adjunction inequalities obtained by Oszváth and Szabó.

1 Introduction

In this paper we study the gluing theory for Seiberg–Witten invariants of 4-manifolds split along the 3-manifold \( Y = \Sigma \times S^1 \), where \( \Sigma \) is a surface of genus \( g \geq 2 \). This produces applications to the determination of the Seiberg–Witten invariants of 4-manifolds which are constructed as connected sums of other 4-manifolds along embedded surfaces, and to obtain constrains for the Seiberg–Witten invariants of 4-manifolds containing an embedded surface of genus \( g \) and nonnegative self-intersection. The seminal work in this direction is provided by [12] leading to a proof of the generalized Thom conjecture. Analysis of this kind on non-trivial circle bundles appears in [14] and [22].

Before stating the results, we set up some notation.

Let \( X \) be a closed, connected, oriented smooth 4-manifold with \( b^+ = \dim H_2^+(X; \mathbb{R}) > 0 \). Fix a Riemannian metric \( g \) and a homology orientation (i.e. an orientation of \( \Lambda^{top} H^1(X; \mathbb{R}) \otimes \Lambda^{top} H_2^+(X; \mathbb{R}) \)). Let \( \mathfrak{s} \) be a SpinC structure on \( X \), i.e., an equivalence class of lifting of the frame bundle of \( X \) to a \( \text{Spin}^C(4) \)-bundle. The Seiberg–Witten invariant ([27], [13], [25]–[26]) is a linear functional

\[ SW_{X,\mathfrak{s}} : A(X) \to \mathbb{Q}, \]

where \( A(X) = \text{Sym}^* H_0(X) \otimes \Lambda^* H_1(X) \) is the free graded algebra generated by the class of the point \( x \in H_0(X) \) together with generators of \( H_1(X) \) (we understand rational coefficients for the homology \( H_1(X) \)). We grade \( A(X) \) by giving degree 2 to \( x \) and degree 1 to the elements of \( H_1(X) \). Note that we can consider integral homology classes and we would obtain that \( SW_{X,\mathfrak{s}} \) takes values in \( \mathbb{Z} \).

The Seiberg–Witten invariants are constructed by studying the moduli space \( \mathcal{M}_X(\mathfrak{s}) \) of solutions \( (A, \Phi) \) modulo gauge transformations to the Seiberg–Witten equations ([27], [13] and [25]). Such equations are for a pair \((A, \Phi)\),

\[
\left\{
\begin{array}{l}
\rho((F_A - \sqrt{-1} \xi)^+) = (\Phi \otimes \Phi^*)_0, \\
D_A \Phi = 0,
\end{array}
\right.
\] (1.1)

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where $\Phi$ is a section of the positive spin bundle $W^+$ associated to $s$, and $A$ is a connection on the determinant line bundle $L = \det W^+ = \det W^-$, $F_A$ is the curvature of $A$, $D_A : \Gamma(W^+) \to \Gamma(W^-)$ is the Dirac operator twisted with the connection $A$, $\rho$ denotes Clifford multiplication, $(\Phi \otimes \Phi^*)_b$ is the trace-free part of $(\Phi \otimes \Phi^*)$, interpreted as an endomorphism of $W^+$, $\xi$ is a (small) closed real two-form introduced as a perturbation.

Note that the invariants are zero on elements whose degree is not $d(s)$ where

\[
d(s) = \frac{c_1(s)^2 - 2\chi(X) + 3\sigma(X)}{4}
\]

is the dimension of $M_X(s)$. Also recall that there is an action of $H^2(X; \mathbb{Z})$ on the set of $\text{Spin}^c$ structures, which we shall denote as $s + h$, for $h \in H^2(X; \mathbb{Z})$. When $b^+ > 1$ the Seiberg–Witten invariants are independent of metrics and perturbations. When $b^+ = 1$ the Seiberg–Witten invariants depend on a chamber structure. Fix one of the two components $\mathcal{K}_0$ of the positive cone

\[
\mathcal{K}(X) = \{ x \in H^2(X; \mathbb{R}) - \{0\} \mid x^2 \geq 0 \}.
\]

Then we say that the Seiberg–Witten invariants are computed in $\mathcal{K}_0$ when the metric $g$ and perturbation $\xi$ satisfy $(c_1(s) + \frac{1}{2}[\xi]) \cdot \omega_\xi < 0$, where $\omega_\xi \in H^2(X; \mathbb{R})$ is the period point for the metric $g$ lying in $\mathcal{K}_0$.

A basic class for $X$ is a $\text{Spin}^c$ structure with non-zero Seiberg–Witten invariant. By analogy with the definitions of simple type in the context of Donaldson invariants [18, Introduction], we give the following

**Definition 1.1** Let $X$ be a $4$-manifold with $b^+ > 1$. We say that

(i) $X$ is of simple type if $SW_{X,s}(z) = 0$ for any $z$ in the ideal generated by $x$ in $A(X)$, for any $s$.

(ii) $X$ is of $H_1$-simple type if $SW_{X,s}(z) = 0$ for any $z$ in the ideal of $A(X)$ generated by $H_1(X)$, for any $s$.

(iii) $X$ is of strong simple type if it is both of simple type and of $H_1$-simple type, i.e. $SW_{X,s}(z) = 0$ whenever $\deg(z) > 0$, for any $s$.

Note that when $X$ has $b_1 = 0$ it is automatically of $H_1$-simple type. There are manifolds not of $H_1$-simple type (for instance any manifold which is a connected sum $X \# S^2 \times S^2$, where $X$ has $b^+ > 1$, see [21, Proposition 2.21]), but it is an open question whether all 4-manifolds with $b^+ > 1$ are of simple type.

Now we are ready to state our main results. On the one hand, we have applications to computing the Seiberg–Witten invariants of connected sums along surfaces (see [15]). We prove the following results in Section 5.

**Theorem 1.2** Let $X_1$ and $X_2$ be closed 4-manifolds with embeddings $\Sigma \hookrightarrow X_i$, $i = 1, 2$, of a surface of genus $g \geq 2$, such that the image $\Sigma_i$ has self-intersection zero and represents a non-torsion homology class. Let $X = X_1 \#_{\Sigma} X_2$ be a connected sum along $\Sigma$ with some identification (see Section 5 for precise definition). Suppose that $\Sigma_1, \Sigma_2$ are both of strong simple type, and let $s_1, s_2$ be $\text{Spin}^c$ structures on $X_1, X_2$ respectively, such that $\langle c_1(s_1), [\Sigma_1] \rangle = \langle c_1(s_2), [\Sigma_2] \rangle \neq 0$ and $d(s_1) = d(s_2) = 0$. Let $s_0$ be a $\text{Spin}^c$ structure on $X$ obtained by gluing $s_1$ and $s_2$ and let $z \in H^2(X)$ such that $d(s_0) = deg(z)$. Then

\[
\sum_{h \in \mathcal{R}im} SW_{X,s_0,h}(z) = \begin{cases} SW_{X_1,s_1}(1) \cdot SW_{X_2,s_2}(1), & z = 1, \quad \langle c_1(s_0), [\Sigma] \rangle = 2g - 2, \\ 0, & z = 1, \quad \langle c_1(s_0), [\Sigma] \rangle < 2g - 2, \\ 0, & \deg(z) > 0, \end{cases}
\]

where $\mathcal{R}im \subset H^2(X; \mathbb{Z})$ is the subspace generated by the rim tori (cf. [6], also Section 5). If any of the manifolds has $b^+ = 1$, then the Seiberg–Witten invariants are computed for the component of the positive cone containing $cPD(\Sigma_i)$, where $\varepsilon = 1$ if $\langle c_1(s_0), [\Sigma] \rangle < 0$ and $\varepsilon = -1$ if $\langle c_1(s_0), [\Sigma] \rangle > 0$.

Moreover, if the connected sum is admissible (see Definition 5.3), then there are no basic classes $s$ of $X$ such that $0 < \langle c_1(s), [\Sigma] \rangle < 2g - 2$, and the basic classes of $X$ with $\langle c_1(s), [\Sigma] \rangle = \pm(2g - 2)$ are in bijection with pairs of basic classes $(s_1, s_2)$ of $\Sigma_1$ and $\Sigma_2$ respectively, such that $\langle c_1(s_1), [\Sigma_1] \rangle = \langle c_1(s_2), [\Sigma_2] \rangle = \pm(2g - 2)$.

This theorem is analogous to [15, Corollary 13] and [15, Corollary 15] for the case of Donaldson invariants. It is generalized to the following analogue of [18, Theorem 9.5].

**Theorem 1.3** Let $X_1$ and $X_2$ be closed 4-manifolds with embeddings $\Sigma \hookrightarrow X_i$, $i = 1, 2$, of a surface of genus $g \geq 2$, such that the image $\Sigma_i$ has self-intersection zero and represents a non-torsion homology class.
Let $X = \mathbb{X}_1 \# \mathbb{X}_2$ be a connected sum along $\Sigma$. Suppose that $\mathbb{X}_1$, $\mathbb{X}_2$ are of $H_1$-simple type, and let $\mathfrak{s}_1$, $\mathfrak{s}_2$ be Spin$^c$ structures on $\mathbb{X}_1$, $\mathbb{X}_2$ respectively, such that $\langle c_1(\mathfrak{s}_1), [\Sigma_1] \rangle = \langle c_1(\mathfrak{s}_2), [\Sigma_2] \rangle \neq 0$. Let $\mathfrak{s}_0$ be a Spin$^c$ structure on $X$ obtained by gluing $\mathfrak{s}_1$ and $\mathfrak{s}_2$ and let $z \in \mathcal{A}(X)$ such that $d(\mathfrak{s}_0) = \deg z$. Then

$$\sum_{h \in Rim} SW_{X, s_0 + h}(z) = 0$$

if $\deg(z) > 0$ or if $\langle c_1(\mathfrak{s}), [\Sigma] \rangle \neq 2g - 2 \pmod{4}$. If $X$ has $b^+ = 1$ then the Seiberg–Witten invariants are computed for the component of the positive containing $\varepsilon\text{P.D.}[\Sigma]$, where $\varepsilon = 1$ if $\langle c_1(\mathfrak{s}_0), [\Sigma] \rangle < 0$ and $\varepsilon = -1$ if $\langle c_1(\mathfrak{s}_0), [\Sigma] \rangle > 0$.

Moreover, if the connected sum is admissible then the basic classes $\mathfrak{s}$ of $X$ with $\langle c_1(\mathfrak{s}), [\Sigma] \rangle \neq 0$ all satisfy $\langle c_1(\mathfrak{s}), [\Sigma] \rangle \equiv 2g - 2 \pmod{4}$.

The restriction $\langle c_1(\mathfrak{s}), [\Sigma] \rangle \neq 0$ in Theorems 1.2 and 1.3 is due to the fact that the gluing theory for Seiberg–Witten invariants used here only works for non-torsion Spin$^c$ structures.

On the other hand, our analysis also leads to a different proof of the higher type adjunction inequalities first obtained by Oszváth and Szabó [21]. Our method of proof is more transparent and parallels that of [19] for proving the higher type adjunction inequalities for Donaldson invariants. Section 6 is devoted to this issue.

**Theorem 1.4** ([21, Theorem 1.4]) Let $X$ be a 4-manifold and let $\Sigma \subset X$ be an embedded surface of genus $g \geq 2$ representing a non-torsion homology class with self-intersection $[\Sigma]^2 \geq 0$. Let $a \in \mathcal{A}(X)$ and $b \in \mathcal{A}(\Sigma) \subset \mathcal{A}(X)$. If $X$ has $b^+ > 1$ and $\mathfrak{s}$ is a Spin$^c$ structure with $SW_{X, \mathfrak{s}}(a, b) \neq 0$ and $\langle c_1(\mathfrak{s}), [\Sigma] \rangle + [\Sigma]^2 > 0$ then we have

$$\|\langle c_1(\mathfrak{s}), [\Sigma] \rangle\| + [\Sigma]^2 + \deg(b) \leq 2g - 2.$$

Furthermore, when $b^+ = 1$ then for each Spin$^c$ structure $\mathfrak{s}$ on $X$ with $-\langle c_1(\mathfrak{s}), [\Sigma] \rangle + [\Sigma]^2 > 0$, for which $SW_{X, \mathfrak{s}}(a, b) \neq 0$, when calculated in the component of $K(X)$ containing P.D.$\mathcal{S}$, we have an inequality

$$-\langle c_1(\mathfrak{s}), [\Sigma] \rangle + [\Sigma]^2 + \deg(b) \leq 2g - 2.$$

**Theorem 1.5** ([21, Theorem 1.1]) Let $X$ be a 4-manifold of $H_1$-simple type (e.g. with $b_1 = 0$) and let $\Sigma \subset X$ be an embedded surface of genus $g \geq 2$ representing a non-torsion homology class with $[\Sigma]^2 \geq 0$. If $X$ has $b^+ > 1$ then for each basic class $\mathfrak{s}$ for $X$ with $\langle c_1(\mathfrak{s}), [\Sigma] \rangle + [\Sigma]^2 > 0$ we have

$$\|\langle c_1(\mathfrak{s}), [\Sigma] \rangle\| + [\Sigma]^2 + 2d(\mathfrak{s}) \leq 2g - 2.$$

If $b^+ = 1$ then for each basic class $\mathfrak{s}$ for the Seiberg–Witten invariants of $X$ calculated in the component of $\mathcal{K}(X)$ which contains P.D.$\mathcal{S}$ such that $-\langle c_1(\mathfrak{s}), [\Sigma] \rangle + [\Sigma]^2 > 0$, we have an inequality

$$-\langle c_1(\mathfrak{s}), [\Sigma] \rangle + [\Sigma]^2 + 2d(\mathfrak{s}) \leq 2g - 2.$$

This is a particular case of the more general result

**Theorem 1.6** ([21, Theorem 1.3]) Let $X$ be a 4-manifold with an embedded surface $i : \Sigma \hookrightarrow X$ of genus $g \geq 2$ representing a non-torsion homology class with $[\Sigma]^2 \geq 0$. Let $l$ be an integer so that there is a symplectic basis $\{\gamma_i\}_{i=1}^{2g}$ of $H_1(\Sigma)$ with $\gamma_1 \cdot \gamma_{g+1} = 1, 1 \leq i \leq g$, satisfying that $\iota_*(\gamma_i) = 0$ in $H_1(X)$ for $i = 1, \ldots, l$. Let $a \in \mathcal{A}(X)$ and $b \in \mathcal{A}(\Sigma)$ be an element of degree $\deg(b) \leq l + 1$. If $X$ has $b^+ > 1$ and $\mathfrak{s}$ is a Spin$^c$ structure such that $SW_{X, \mathfrak{s}}(a, b) \neq 0$ and $\langle c_1(\mathfrak{s}), [\Sigma] \rangle + [\Sigma]^2 > 0$ then we have

$$\|\langle c_1(\mathfrak{s}), [\Sigma] \rangle\| + [\Sigma]^2 + 2\deg(b) \leq 2g - 2.$$

Furthermore, when $b^+ = 1$ then for each Spin$^c$ structure $\mathfrak{s}$ on $X$ with $-\langle c_1(\mathfrak{s}), [\Sigma] \rangle + [\Sigma]^2 > 0$, for which $SW_{X, \mathfrak{s}}(a, b) \neq 0$, when calculated in the component of $\mathcal{K}(X)$ containing P.D.$\mathcal{S}$, we have

$$-\langle c_1(\mathfrak{s}), [\Sigma] \rangle + [\Sigma]^2 + 2\deg(b) \leq 2g - 2.$$
Again the restriction $|c_1(s) + [\Sigma]| + [\Sigma]^2 > 0$ is due to the fact that we only use the Seiberg–Witten–Floer theory for non-torsion Spin$^C$ structures (see Section 2).

In order to prove these results, we determine completely the structure of the Seiberg–Witten–Floer (co)homology of the three manifold $Y = \Sigma \times S^1$, where $\Sigma$ is a closed surface of genus $g \geq 2$, for any Spin$^C$ structure with non-zero first Chern class. We use the Seiberg–Witten–Floer groups as defined in [4], since they satisfy a gluing theorem for Seiberg–Witten invariants. We prove

**Theorem 1.7** Let $s_Y$ be a Spin$^C$ structure on $Y = \Sigma \times S^1$ with $c_1(s_Y) \neq 0$. If $c_1(s_Y) \neq 2rP.D.[S^1]$, with $-(g-1) \leq r \leq g-1$, then the Seiberg–Witten–Floer (co)homology is $\text{HFSW}^*(Y, s_Y) = 0$. Let $s_r$ be the (unique) Spin$^C$ structure on $Y$ with $c_1(s_r) = 2rP.D.[S^1]$, $-(g-1) \leq r \leq g-1$, $r \neq 0$, and set $d = g - 1 - |r|$. Then there is an isomorphism of vector spaces $\text{HFSW}^*(Y, s_r) \cong H^*(\text{Sym}^d(\Sigma))$, where $\text{Sym}^d(\Sigma)$ is the $d$-th symmetric product of $\Sigma$.

Theorem 1.7 will follow from Theorem 4.9. This theorem will follow from Theorem 4.9. Let $-(g-1) \leq r \leq g-1$, $r \geq 0$, and set $d = g - 1 - |r|$. Then there is a presentation

$$\text{HFSW}^*(\Sigma \times S^1, s_r) = \bigoplus_{k=0}^{d} \Lambda^k_0 \otimes \mathbb{Q}[\eta, \theta] / (\mathcal{R}^q_k, \theta \mathcal{R}^q_{k+1}, \eta^{d+1}, \theta^{d+1}),$$

where $\theta \in \Lambda^2 H_1(\Sigma)$ represents the intersection form in $H^1(\Sigma)$ and

$$\Lambda^0_0 = \Lambda^0 H_1(\Sigma) = \ker (\theta^{g-k+1} : \Lambda^k H_1(\Sigma) \rightarrow \Lambda^{2g-k+2} H_1(\Sigma))$$

is the primitive part. The polynomials $\mathcal{R}^q_k$ are defined as

$$\mathcal{R}^q_k = \sum_{i=0}^{[\frac{d-k-\alpha+1}{2}]} \frac{[d-k-i]}{[\frac{d-k}{2}]} (-1)^i \eta^{\alpha-i} \theta^i,$$

where $\alpha = \lfloor \frac{d-k}{2} \rfloor + 1$, $0 \leq k \leq d$, and $\mathcal{R}^q_{d+1} = 1$ (here $[x]$ denotes the integer part of $x$).

This theorem will follow from Theorem 4.9.

2 Seiberg–Witten–Floer homology for non-torsion Spin$^C$ structures

Let $Y$ be an oriented 3-manifold with first Betti number $b_1 > 0$ and a Spin$^C$ structure $s_Y$ on $Y$ with $c_1(s_Y)$ non-torsion. We are going to review the construction of the Seiberg–Witten–Floer (co)homology groups $\text{HFSW}_* (Y, s_Y)$ of $(Y, s_Y)$ from [4].

2.1 Definition of $\text{HFSW}_* (Y, s_Y)$

We endow $Y$ with a metric $g_Y$ and fix a connection $A_0$ on the determinant line bundle $L_Y = \det W_Y$, where $W_Y$ stands for the spin bundle associated to the Spin$^C$ structure $s_Y$.

Let $A_Y$ be the (affine) space of pairs $(A, \psi)$ formed by a connection $A$ on $L_Y$ and a section $\psi$ of $W_Y$. This is acted upon by the group of gauge transformations $G_Y = \text{Map}(Y, S^1)$. Fix a coexact one-form $\eta$ on $Y$. Then there is a Chern-Simons-Dirac functional defined on the configuration space $A_Y / G_Y$ (taking values in $\mathbb{R}/\mathbb{Z}$) given by

$$C_\eta(A, \psi) = -\frac{1}{2} \left( \int_Y (A - A_0) \wedge (F_A + F_{A_0} - 2 \ast \sqrt{-1} \eta) + \langle \psi, \partial_A \psi \rangle \right).$$
where $\hat{\theta}_A$ stands for the Dirac operator on $W_Y$ coupled with $A$ and $*$ is the Hodge operator on $Y$. The critical points of $C_Y$ correspond to translation invariant solutions to (1.1) on the tube $Y \times \mathbb{R}$ for the perturbation $\xi = *\eta$. The equations we are led to are

$$
\begin{align*}
*F_A &= q(\psi) + \eta, \\
\hat{\theta}_A \psi &= 0,
\end{align*}
$$

(2.1)

where $q(\psi)$ is the quadratic form defined by minus the Clifford multiplication map $\rho : T^*Y \to \text{End}(W_Y)$.

When $c_1(s_Y)$ is a non-torsion element in $H^2(Y, \mathbb{Z})$, we can choose a generic perturbation parameter $\eta$ to get a finite collection of non-degenerate irreducible (oriented) critical points of $C_Y$ [4]. Denote by $M_Y(s_Y, \eta)$ the set of solutions to the perturbed Seiberg–Witten equations (2.1) on $(Y, s_Y)$. There is also a well-defined relative index on $M_Y(s_Y, \eta)$ taking values in $\mathbb{Z}/N\mathbb{Z}$:

$$
\text{ind}(\cdot, \cdot) : M_Y(s_Y, \eta) \times M_Y(s_Y, \eta) \to \mathbb{Z}/N\mathbb{Z},
$$

where

$$
N = \text{G.C.D.} \{ \langle c_1(s_Y), \sigma \rangle \mid \sigma \in H_2(Y, \mathbb{Z}) \}.
$$

The Seiberg–Witten–Floer chain complex $CFSW_\ast(Y)$ is the vector space (over the rational coefficient $\mathbb{Q}$) generated by the gauge classes of solutions to (2.1) with this relative grading.

The downward flow equation of $C_Y$ is the 4-dimensional Seiberg–Witten equation on $(Y \times \mathbb{R}, g_Y + dt^2)$ with the pull-back $\text{Spin}^c$ structure under the temporal gauge:

$$
\begin{align*}
\frac{\partial A(t)}{\partial t} &= -*F_A + q(\psi) + \eta, \\
\frac{\partial \psi(t)}{\partial t} &= -\theta_A \psi.
\end{align*}
$$

(2.2)

In general, in order to achieve the transversality property for the moduli space of the Seiberg–Witten solutions on $Y \times \mathbb{R}$ to (2.2) without destroying the $\mathbb{R}$-translation action and obtaining a natural compactification, we have to choose a suitable perturbation of $C_Y$ supported away from the set of critical points. See [4] for a detailed discussion. (Similar kind of perturbations were first constructed in [7].)

Let $a$ and $b$ in $M_Y(s_Y, \eta)$ be two gauge classes of irreducible solutions to (2.1). For generic perturbations as in [4], any connected component of the moduli space $\mathcal{M}(a, b)$ of gauge classes of solutions to the perturbed Seiberg–Witten equations on the tube $Y \times \mathbb{R}$ with asymptotic limits $a$ and $b$ respectively at two ends, is smooth, orientable and admits a free $\mathbb{R}$-action. We shall denote by $\mathcal{M}^0(a, b)$ the union of components of dimension $D$ in the quotient space of $\mathcal{M}(a, b)$ by the $\mathbb{R}$-action. Note that $D \equiv \text{ind}(a, b) - 1 \pmod{N}$. We define a boundary map

$$
\partial : CFSW_i(Y) \to CFSW_{i-1}(Y)
$$

$$
a \mapsto \sum_{\substack{b \in \mathcal{M}_Y(s_Y, \eta) \\text{ind}(a, b) = 1 \pmod{N}}} \#\mathcal{M}^0(a, b) b
$$

The compactifications of the moduli space of the trajectory flow lines ensure that $\partial$ is well-defined and $\partial^2 = 0$ [4] so we obtain the Seiberg–Witten–Floer cohomology, denoted by $HFSW^\ast(Y, s_Y)$, which is a $\mathbb{Z}_N$-graded vector space.

Notice that the first Chern class of the non-torsion $\text{Spin}^c$ structure $s_Y$ defines a homomorphism $c_1(s_Y) : H^1(Y, \mathbb{Z}) \to \mathbb{Z}$ by $c_1(s_Y)([u]) = \langle [u] \cup c_1(s_Y), [Y] \rangle$ for any element $[u] \in H^1(Y, \mathbb{Z})$. For any subgroup $K$ in

$$
\ker(c_1(s_Y)) = \{ [u] \in H^1(Y, \mathbb{Z}) \mid \langle [u] \cup c_1(s_Y), [Y] \rangle = 0 \},
$$

there is a subgroup $G_Y^K$ of the full gauge transformation group $G_Y$, whose elements lie in the connected components determined by $K$ (since the group of connected components of $G_Y$ is $H^1(Y, \mathbb{Z})$). Consider the Chern-Simons-Dirac functional on the space $A_Y$ modulo the gauge group $G_Y^K$. The critical point set, denoted by...
\( M_{Y,K}(s_Y, \eta) \), is a covering space

\[
\pi_K : M_{Y,K}(s_Y, \eta) \rightarrow M_Y(s_Y, \eta),
\]

whose fiber is a \( H^1(Y,\mathbb{Z})/K \)-homogeneous space.

There is a variant of the Seiberg–Witten–Floer chain complex whose generators are elements in \( M_{Y,K}(s_Y, \eta) \) with relative \( \mathbb{Z} \)-graded indices and boundary operator \( \partial^K \) given by counting the gradient flow lines of the perturbed Chern-Simons-Dirac functional on \( \mathcal{A}_Y/\mathcal{G}_Y^K \) connecting two critical points with relative index 1. We denote the Seiberg–Witten–Floer homology in this setting by \( HFSW_{*,[K]}(Y, s_Y) \).

The following theorem was established in [4] regarding various properties of these Seiberg–Witten–Floer homologies of \( (Y, s_Y) \).

**Theorem 2.1** (Theorem 1.1 in [4]) For any closed oriented 3-manifold \( Y \) with \( b_1(Y) > 0 \) and a \( \text{Spin}^C \) structure \( s_Y \) such that \( c_1(s_Y) \) is non-torsion. Let \( N \) be the divisibility of \( c_1(s_Y) \) in \( H^2(Y;\mathbb{Z})/\text{torsion} \), that is, \( N = \text{G.C.D.} \{ c_1(s_Y), \sigma \} \mid \sigma \in H_2(Y,\mathbb{Z}) \). Then there exists a finitely generated Seiberg–Witten–Floer complex whose homology \( HFSW_*(Y, s_Y) \) satisfies the following properties:

1. \( HFSW_*(Y, s_Y) \) is a topological invariant of \( (Y, s_Y) \) and it is a \( \mathbb{Z}_N \)-graded (rational) vector space.
2. There is an action of \( \text{ker}(c_1(s_Y)) \subset H^1(Y,\mathbb{Z}) \) on \( HFSW_*(Y, s_Y) \) with elements in \( H_0(Y) \) and \( H_1(Y) \) decreasing degrees in \( HFSW_*(Y, s_Y) \) by 2 and 1 respectively. We denote the action as \( z \cdot \Xi \), for \( z \in H_1(Y) \) and \( \Xi \in HFSW_*(Y, s_Y) \).
3. For \( (-Y, -s_Y) \), where \( -Y \) is \( Y \) with the reversed orientation and \( -s_Y \) is the induced \( \text{Spin}^C \) structure, the corresponding Seiberg–Witten–Floer complex \( C_\ast(-Y, -s_Y) \) is the dual complex of \( C_\ast(Y, s_Y) \). There is a natural pairing

\[
\langle \ , \ \rangle : HFSW_*(Y, s_Y) \times HFSW_{*-1}(-Y, -s_Y) \rightarrow \mathbb{Q}
\]

such that \( \langle z \Xi_1, z \Xi_2 \rangle = \langle \Xi_1, z \Xi_2 \rangle \) for \( z \in \text{ker}(c_1(s_Y)) \equiv \mathbb{A}(-Y) \subseteq HFSW_*(Y, s_Y) \) and \( \Xi_1 \in HFSW_*(Y, s_Y) \) and \( \Xi_2 \in HFSW_{*-1}(-Y, -s_Y) \).

4. For any subgroup \( K \subset \ker(c_1(s_Y)) \subset H^1(Y,\mathbb{Z}) \), there is a variant of Seiberg–Witten–Floer homology denoted by \( HFSW_{*,[K]}(Y, s_Y) \), which is a topological invariant and a \( \mathbb{Z}_2 \)-graded \( \mathcal{A} \) module. If \( K_1 \subset K_2 \) are two subgroups in \( \ker(c_1(s_Y)) \), there is a \( \mathcal{A} \) equivariant homomorphism

\[
HFSW_{*,[K_1]}(Y, s_Y) \rightarrow HFSW_{*,[K_2]}(Y, s_Y).
\]

Moreover, for any \( m \in \mathbb{Z} \), there is an isomorphism

\[
HFSW_{m,[\ker(c_1(s_Y))]}(Y, s_Y) \cong HFSW_{m,([\mod N]}(Y, s_Y).
\]

2.2 Relative Seiberg–Witten invariants and gluing formula

Let \( X_1 \) be an oriented, connected 4-manifold furnished with a cylindrical end of the form \( Y \times [0, \infty) \). Suppose we have a \( \text{Spin}^C \) structure \( s_Y \) over \( X_1 \) whose restriction to \( Y \) is a non-torsion \( \text{Spin}^C \) structure \( s_Y \). Consider finite energy solutions to the Seiberg–Witten equations on \( X_1 \) with finite variations of the perturbed Chern-Simons-Dirac functional on the end. There is an associated boundary asymptotic value map

\[
\partial_\infty : M_{X_1}(s_Y, \xi) \rightarrow M_{Y,X_1}(s_Y, \eta)
\]

where \( M_{Y,X_1}(s_Y, \eta) \) is the quotient of solutions to the perturbed Seiberg–Witten equations on \( (Y, s_Y) \) by the action of those gauge transformations which can be extended to \( X_1 \). In fact,

\[
\pi_1 : M_{Y,X_1}(s_Y, \eta) \rightarrow M_Y(s_Y, \eta)
\]

is a covering map with fiber an \( H^1(Y,\mathbb{Z})/\text{im}(i^\ast_1) \)-homogeneous space. Here \( K_1 = \text{im}(i^\ast_1) \subset \ker(c_1(s_Y)) \) is the image of the map \( i^\ast_1 : H^1(X_1,\mathbb{Z}) \rightarrow H^1(Y,\mathbb{Z}) \) induced from the boundary embedding map \( i_1 : Y \hookrightarrow X_1 \).

Generically, the fiber of \( \partial_\infty \) is an oriented, smooth manifold of dimension given by Atiyah-Patodi-Singer index theorem, and it can be compactified to a smooth manifold with corners. See [4] for the detailed discussion.
The relative Seiberg–Witten invariant of \((X_1, s_1)\), as defined in [4], takes values in the Seiberg–Witten–Floer homology \(HFSW_{\ast,[K]}(Y, \ast_Y)\), and defines an \(\mathcal{A}(Y)\)-equivariant linear map

\[
\phi_{X_1}^{SW}(s_1, \cdot) : \mathcal{A}(X_1) \rightarrow HFSW_{\ast,[K]}(Y, \ast_Y).
\]

Here the \(\mathcal{A}(Y)\)-action on \(\mathcal{A}(X_1)\) is induced from the homomorphism \((i_1)_* : \mathcal{A}(Y) \rightarrow \mathcal{A}(X_1)\). For any \(z_1 \in \mathcal{A}(X_1)\) of degree \(d\), \(\phi_{X_1}^{SW}(s_1, z_1)\) can be expressed in terms of the Seiberg–Witten invariant from the components of dimension \(d\) in \(M_{X_1}(s_1, \xi)\).

For a second 4-manifold \(X_2\) with a cylindrical end \((-Y) \times [0, \infty)\), we construct \(X = X_1 \cup_Y X_2\) by cutting the ends and gluing along the common boundary \(Y\). The resulting manifold may depend on the isotopy class of the diffeomorphism identifying the boundaries, but we shall not make the dependence explicit. If there is a Spin\(^C\) structure \(s_2\) on \(X_2\) with \(s_2|_Y \cong s_Y\), then we can glue \(s_1\) and \(s_2\). The indeterminacy for the gluing is parametrized by \(\text{coker}(H^1(X_1; \mathbb{Z}) \oplus H^1(X_2; \mathbb{Z}) \rightarrow H^1(Y; \mathbb{Z}))\). The following gluing formula is taken from [4].

**Theorem 2.2** (Theorem 1.2 in [4]) Let \(X\) be a closed manifold with \(b^+ \geq 1\) which is written as \(X = X_1 \cup_Y X_2\), where \(X_1\) and \(X_2\) are 4-manifolds with boundary and \(\partial X_1 = -\partial X_2 = -\partial Y\). Suppose that we have Spin\(^C\) structures \(s_1\) and \(s_2\) on \(X_1\) and \(X_2\) respectively such that \(s_1|_Y = s_2|_Y = s_Y\) is a non-torsion Spin\(^C\) structure on \(Y\). Let \(K_1 = \text{im}(i_1^*)\) and \(K_2 = \text{im}(i_2^*)\) and denote by \(\pi_i : HFSW_{\ast,[K]}(\pm Y, \pm s_Y) \rightarrow HFSW_{\ast}(\pm Y, \pm s_Y)\) the maps of Theorem 2.1.

For \(z_1 \in \mathcal{A}(X_1), i = 1, 2\), let \(S\) be the set of Spin\(^C\) structures \(s\) on \(X\) with \(s|_{X_i} = s_i, i = 1, 2\) and \(d(s) = \text{deg}(z_1) + \text{deg}(z_2)\). Then

\[
\sum_{s \in S} SW_{X,s}(z_1, z_2) = \left< \pi_1(\phi_{X_1}^{SW}(s_1, z_1)), \pi_2(\phi_{X_2}^{SW}(s_2, z_2)) \right>,
\]

where the pairing on the right-hand side is the pairing \(HFSW_{\ast}(Y, s_Y) \times HFSW_{\ast}(\pm Y, \pm s_Y) \rightarrow \mathbb{Q}\). When \(b^+ = 1\), the Seiberg–Witten invariants on \(X\) correspond to a metric giving a long neck.

### 3 Seiberg–Witten–Floer homology of \(\Sigma \times S^1\)

From now on we shall consider the three-manifold \(Y = \Sigma \times S^1\), which is the central object of our study. As \(H^2(Y; \mathbb{Z})\) has no 2-torsion, the Spin\(^C\) structures \(s_Y\) on \(Y\) are determined by the determinant line bundle \(L_Y = c_1(s_Y)\). As \(c_1(s_Y)\) reduces to \(w_2(Y) = 0\) modulo 2, it has to be an even class in \(H^2(Y; \mathbb{Z})\).

**Proposition 3.1** Let \(s_Y\) be a Spin\(^C\) structure on \(Y\). Let \(\mathcal{M}\) be the moduli space of solutions to (2.1) with zero perturbation. Then \(\mathcal{M}\) is empty unless \(c_1(s_Y) = 2rPD_\mathbb{Z}[S^1]\), with \(-g - 1 \leq r \leq g - 1\). For \(c_1(s_Y) = 2rPD_\mathbb{Z}[S^1]\), with \(-g - 1 \leq r \leq g - 1\) and \(r \neq 0\), \(\mathcal{M}\) is Morse-Bott irreducible and isomorphic to the d-th symplectic product \(\text{Sym}^d(\Sigma)\) with \(d = g - 1 - |r|\).

**Proof.** We choose a rotation invariant metric for \(Y\) of the form \(g_\Sigma + d\theta \otimes d\theta\), where \(g_\Sigma\) is a metric on \(\Sigma\) with unit area and scalar curvature \(-4\pi(2g - 2)\), and \(\theta\) is the coordinate on \(S^1 = \mathbb{R}/\mathbb{Z}\). Think of \(\Sigma \times S^1\) as \(\Sigma \times [0, 1]\) with the boundaries \(\Sigma \times \{0\}\) and \(\Sigma \times \{1\}\) identified by the identity. The line bundle \(L_Y\) is constructed as the pull back under the projection \(\Sigma \times [0, 1] \rightarrow \Sigma\) of a line bundle \(L_\Sigma\) on \(\Sigma\) by gluing along the boundaries with an isomorphism \(\sigma \in \text{G}_\Sigma = \text{Map}(\Sigma, S^1)\). Then \(c_1(L_Y) = c_1(L_\Sigma) + [\sigma] \otimes [S^1]\), where \([\sigma]\) is the class of \(\sigma\) in \([\Sigma; S^1] \cong H^1(\Sigma; \mathbb{Z})\). The Spin\(^C\) structure \(s_Y\) induces a Spin\(^C\) structure \(s_\Sigma\) on \(\Sigma\) with determinant line bundle \(L_\Sigma\). The spin bundle of \(s_\Sigma\) is of the form \(W_\Sigma = (\Lambda^0 \oplus \Lambda^{0,1}) \otimes \mu\), for a line bundle \(\mu\) such that \(L_\Sigma = K_\Sigma^{-1} \otimes \mu^2\), where \(K_\Sigma\) stands for the canonical bundle of \(\Sigma\).

Now consider any solution \((A, \psi)\) to (2.1) with \(\eta = 0\). In \(\Sigma \times [0, 1]\), we can kill, by a gauge transformation, the \(d\theta\) component of \(A\), i.e., we can suppose that we have a family \(\alpha_\theta, \theta \in [0, 1]\), of connections on \(L_\Sigma\) (up to a constant gauge) with the boundary condition \(A_\eta = \sigma^* (A_0)\), for some \(\sigma \in \text{G}_\Sigma\) in the homotopy class determined by \(L_Y\). So \((A, \psi)\) is interpreted as a path \((A_\theta, \alpha_\Psi, \beta_\theta), \theta \in [0, 1]\), where \(\alpha_\theta \in \Lambda^0 \otimes \mu\) and \(\beta_\theta \in \Lambda^{0,1} \otimes \mu\). Let us rewrite equations (2.1) in this set-up. Clearly \(*F_A = \Lambda F_{A_\theta} d\theta + \ast_\Sigma \left( \frac{\partial A}{\partial \theta} \right)\), and the map \(q\) and the Dirac operator are as follows

\[
q(\psi) = -\frac{(\alpha_\theta^2 + \beta_\theta^2)}{2} + \frac{|\alpha|^2 - |\beta|^2}{2} \, d\theta,
\]
$$D_A = \left( \begin{array}{cc} -\sqrt{-1} \frac{\partial}{\partial \theta} & \sqrt{2} \frac{\partial}{\partial r} \\ \sqrt{2} \frac{\partial}{\partial A} & -\sqrt{-1} \frac{\partial}{\partial \theta} \end{array} \right) : (\Lambda^0 \otimes \mu) \oplus (\Lambda^{0,1} \otimes \mu) \longrightarrow (\Lambda^0 \otimes \mu) \oplus (\Lambda^{0,1} \otimes \mu).$$

So the solutions to (2.1) correspond to solutions to

$$\begin{cases}
\frac{\partial \alpha}{\partial \theta} = -\sqrt{-1} \sqrt{2} \partial_{A_\theta} \beta, \\
\frac{\partial \beta}{\partial \theta} = \sqrt{-1} \sqrt{2} \partial_{A_\theta} \alpha, \\
2 \frac{\partial A_\theta}{\partial \theta} = -\sqrt{-1} (\alpha \overline{\gamma} + \beta \overline{\eta}), \\
2 \sqrt{-1} \Lambda F_A = -|\alpha|^2 + |\beta|^2.
\end{cases} \quad (3.1)$$

We can write $A_\theta = \partial_{A_\theta} + \partial_{A_\theta}$, so the third line is $\frac{\partial}{\partial \theta}(\partial_{A_\theta}) = -\sqrt{-1} \frac{\partial}{\partial \theta} \alpha \beta$, $\frac{\partial}{\partial \theta}(\partial_{A_\theta}) = -\sqrt{-1} \frac{\partial}{\partial \theta} \beta$. Now suppose we have a solution to (3.1). Then we work out the following expression (using $\sqrt{-1} \partial^* = \Lambda \partial$ on $(0,1)$-forms and $|\beta|^2 = -\sqrt{-1} \Lambda \beta \wedge \overline{\beta}$)

$$\frac{\partial}{\partial \theta} (\partial^* \beta) = -\frac{\partial}{\partial \theta} (\sqrt{-1} \Lambda \partial) \beta + \partial^* \left( \frac{\partial}{\partial \theta} \right),$$

with the given equalities to get

$$-\frac{1}{\sqrt{2} \sqrt{-1}} \frac{\partial^2 \alpha}{\partial \theta^2} = \sqrt{-1} \Lambda \frac{\sqrt{-1}}{2} \alpha \beta - \partial^* \left( -\sqrt{-1} \sqrt{2} \partial \alpha \right),$$

$$-\frac{\partial^2 \alpha}{\partial \theta^2} + \sqrt{2} \alpha |\beta|^2 + 2 \partial^* \partial \alpha = 0.$$

Take scalar product with $\alpha$ and integrate along $\Sigma$ by parts to get

$$-\int_{\Sigma} \left( \frac{\partial^2 \alpha}{\partial \theta^2} \right) + \frac{\sqrt{2}}{2} \int_{\Sigma} |\alpha|^2 |\beta|^2 + 2 \int_{\Sigma} |\partial \alpha|^2 = 0,$$

for every $\theta \in [0,1]$. This equation makes sense in $S^1$, since the values for $\theta = 0$ and $\theta = 1$ coincide. Then we can integrate again by parts to get

$$\left\| \frac{\partial}{\partial \theta} \right\|^2 + \frac{\sqrt{2}}{2} \|\alpha \beta\|^2 + 2 \|\partial \alpha\|^2 = 0.$$

So either $\alpha = 0$ or $\beta = 0$. In any case, $A_\theta$, $\alpha_\theta$ and $\beta_\theta$ are constant, i.e. if the line bundle $L_Y$ admits solutions to (2.1) then it is pulled-back from $\Sigma$ and any solution is invariant under rotations in the $S^1$ factor.

Assume now that $c_1(L_Y) = 2r \text{P.D.}[S^1]$. For any solution to Equations (3.1), either $\alpha = 0$, $\partial_{A_\theta} \beta = 0$ or $\beta = 0$, $\partial_{A_\theta} \alpha = 0$. Also $2r = c_1(L_\Sigma) = -\frac{\sqrt{2}}{2\pi} \int_{\Sigma} F_A = -\frac{1}{2\pi} \int_{\Sigma} (|\beta|^2 - |\alpha|^2)$. If $r < 0$ then $\beta = 0$ and the solutions to Equations (3.1) are equivalent to the solutions to

$$\begin{cases}
\partial_{A_\theta} \alpha = 0, \\
2 \sqrt{-1} \Lambda F_A = -|\alpha|^2,
\end{cases}$$

on $\Sigma$. These are the typical vortex equations. The space of solutions is the $d$-th symmetric product $\text{Sym}^d(\Sigma)$, where $d = g - 1 + r$. If $r < -(g - 1)$ then there are no solutions. The case $r > 0$ is analogous.

**Theorem 3.2** Let $\mathfrak{s}_Y$ be a Spin$^C$ structure on $Y$ with $c_1(\mathfrak{s}_Y) \neq 0$. Then $\text{HFSW}^r(Y, \mathfrak{s}_Y) = 0$ unless $c_1(\mathfrak{s}_Y) = 2r \text{P.D.}[S^1]$, with $-(g - 1) \leq r \leq g - 1$. Let $\mathfrak{s}_r$ be the Spin$^C$ structure on $Y$ with $c_1(\mathfrak{s}_r) = 2r \text{P.D.}[S^1]$, $-(g - 1) \leq r \leq g - 1$, $r \neq 0$. Put $d = g - 1 - |r| \geq 0$, then dim $\text{HFSW}^r(Y, \mathfrak{s}_r) \leq \dim H^r(\text{Sym}^d(\Sigma))$. 

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Proof. The first claim is a direct consequence of Proposition 3.1. Also from Proposition 3.1, we know that for $s_\ast$, the unperturbed Chern–Simons Seiberg–Witten functional already has non-degenerate critical manifolds. As in [8] and [2, Proposition 6], we can choose a perturbation modelled on the finite dimensional critical manifold $\text{Sym}^d(\Sigma)$. Choose a positive and perfect Morse function $f$ on $\Sigma$, i.e. $f$ has one critical point of index 0, 2$g$ critical points of index 1 and one critical point of index 2. For any point $(x_1, x_2, \ldots, x_d) \in \Sigma,$ define $F(x_1, x_2, \ldots, x_d) = \prod_{i=1}^{d} f(x_i)$, then it is easy to check that $F$ is a Morse function on $\text{Sym}^d(\Sigma)$. The critical points of $F$ consist of those $(x_1, x_2, \ldots, x_d)$ where $x_i$ is a critical point of $f$, and the Morse index of $(x_1, x_2, \ldots, x_d)$ is the sum of the Morse indices of the $x_i$'s. Therefore, the number of the critical points of $F$ with Morse index $i$ is given by

$$\left(\frac{2g}{i}\right) + \left(\frac{2g}{i-2}\right) + \ldots + \left(\frac{2g}{i-2[i/2]}\right),$$

which is exactly the $i$-th Betti number of $\text{Sym}^d(\Sigma)$ (see [9]). Hence, $F(x_1, x_2, \ldots, x_d)$ is a perfect Morse function on $\text{Sym}^d(\Sigma)$. Then we can perturb the Chern–Simons Seiberg–Witten functional such that there exists a one-to-one correspondence between the perturbed Seiberg–Witten monopoles on $\Sigma \times S^1$ and the critical points of $F$ on $\text{Sym}^d(\Sigma)$. Both sets of critical points are non-degenerate and have the same relative indices modulo $2|\tau|$. This implies that $\dim HFSW^\ast(Y, s_\ast) \leq \dim H^\ast(\text{Sym}^d(\Sigma))$.

To shorten the notation, we shall write from now on

$$V_r = HFSW^\ast(Y, s_\ast),$$

for $-(g-1) \leq r \leq g-1$, $r \neq 0$. In this section we will study the finite dimensional vector spaces $V_r$ for $r \neq 0$. They have a natural $\mathbb{Z}/2|\tau|\mathbb{Z}$-grading. The only tools we shall use are the bound on the dimension provided by Theorem 3.2 and the gluing Theorem 2.2. First, it is easily seen that the diffeomorphism $f \times c : \Sigma \times S^1 \rightarrow \Sigma \times S^1$, where $f : \Sigma \rightarrow \Sigma$ is an orientation reversing diffeomorphism and $c : S^1 \rightarrow S^1$ is the conjugation, induces an isomorphism $V_r \cong V_{-r}$. Henceforth we shall suppose $r > 0$ in (3.2).

Let

$$Z = \Sigma \times D^2$$

denote the 4-manifold given by the product of $\Sigma$ and a 2-dimensional disc, so that $\partial Z = \Sigma \times S^1$. Let $i : Y \hookrightarrow Z$ and let $\Delta = pt \times D^2 \subset Z$. The Spin$^c$ structures on $Z$ are parametrized by $H^2(Z; \mathbb{Z}) = H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}$. We write $s_\ast$ for the Spin$^c$ structure on $Z$ with $c_1(s_\ast) = 2r \text{P.D.}[\Delta]$, where $\text{P.D.}[\Delta]$ is the Poincaré dual of the relative 2-cycle $[\Delta]$. We use the same notation $s_\ast$ for Spin$^c$ structures on $Y$ and on $Z$. No confusion should arise from this, as they are compatible in the sense that $s_\ast|_Y = s_\ast$.

Note that $\ker(c_1(s_\ast)) = \text{im}(i^* : H^1(Z; \mathbb{Z}) \rightarrow H^1(Y; \mathbb{Z}))$ and $HFSW^\ast(Y, s_\ast) \cong HFSW^\ast_{(\ker(c_1(s_\ast)))}(Y, s_\ast)$ from Theorem 2.1. The relative Seiberg–Witten–Floer homology of $Z$ give a map

$$A(Z) \cong A(\Sigma)$$

for $z = HFSW^\ast(Y, s_\ast)$

$$\phi^Z_{SW}(s_\ast, z)$$

As $Z = Y \cup_Y Z = \Sigma \times S^2$, the gluing Theorem 2.2 yields

$$\sum_{n \in \mathbb{Z}} SW_{s_\ast + n[\Sigma]}(z_1, z_2) = \langle \phi^Z_{SW}(s_\ast, z_1), \phi^Z_{SW}(s_\ast, z_2) \rangle,$$

for any $z_1, z_2 \in A(\Sigma)$, where the Spin$^c$ structure $s_\ast$ on $S$ is the one with $c_1(s_\ast) = 2r \text{P.D.}[S^2]$. The metric that we must use for the Seiberg–Witten–Floer invariants in the left-hand side of (3.4) is one giving a long neck, i.e. with period point $\omega_0$ close to $[\Sigma]$ in $K_0 = \{ a|S^2| + b[\Sigma] \mid a, b > 0 \}$. This implies that $c_1(s_\ast + n[\Sigma]) \cdot \omega_0 > 0$ as $r > 0$, so the invariants are calculated in the component $-K_0$ of the positive cone.

As $r \neq 0$, there is at most one $n \in \mathbb{Z}$ that contributes to the left-hand side in (3.4), since $c_1(s_\ast + n[\Sigma]) = 2r|S^2| + 2n[\Sigma]$ and

$$d(s_\ast + n[\Sigma]) = 2rn + 2(g - 1).$$

(3.5)
It is thus important to know the Seiberg–Witten invariants of $S = \Sigma \times S^2$ for the component $-K_0$, which we describe now. We fix the homology orientation given by the usual orientation of $H^1(S) = H^1(\Sigma)$ and the orientation of $H^2(S) = \mathbb{R}\omega_0$ determined by $-\omega_0$.

Fix a symplectic basis $\{\gamma_1, \ldots, \gamma_2g\}$ of $H_1(\Sigma; \mathbb{Z})$ with $\gamma_i \cap \gamma_{i+g} = pt$, for $1 \leq i \leq g$. Then

$$A(\Sigma) = \mathbb{Q}[x] \otimes \Lambda^*(\gamma_1, \ldots, \gamma_{2g})$$

and there is an action of the mapping class group of $\Sigma$, $\pi_0(\text{Diff}(\Sigma))$, factoring through an action of the symplectic group $\text{Sp}(2g, \mathbb{Z})$, on both $A(\Sigma)$ and $V_r$, making the map (3.3) equivariant. Define

$$\theta = \sum_{i=1}^{g} \gamma_i \wedge \gamma_{g+i} \in \Lambda^2 H_1(\Sigma).$$

Then the invariant part $A(\Sigma)_i$ of $A(\Sigma)$ is generated by $x$ and $\theta$. We decompose $A(\Sigma)$ in irreducible $\text{Sp}(2g, \mathbb{Z})$-representations as

$$A(\Sigma) = \bigoplus_{k=0}^{g} \Lambda_0^k \otimes \mathbb{Q}[x, \theta]/(\theta^{g+k+1}),$$

where

$$\Lambda_0^k = \Lambda_0^k H_1(\Sigma) = \ker(\theta^{g-k+1}: \Lambda^k H_1(\Sigma) \to \Lambda^{2g-k+2} H_1(\Sigma))$$

is the primitive component of $\Lambda^k H_1(\Sigma)$, for $0 \leq k \leq g$. Then, as the Seiberg–Witten invariant $SW_{S,s}(z)$ is invariant under the action of $\text{Diff}(\Sigma)$, $SW_{S,s}(z) = 0$ for any $z \in \bigoplus_{k=1}^{g} \Lambda_0^k \otimes \mathbb{Q}[x, \theta]/(\theta^{g+1-k})$. It only matters to compute $SW_{S,s}(z)$ for $z = x^n \theta^\rho$.

Lemma 3.3 Fix $0 < r \leq g - 1$ and $n \in \mathbb{Z}$. Set $d = \frac{g-1-r}{r}$. Then $SW_{S,s+r\mathbb{Z}[\Sigma]} = 0$ unless $n \leq -1$ and $D = rn + g + 1 \geq 0$ (there is only a finite number of possible values for $n$). In that case

$$SW_{S,s+n[\Sigma]}(x^n \theta^b) = \frac{g!}{(g-b)!}(-n)^{g-b},$$

for $a + b = D$, $0 \leq b \leq g$. Note that $D \leq d$ and $D \equiv d \pmod{r}$. As a consequence, for $n = -1$ (i.e., when $D = d$) we have $SW_{S,s+n[\Sigma]}(z) = \langle z, \text{Sym}^{d}(\Sigma) \rangle$, for any $z \in A(\Sigma)$ of degree $2d$.

Proof. Let $L$ be the determinant bundle of $s_r + n[\Sigma]$, so that $c_1(L) = 2r[\mathbb{S}^2] + 2n[\Sigma]$. Let $H = [\Sigma] + \epsilon[\mathbb{S}^2]$ be a polarization close to $[\Sigma]$, i.e., $\epsilon > 0$ small. Then $\deg_{H} L = 2r + 2n \epsilon > 0$, so by [3, Proposition 27] the non-perturbed Seiberg–Witten moduli space on $S$ is $\mathbb{P}(H^0(K \otimes L^*)^*)$, where $-K + 2L = L$, so $K - L = \frac{1}{2}\omega_0 \equiv (g-1-r)[\mathbb{S}^2] + (-1-n)[\Sigma]$. For $n \geq 0$ this is empty and hence $SW_{S,s+n[\Sigma]} = 0$.

For $n \leq -1$, $d(s_r + n[\Sigma]) = 2(rn + g - 1)$. Let $H_0 = [\Sigma] + \epsilon[\mathbb{S}^2]$ be a polarization close to $[\mathbb{S}^2]$, i.e., $\epsilon > 0$ small. Then $\deg_{H_0} L = 2\epsilon + 2n \leq 0$, so by [3, Proposition 27] the non-perturbed Seiberg–Witten moduli space on $S$ is $\mathbb{P}(H^0(L)^*)$, where $L = \frac{1}{2}\omega_0 \equiv (g-1+r)[\mathbb{S}^2] + (-1-n)[\Sigma]$. Hence the moduli space is empty and the Seiberg–Witten invariant for this polarisation, is zero. The Seiberg–Witten invariant $SW_{S,s+r[\Sigma]}$ is obtained via wall-crossing from [20]. With the notations therein, $u_r \in \Lambda^2 H_1(S; \mathbb{Z})$ is given by $u_r(\gamma_i \wedge \gamma_j) = \frac{1}{2}(\gamma_i \cup \gamma_j, c_1(L))$, i.e. $u_r = r\theta$, and

$$SW_{S,s+n[\Sigma]}(x^n \theta^b) = \left\langle \theta^b \left(\frac{u_r}{(g-b)!}\right)^{g-b}, [\text{Jac} S] \right\rangle = \frac{g!}{(g-b)!}(-n)^{g-b}.$$
Recall our basic set up. We have the three manifold

$$H_{FSW}^4(\Sigma) \cong H^{2d-i}(\text{Sym}^d(\Sigma)) \to \mathbb{Q}.$$  

So $\text{det} A_i = 0$, for $0 \leq i \leq 2d$. By the formula (3.4) and Lemma 3.3, $\langle e_i, e_j \rangle = 0$ if $\deg z_i + \deg z_j > 2d$ and $\langle e_i, e_j \rangle = (z_i, z_j)$ if $\deg z_i + \deg z_j = 2d$. Therefore the intersection matrix $\langle (e_i, e_j) \rangle$ is of the form

$$\begin{pmatrix}
0 & \cdots & * \\
0 & \cdots & A_1 \\
\vdots & \ddots & \vdots \\
A_{2d} & \cdots & 0
\end{pmatrix},$$

where $A_i$ are the sub-matrices corresponding to the pairing $H^*(\text{Sym}^d(\Sigma)) \otimes H^{2d-i}(\text{Sym}^d(\Sigma)) \to \mathbb{Q}$. So $\text{dim } V_r = \text{dim } H^*(\text{Sym}^d(\Sigma))$. As we already have the opposite inequality from Theorem 3.2, it must be $\text{dim } V_r = \text{dim } H^*(\text{Sym}^d(\Sigma)) \text{ and } \{z_i\}_{i \in I}$ is a basis for $V_r$. 

The proof of this proposition shows that the map (3.3) is surjective. We have the following

**Proposition 3.5** Let $Z = \Sigma \times D^2$, $S = \Sigma \times S^2$, $z \in \mathcal{A}(\Sigma)$ and $0 < |r| \leq g - 1$. Then the following are equivalent:

(i) $\phi^*_Z(s_r, z) = 0$.

(ii) $\text{SW}_{S,s_r+n[2]}(zz) = 0$, for all $i \in I$ and integer $n$.

(iii) $\text{SW}_{S,s_r+n[2]}(zz') = 0$, for all $z' \in \mathcal{A}(\Sigma)$ and integer $n$.

### 4 Ring structure of $HFSW^*(\Sigma \times S^1, s_r)$

Recall our basic set up. We have the three manifold $Y = \Sigma \times S^1$ together with the Spin$^C$ structure $s_r$ with $c_1(s_r) = 2rP.D.[S^1] \in H^2(Y; \mathbb{Z})$, $0 < |r| \leq g - 1$, and put $d = g - 1 - |r|$. We can define a product on $V_r = HFSW^*(Y, s_r)$ as follows. By Proposition 3.5,

$$T_g = \{ z \in \mathcal{A}(\Sigma) | \phi^*_Z(s_r, z) = 0 \}$$

is an ideal of $\mathcal{A}(\Sigma)$. So we define an associator and graded commutative ring structure on $V_r$ by

$$\phi^*_Z(s_r, z_1) \cdot \phi^*_Z(s_r, z_2) = \phi^*_Z(s_r, z_1 z_2),$$

for any $z_1, z_2 \in \mathcal{A}(\Sigma)$. We recall that $Z = \Sigma \times D^2$. Therefore $V_r = \mathcal{A}(\Sigma)/T_g$. This makes the map (3.3) an epimorphism of rings.

**Lemma 4.1** Let $Z = \Sigma \times D^2$, let $X_1$ be a 4-manifold with boundary $\partial X_1 = Y$ and let $s$ be a Spin$^C$ structure on $X_1$ such that $\phi|_Y = s_r$. Then for any $z_1 \in \mathcal{A}(X_1)$ and $z \in \mathcal{A}(\Sigma)$ we have

$$\phi^*_Z(s_r, z) \cdot \phi^*_Z(s_r, z_1) = \phi^*_Z(s_r, z z_1).$$

**Proof.** First, for any $\phi \in V_r$ we have $\langle \phi \cdot \phi^*_Z(s_r, z); \phi^*_Z(s_r, z) \rangle = \langle \phi, \phi^*_Z(s_r, z z) \rangle$. This is true since by the very definition of the product it holds for the elements $\phi = \phi^*_Z(s_r, z)$, which generate $V_r$.

Now for any $z \in \mathcal{A}(\Sigma)$ we have

$$\langle \phi^*_Z(s_r, z); \phi^*_Z(s_r, z) \rangle = \langle \phi^*_Z(s_r, z_1), \phi^*_Z(s_r, z z) \rangle = \text{SW}_{X_{s_r+n[2]}(z_1 z z)} = \langle \phi^*_Z(s_r, z_1 z z); \phi^*_Z(s_r, z) \rangle,$$

where $X = X_1 \cup Y$ and $n$ is a suitable integer determined by (3.5). By Proposition 3.5 we have the result. 

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Note that the isomorphism $V_r \cong V_{-r}$ intertwines the ring structures, so we may restrict to the case $r > 0$. Recall that $d = g - 1 - r$.

**Theorem 4.2** Denote by $\cdot$ the product induced in $H^*\big(\text{Sym}^d(\Sigma)\big)$ by the product in $V_r$ under the isomorphism of Proposition 3.4. Then $\cdot$ is a deformation of the cup product graded modulo $2r = 2(g - 1 - d)$, i.e. for $f_1 \in H^i\big(\text{Sym}^d(\Sigma)\big)$, $f_2 \in H^j\big(\text{Sym}^d(\Sigma)\big)$, it is $f_1 \cdot f_2 = \sum_{m \geq 0} \Phi_m(f_1, f_2)$, where $\Phi_m \in H^{i+j+2mr}\big(\text{Sym}^d(\Sigma)\big)$ and $\Phi_0 = f_1 \cup f_2$.

**Proof.** By Lemma 3.3 and Proposition 3.4, for any $i, j \in I$, $(e_i, e_j)$ is zero unless $\deg z_i + \deg z_j = 2d - 2mr$, with $m \geq 0$. Moreover, when $\deg z_i + \deg z_j = 2d$, it is $(e_i, e_j) = \langle z_i, z_j \rangle$. Now the same argument as in [16, Theorem 5] accomplishes the result, the only difference being that, in our present case, the deformation produces terms of increasing degrees.

**Corollary 4.3** Let $f \in A(\Sigma)$ be an homogeneous element of degree strictly bigger than $2d$. Then $f$ is zero in $V_r$.

The last ingredient that we need in order to describe $V_r$ is the cohomology $H^*\big(\text{Sym}^d(\Sigma)\big)$ of the $d$-th symmetric product $\text{Sym}^d(\Sigma)$ of the surface $\Sigma$. Here $d = g - 1 - r$, so $d$ is in the range $0 \leq d < g - 1$. This cohomology ring was initially described in [9] and revisited in [19, Section 4] where it was described as $\text{Sp}(2g, \mathbb{Z})$-representation, which is the form well suited for our purposes. Note that over the rational coefficients, $\Lambda(\Sigma) = \mathbb{Q}[x] \otimes \Lambda^*(\gamma_1, \ldots, \gamma_2g)$.

Put an auxiliary complex structure on $\Sigma$ and interpret $\text{Sym}^d(\Sigma)$ as the moduli space of degree $d$ effective divisors on $\Sigma$. Let $D \subset \text{Sym}^d(\Sigma) \times \Sigma$ be the universal divisor. Then from the slant product,

$$
\left\{
\begin{aligned}
\eta &= c_1(D)/x \in H^2(\text{Sym}^d(\Sigma)) , \\
\psi_i &= c_1(D)/\gamma_i \in H^1(\text{Sym}^d(\Sigma)) ,
\end{aligned}
\right. \quad 1 \leq i \leq 2g ,
$$

are generators of the ring $H^*(\text{Sym}^d(\Sigma))$, i.e. there is a graded $\text{Sp}(2g, \mathbb{Z})$-equivariant epimorphism

$$
A(\Sigma) \cong \mathbb{Q}[\eta] \otimes \Lambda^*(\psi_1, \ldots, \psi_{2g}) \rightarrow H^*(\text{Sym}^d(\Sigma)) .
$$

We set $\theta = \sum_{i=1}^{2g} \psi_i \psi_{-i} \in H^2(\text{Sym}^d(\Sigma))$, abusing a little bit notation since it correspond to the element $\theta = \sum_{i=1}^{2g} \gamma_i \wedge \gamma_{-i} \in A(\Sigma)$ under (4.1). Also we identify $A^0 = A^0(\psi_1, \ldots, \psi_{2g})$ under the same map. Clearly $\eta$ and $\theta$ generate the invariant part $H^*(\text{Sym}^d(\Sigma))$. The description of $H^*(\text{Sym}^d(\Sigma))$ as $\text{Sp}(2g, \mathbb{Z})$-representation is given in the following

**Proposition 4.4** ([19, Proposition 3.5]) For $0 \leq d \leq g - 1$ there is a presentation

$$
H^*(\text{Sym}^d(\Sigma)) = \bigoplus_{k=0}^{d} \Lambda^k \otimes \mathbb{Q}[\eta, \theta]/J^0_k ,
$$

where $J^0_k = (R^0_k, \theta R^0_{k+1}, \theta^2 R^0_{k+2}, \ldots, \theta^{d+1-k})$, $0 \leq k \leq d$, and

$$
R^0_k = \sum_{i=0}^{d-k} \binom{d-k}{i} (\theta^i)^{\alpha - i} ,
$$

for $0 \leq k \leq d$, with $\alpha = d + 1$ (for consistency, $R^0_{d+1} = 1$). Actually $J^0_k = (R^0_k, \theta R^0_{k+1})$. A basis for $\mathbb{Q}[\eta, \theta]/J^0_k$ as vector space is given by $\eta^a \theta^b$, with $2a + b \leq d - k$.

For the space $V_r$ we set

$$
\left\{
\begin{aligned}
\eta &= \phi^{SW}_r(x, x) \in V_r , \\
\psi_i &= \phi^{SW}_r(\delta_r, \gamma_i) \in V_r ,
\end{aligned}
\right. \quad 1 \leq i \leq 2g ,
$$

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where \( \eta \) has degree 2 and \( \psi_i \) degree 1, \( Z = \Sigma \times D^2 \). (We name with the same letters elements in \( V_r \) and in \( H^*(\text{Sym}^d(\Sigma)) \) as they are obviously related by (3.3) and (4.1).) These elements are generators of \( V_r \) as algebra. This means that (3.3) is a \( \text{Sp}(2g, \mathbb{Z}) \)-equivariant epimorphism

\[
A(\Sigma) \cong \mathbb{Q}[\eta] \otimes A^*(\psi_1, \ldots, \psi_{2g}) \twoheadrightarrow V_r.
\]

Clearly \( \eta \) and \( \theta = \sum_{i=1}^g \psi_i \psi_{g+i} \) generate the invariant part of \( V_r \). Now we are going to relate the ring structure of \( H^*(\text{Sym}^d(\Sigma)) \) with that of \( V_r \).

**Proposition 4.5** Let \( 0 < r \leq g-1 \) and set \( d = g-1-r \). Then there is a presentation

\[
V_r = \bigoplus_{k=0}^d \mathbb{A}_k^k \otimes \mathbb{Q}[\eta, \theta] / I_k^g
\]

where \( I_k^g = (\mathcal{R}_k^g, \theta \mathcal{R}_k^g, 1) \subset \mathbb{Q}[\eta, \theta] \) are ideals (dependent on \( g, k \) and \( r \)) such that

\[
\mathcal{R}_k^g = \sum_{i=0}^{\alpha + mr} \frac{a_{im}}{\mathcal{a}_{g}((g-k) - i)} \eta^{\alpha + mr - i} \theta^i,
\]

(4.2)

where \( \alpha = \left[ \frac{d-k}{2} \right] + 1, \mathcal{R}_k^g \) are given in Proposition 4.4 and \( a_{im} \) are some rational numbers (dependent on \( g, k, r \)). A basis for \( \mathbb{Q}[\eta, \theta] / I_k^g \) is given by \( \eta^{a} \theta^{b} \), with \( 2a + b \leq d - k \).

**Proof.** Let \( \{ z_i^{(k)} \} \) be a basis for \( \mathcal{A}_0^k \). Then by Proposition 4.4, \( z_i^{(k)} x^{a} \theta^{b} \), \( 2a + b + k \leq d \), form a basis for \( H^*(\text{Sym}^d(\Sigma)) \). We use this basis in Proposition 3.4 to construct a (\( \text{Sp}(2g, \mathbb{Z}) \)-equivariant) isomorphism \( H^*(\text{Sym}^d(\Sigma)) \cong V_r \). The fact that \( \mathcal{R}_k^g \subset J_k^g \) means that \( z_i^{(k)} \mathcal{R}_k^g = 0 \) in \( H^*(\text{Sym}^d(\Sigma)) \). Fix \( z_0^{(k)} = \psi_1 \ldots \psi_k \in \mathcal{A}_0^k \), then \( \mathcal{A}_0^g = \text{Span} \{ \text{Sp}(2g, \mathbb{Z}) z_i^{(k)} \} \). Rewriting \( z_0^{(k)} \mathcal{R}_k^g \) in terms of the product of Theorem 4.2, and using the arguments of [24, Section 2] (and the fact that the action of \( \text{Sp}(2g, \mathbb{Z}) \) is compatible with the ring structure on \( V_r \)), we get that

\[
z_0^{(k)} \mathcal{R}_k^g + \sum_{m>0} z_i^{(k)} \mathcal{R}_{km}^g = 0
\]

(4.3)

in \( V_r \), where \( \deg \mathcal{R}_{km}^g = \deg \mathcal{R}_k^g + mr = \alpha + mr, \) and \( \mathcal{R}_{km}^g \) can be expressed in terms of the chosen basis, i.e. as a linear combination of the monomials of the form \( \eta^{a + mr - j} \theta^j \) with \( 2a + 2mr - (d-k) \leq j \leq a + mr \). As in the proof of [17, Proposition 16], we have that the only nonvanishing \( \mathcal{R}_{km}^g \) in (4.3) correspond to \( z_0^{(k)} \) (otherwise one can find an element of \( \text{Sp}(2g, \mathbb{Z}) \) only fixing \( z_0^{(k)} \), which would produce a relation between the elements of the basis of \( V_r \), which is impossible), so (4.3) reduces to \( z_0^{(k)} (R_k^g + \sum_{m>0} \mathcal{R}_{km}^g) = 0 \) in \( V_r \). This produces the relation

\[
R_k^g = \mathcal{R}_k^g + \sum_{m>0} \mathcal{R}_{km}^g
\]

as stated in (4.2).

Also \( \theta \mathcal{R}_{k+1}^g \subset \mathcal{I}_k^g \) since \( \theta \mathcal{I}_{k+1}^g \subset \mathcal{I}_k^g \). Now \( I_k^g \) is generated by these two elements since \( J_k^g \) is generated by \( R_k^g \) and \( \theta R_{k+1}^g \) (see [24, Section 2]). \hfill \Box

**Remark 4.6** Note that for \( d \) odd, \( \mathcal{R}_k^g \) is the relation uniquely determined by expressing \( \eta^a \in \mathbb{Q}[\eta, \theta] / I_k^g \), \( a = \left[ \frac{d-k}{2} \right] + 1 \), in terms of the monomials of the basis of \( \eta^a \theta^b \), \( 2a + b \leq d - k \). For \( d \) even, \( \mathcal{R}_k^g \) is the relation uniquely determined by expressing \( \eta^a \in \mathbb{Q}[\eta, \theta] / I_k^g \), \( a = \left[ \frac{d-k}{2} \right] + 1 \), \( \eta^{a-1} \theta \) in terms of the monomials of the basis.

**Corollary 4.7** There is an isomorphism of associated graded rings

\[
\text{Gr}_g V_r \cong \text{Gr}_g H^*(\text{Sym}^d(\Sigma)),
\]

where \( d = g - 1 - r \). Let \( H^*(\Sigma \times S^1) \) be the instanton Floer homology of \( Y = \Sigma \times S^1 \) with \( \text{SO}(3) \)-bundle with \( w_2 = P.D. \mathbb{S}^1 \), which was computed in [16]. \( H^*(\Sigma \times S^1) \) can be decomposed [16, Proposition 20] according to the eigenvalues of \( \alpha = 2\mu(\Sigma) \) as

\[
H^*(\Sigma \times S^1) = \bigoplus_{r=-g-1}^{-} H_r^g.
\]
where \( \alpha \) has eigenvalue \( \lambda r \) (if \( r \) is odd) or \( 4\sqrt{-1} r \) (if \( r \) is even) on \( H_r \), \(-(g-1) \leq r \leq g-1 \). Then \([19, \text{Corollary 3.7}] \) gives an isomorphism

\[
Gr_0 HFSW^\ast(\Sigma \times S^1, s_r) \cong Gr_\gamma H_r ,
\]

where \( \gamma = -2 \sum \mu(\gamma_i) \mu(\gamma_{r+i}) \) as in [16].

**Lemma 4.8** Fix \( r > 0 \). Then the ideals of Proposition 4.5 satisfy the recursion \( I^g_k = I^g_{k-1} \), for \( k > 0 \) and \( r \leq g-2 \). Equivalently, \( \mathcal{R}_k^g = \mathcal{R}_{g-k}^g \).

**Proof.** By the computation of the Seiberg–Witten invariants of \( S \) in Lemma 3.3 and the invariance under the action of \( \text{Sp}(2g, \mathbb{Z}) \), we have

\[
SW_{S,s_r+n[S]}(\gamma_1 \cdots \gamma_k \gamma_{g+1} \cdots \gamma_{g+k} x^a \theta^b) = \frac{SW_{S,s_r+n[S]}}{(g-k)!}\left(\frac{g}{k}\right)^{-1} \theta^k x^a \theta^b
\]

for \( a + b = g - k - 1 - rn \). Therefore for any \( R \in A(\Sigma), z \in A(\Sigma), \)

\[
\langle \psi_1 \cdots \psi_{k-1} \psi_g R, \psi_{2g} z \rangle_g = \langle \psi_1 \cdots \psi_{k-1} R, z \rangle_{g-1} ,
\]

where the subindex means the genus of the surface \( \Sigma \). This implies the statement by Proposition 3.5. The last part follows from Remark 4.6.

Now we aim to compute the coefficients \( a_{im} \) of \( \mathcal{R}_0 = \mathcal{R}_0^g \) in (4.2). Let \( m > 0 \). We collect the coefficients together in a polynomial

\[
p_m(x) = \sum a_{im} x^{g-i} ,
\]

where we consider \( a_{im} = 0 \) for \( i \notin \{2\alpha + 2mr - d, \alpha + mr \} \cap \mathbb{Z}, \alpha = \left[ \frac{d}{2} \right] + 1 \). Note that there are a finite number of non-zero polynomials. By analogy we consider

\[
p_0(x) = (x-1)^{d-\alpha+1} x^d (d-\alpha-1) ,
\]

so that \( \mathcal{R}_0 = \sum \frac{a_{im}}{\binom{g}{i}} \theta^{g-i} \), as given in Proposition 4.4. By definition \( \mathcal{R}_0 = 0 \in V_r \), therefore we have

\[
\langle \mathcal{R}_0, \theta^a \theta^b \rangle = 0 ,
\]

whenever \( \alpha + a + b = d - kr, k \geq 0 \). Now using the computation of the invariants of \( S \) in Lemma 3.3, we get

\[
\sum_{m=0}^k \frac{a_{im}}{\binom{g}{i}} \theta^{g-i} = \sum_{m=0}^k \frac{(g-i)!}{(g-b-i)!} (k-m+1)^{g-b-i} = 0 ,
\]

for all \( k \geq 0 \) and \( 0 \leq b \leq d - \alpha - kr \). So

\[
\sum_{m=0}^k \frac{d^n}{dx^n} p_m(x) \bigg|_{x=k-m+1} = 0
\]

for all \( k \geq 0 \) and \( 0 \leq b \leq d - \alpha - kr \). By Taylor expansion, this is equivalent to saying that

\[
p_k(x) \equiv -(p_0(x+k) + p_1(x+k-1) + \ldots + p_{k-1}(x+1)) \pmod{(x-1)^{d-\alpha-kr+1}} .
\]

This condition, together with the fact that \( p_k(x) \) has degree \( g - (2\alpha + 2kr - d) \) and it is divisible by \( x^{g-(\alpha+kr)} \), uniquely determines \( p_k(x) \) by recursion.
For instance, let us calculate explicitly $p_1(x)$. From (4.5) we have that

\[
p_0(x + 1) = x^{d - a + 1}(x + 1)^{g - (d - a - 1)} = x^{g - r}(x + 1)^{a + r} = x^{g - r - r} \left( \sum_{k=0}^{a+r} \binom{\alpha + r}{k} 2^k (x - 1)^{\alpha + r - k},
\right.
\]

using that $d = g - 1 - r$. Now $p_1(x)$ is divisible by $x^{g - r}$, has degree $(g - \alpha - r) + (d - \alpha - r)$ and $p_1(x) \equiv -p_0(x + 1) \pmod{(x - 1)^{d - \alpha - r + 1}}$. Therefore

\[
p_1(x) = -x^{g - r - r} \sum_{k=2a+2r-d}^{a+r} \binom{\alpha + r}{k} 2^k (x - 1)^{\alpha + r - k}.
\]

This time we do not restrict the range for $i$. We can compute the rest of the coefficients $a_{im}$ for $m > 1$, by recurrence using this method but the result is a collection of rather cumbersome formulae which do not shed light on the ring structure of $V_r$. This is to no surprise: the shape of the relations $R_k^g$ depends on the basis of $Q[\eta, \theta]/I_k^g$ that we have chosen in Proposition 4.5, and this basis has been chosen rather arbitrarily. We shall present now a slightly modified version of the previous argument which computes explicitly (a full set of) relations for $V_r$, by just not fixing any basis for $Q[\eta, \theta]/I_k^g$. This leads to a closed formula for generators of the ideals $I^g_k$.

**Theorem 4.9** Let $0 < r \leq g - 1$ and set $d = g - 1 - r$. Then there is a presentation

\[
V_r = \bigoplus_{k=0}^{d} A_k^g \otimes \frac{Q[\eta, \theta]}{(\tilde{R}_k^g, \theta, R_{k+1}^g, \eta^{d+1}, \theta^{d+1})},
\]

where

\[
\tilde{R}_k^g = \sum_{i=0}^{\alpha} \frac{(d-k-\alpha+1)}{i!(\alpha - i)!} (-1)^i \eta^{a - i} \theta^i - \sum_{i=0}^{\alpha + r} \frac{(\alpha + r)}{i!(\alpha - i)!} \eta^{\alpha + r - i} \theta^i,
\]

where $\alpha = \left[ \frac{d-k}{2} \right] + 1$, for $0 \leq k \leq d$, and $\tilde{R}_{d+1}^g = 1$.

**Proof.** By Lemma 4.8 it is enough to find a relation for $k = 0$,

\[
\tilde{R}_0 = R_0 + \sum_{m=0}^{\infty} a_{im} \frac{\eta^{m+r-i} \theta^i}{i!(\alpha - i)!}.
\]

This time we do not restrict the range for $i$. We only note that we can suppose $a_{im} = 0$ if $i > g$, since $\theta^{g+1} = 0$. As before, we collect the coefficients $a_{im}$ of (4.7) in a polynomial $p_m(x) = \sum a_{im} x^{g-i}$, where $a_{im} = 0$ for
We want to show, as a first application, how the knowledge of the previous sections can be used to compute the Seiberg–Witten invariants of connected sums along surfaces.

Let \( \mathcal{R}_0 \) be a relation that \( R_k \) is an element in the image of \( \mathcal{R}_0 \). This implies that \( b^+ > 0 \). Now take small closed tubular neighbourhoods \( N_{\Sigma_i} \) of \( \Sigma_i \) which are diffeomorphic to \( \Sigma = \Sigma \times D^2 \). Let \( X_i \) be the closure of \( \overline{X}_i - N_{\Sigma_i}, i = 1, 2 \). Then \( X_i \) is a 4-manifold with boundary \( \partial X_i = \Sigma = \Sigma \times S^1 \) and \( \overline{X}_i = X_i \cup Y \subset Z \).

Take an identification \( \phi : \partial X_1 \rightarrow -\partial X_2 \) (i.e. an orientation reversing bundle isomorphism). We define the connected sum of \( \overline{X}_1 \) and \( \overline{X}_2 \) along \( \Sigma \) as

\[
X = X(\phi) = X_1 \cup_{\phi} X_2.
\]

The resulting 4-manifold depends in general on the isotopy class of \( \phi \), but we shall drop \( \phi \) from the notation when there is no danger of confusion, and write then \( X = \overline{X}_1 \#_{\Sigma} \overline{X}_2 \). Note that \( \Sigma \) defines an embedded surface \( \Sigma \subset X \) of genus \( g \) and self-intersection zero.

Consider Spin\(^c\) structures \( s_i \) on \( X_i \) such that \( s_1|_{Y} \cong -s_2|_{Y} \cong s_Y \), with \( c_1(s_Y) \neq 0 \), so that they can be glued together to get a Spin\(^c\) structure \( s_0 \) on \( X \). The Spin\(^c\) structures \( s \) such that \( s|_{X_i} = s_i, i = 1, 2 \), are those of the form \( s_0 + h \), where \( h \) is an element in the image of \( H_2(Y; \mathbb{Z}) \rightarrow H_2(X; \mathbb{Z}) \cong H^2(X; \mathbb{Z}) \), where the last map is Poincaré duality. Let \( \mathcal{R}im \subset H^2(X; \mathbb{Z}) \) be the subspace generated by the rim tori [6], i.e. the image of \( H_1(\Sigma; \mathbb{Z}) \oplus [S^1] \subset H_2(Y; \mathbb{Z}) \) in \( H^2(X; \mathbb{Z}) \). Then any Spin\(^c\) structure \( s \) such that \( s|_{X_i} = s_i, i = 1, 2 \), is of the form \( s = s_0 + h + n[S^1] \), where \( h \in \mathcal{R}im, n \in \mathbb{Z} \).

If \( s_Y \) has \( c_1(s_Y) \neq 0 \) or \( P.D.[S^1] \), for any \( -(g-1) \leq r \leq g-1 \), then Theorem 3.2 tells us that \( SW_{X,s} = 0 \). Now consider the case \( s_Y = s_0 \), with \( -(g-1) \leq r \leq g-1, r \neq 0 \). Set \( d = g-1 - |r| \) as usual.

**Theorem 5.1** Fix \( z_i \in H(\Sigma), i \in I \), homogenous elements such that \( \{z_i\}_{i \in I} \) is a basis for \( H^*(s^4\Sigma) \). Then there exists a universal matrix \((m_{ij})_{i,j \in I}\) such that for every connected sum \( X = \overline{X}_1 \#_{\Sigma} \overline{X}_2 \) along a surface of genus \( g \), Spin\(^c\) structures \( s_i \) on \( X_i \) with \( (c_1(s_i), [\Sigma_i]) = 2r \) and Spin\(^c\) structure \( s_0 \) on \( X \) obtained by gluing \( s_1 \) and \( s_2 \), we have

\[
\sum_{h \in \mathcal{R}im} SW_{X,s_0 + h, (z_1 z_2)} = \sum_{n,m \in \mathbb{Z}, i,j \in I} m_{ij} SW_{\overline{X}_1,s_1 + n[\Sigma_1]}(z_1 z_i) \cdot SW_{\overline{X}_2,s_2 + m[\Sigma_2]}(z_2 z_j),
\]
for any \( z_1 \in A_t(\overline{X}_1) \) and \( z_2 \in A_t(\overline{X}_2) \) with \( d(s) = \deg z_1 + \deg z_2 \) (note that at most one \( n \) and one \( m \) appear in every summand of the right-hand side). If any of the manifolds involved has \( b^+ = 1 \) then its Seiberg–Witten invariants are computed for the component of the positive cone containing \( -rP.D.[\Sigma] \).

**Proof.** Let \( s_i = z_i|_{\overline{X}_i} \), \( i = 1, 2 \). By Proposition 3.4, the elements \( e_i = \phi_{\overline{X}_1}^SW(s_i, z_1) \), \( \in I \), form a basis for \( V_s = HFSW^*(Y, s_i) \). Therefore the map \( V_s \to \mathbb{R}[I], \) given by \( \phi \mapsto \left( \langle \phi, \phi_{\overline{X}_1}^SW(s_i, z_1) \rangle \right)_{i \in I}, \) is an isomorphism such that

\[
\phi_{\overline{X}_1}^SW(s_1, z_1) \mapsto \left( \sum_{n \in \mathbb{Z}} \delta_{X_1, s_1 + n[\Sigma]}(z_1 z_i) \right)_{i \in I}.
\]

Theorem 2.2 says that

\[
\sum_n \delta_{X_1, s_1 + n[\Sigma]}(z_1 z_i) = \left( \phi_{\overline{X}_1}^SW(s_1, z_1), e_i \right),
\]

\[
\sum_m \delta_{X_2, s_2 + m[\Sigma]}(z_2 z_i) = \left( \phi_{\overline{X}_2}^SW(s_2, z_2), e_j \right)
\]

and

\[
\sum_{\{s, s\} \cup I = 1, 2} \delta_{X_1, s_1 + n[\Sigma]}(z_1 z_2) = \left( \phi_{\overline{X}_1}^SW(s_1, z_1), \phi_{\overline{X}_2}^SW(s_2, z_2) \right).
\]

Only the \( \text{Spin}^C \) structures of the form \( s = s_o + h, h \in \mathcal{R}_m \), satisfy \( d(s) = \deg z_1 + \deg z_2 \). The result follows from (\( m_{ij} \)) being the inverse of the intersection matrix for the basis \( \{e_i\}_{i \in I}. \) Note that this matrix is explicitly computable, since by Lemma 3.3 the products \( \langle e_i, e_j \rangle \) are known. 

**Corollary 5.2** If either of \( \overline{X}_1 \) has simple type then \( \sum_{\{s, s\} \cup I = 1} \delta_{X_1, s_1 + h}(x z) = 0 \), for any \( z \in A(X) \) and \( \text{Spin}^C \) structure \( s_o \) on \( X \) with \( \langle c_1(s_o), [\Sigma] \rangle \neq 0 \). Analogously, if either of \( \overline{X}_1 \) has strong simple type then

\[
\sum_{\{s, s\} \cup I = 1} \delta_{X_1, s_1 + h}(z) = 0,
\]

for any \( z \in A(X) \) with \( \deg(z) > 0 \) and any \( \text{Spin}^C \) structure \( s_o \) on \( X \) with \( \langle c_1(s_o), [\Sigma] \rangle \neq 0 \).

In order to remove the summation over the subspace \( \mathcal{R}_m \) in Corollary 5.2 we need an extra condition.

**Definition 5.3** A connected sum \( X = X_1 \#_\Sigma X_2 \) is admissible if \( \mathcal{R}_m \) has no torsion and there exists a subspace \( V \subset H^2(X; \mathbb{Z}) \) such that \( H^2(X, \mathbb{Z}) = V \oplus \mathcal{R}_m \) and \( c_1(s) \in V \) for every basic class \( s \) of \( X \).

**Remark 5.4** Suppose that \( \overline{X}_1 \) and \( \overline{X}_2 \) are Kähler surfaces and \( \Sigma_1 \subset \overline{X}_1 \) are smooth complex curves of genus \( g \), isomorphic as complex curves, with (holomorphic) trivial bundle, such that there is a deformation Kähler family \( \mathcal{Z} \xrightarrow{\pi} D^2 \subset \mathbb{C} \) with fiber \( Z_t = \pi^{-1}(t), t \neq 0 \), smooth and \( Z_0 = \pi^{-1}(0) = \overline{X}_1 \cup_\Sigma \overline{X}_2 \), the union of \( \overline{X}_1 \) and \( \overline{X}_2 \) along \( \Sigma_1 = \Sigma_2 \), with a normal crossing. Then the general fiber \( Z_t = Z_0 \) is the connected sum \( X = X_1 \#_\Sigma X_2 \) with identification given by the isomorphism between the normal bundles of \( \Sigma_1 \) and \( \Sigma_2 \). If \( H^2(X, \mathbb{Z}) \) has no torsion then this identification is admissible, since for any basic class \( s \) one has \( c_1(s) \in H^{1,1} \) and this space is orthogonal to \( \mathcal{R}_m \), as for any \( T \in \mathcal{R}_m \), it is \( T^2 = 0 \) and \( [\omega] \cdot T = 0 \) (\( \omega \) standing for the Kähler form). This implies that \( T \notin H^{1,1} \) unless \( T = 0 \).

**Remark 5.5** In [11, Definition 4.1], Morgan and Szabó define admissible identification when there exists a collection of primitive embedded \( (-2) \)-spheres in \( X \) (obtained by pasting embedded \( (-1) \)-discs in \( X_1 \) and \( X_2 \)) generating a subspace \( V \subset H_2(X) \) such that \( H_2(X) = H \oplus V \), where \( H = \{ D \in H_2(X) | D|_Y = k[\Sigma] \}, \) some \( k \).

Then \( c_1(s) \cdot V = 0 \) for any basic class \( s \), and this implies admissibility in the sense of Definition 5.3 (assuming again that \( H^2(X, \mathbb{Z}) \) has no torsion).

**Corollary 5.6** Suppose that the connected sum \( X = X_1 \#_\Sigma X_2 \) is admissible. If either of \( \overline{X}_1 \) has simple type then \( \delta_{X_1, s_1 + h}(x z) = 0 \) for any \( z \in A(X) \) and any \( \text{Spin}^C \) structure \( s \) with \( \langle c_1(s), [\Sigma] \rangle \neq 0 \).

If either of \( \overline{X}_1 \) has strong simple type then \( \delta_{X_1, s_1 + h}(z) = 0 \) for any \( z \in A(X) \) with \( \deg(z) > 0 \) and any \( \text{Spin}^C \) structure \( s \) with \( \langle c_1(s), [\Sigma] \rangle \neq 0 \).
The formula in Theorem 5.1 becomes simpler when both $\mathcal{X}_i$ have $b_i = 0$. We have the following result

**Theorem 5.7** Let $X = \mathcal{X}_1 \# \Sigma \mathcal{X}_2$ be a connected sum along a surface of genus $g$ where both $\mathcal{X}_1$ and $\mathcal{X}_2$ are of $H_1$-simple type. Let $\delta_i$ be Spin$^c$ structures on $\mathcal{X}_i$ with $|c_1(\delta_i), [\Sigma_i]| = 2r \neq 0$ and let $\delta_o$ be a Spin$^c$ structure on $X$ obtained by gluing $\delta_1$ and $\delta_2$. Suppose $d = g - 1 - |r| \geq 0$. Let $z \in A(X)$ with $\deg(z) = d(\delta_o)$. Then

$$\sum_{h \in R_{im}} \SW_{\mathcal{X}_1, \delta_i + h(z)} = \begin{cases} \sum_{n,m \in \mathbb{Z}} (-1)^{d/2} (g-1) \SW_{\mathcal{X}_1, \delta_i + [\Sigma_i]} (x^{d/2}) \cdot \SW_{\mathcal{X}_2, \delta_2 + m[\Sigma_2]} (x^{d/2}), & z = 1, \ d \text{ even} \\ 0, & \text{otherwise}. \end{cases}$$

**Note that at most one $n$ and one $m$ contribute to this formula.**

**Proof.** By Lemma 4.1, $\phi^\SW_{\mathcal{X}_1}(z_1, z_1) = 0$, for $j = 1, \ldots, 2g$. Therefore

$$\phi^\SW_{\mathcal{X}_1}(z_1, z_1) \in K = \bigcap_{1 \leq j \leq 2g} \ker \psi_j,$$

for any $z_1 \in A(X_1)$. Since $K \subset V_r$ is invariant under the action of $Sp(2g, \mathbb{Z})$, we have that $K \subset \mathbb{Q}[\eta, \theta]/I_0$ by Proposition 4.5. Now $f \in K$ if and only if $\psi_j f = 0$, for $j = 1, \ldots, 2g$. This means that $f \in I_1^g$ in the notation of Proposition 4.5. So $K = I_1^g/I_0^g$, where the generators of $I_1^g$ are $R_1^g$ and $\theta R_1^g$. The intersection pairing $\langle \cdot, \cdot \rangle : K \otimes K \to \mathbb{Q}$ is the restriction of the pairing of $V_r$. Now by Lemma 3.3, $\langle c, c \rangle = 0$ if $\deg z_1 + \deg z_2 > 2d$. For $d$ odd, all the homogeneous components of all the elements in $I_1^g$ have degree strictly bigger than $d$ (note that the component $R_1^g$ of $R_1^g$ has degree $2\left(\frac{d+1}{2}\right) + 1 = d + 1$ and it is the component of lowest degree). So $K \otimes K \to \mathbb{Q}$ is the zero map for $d$ odd, which proves the second line.

For $d$ even, all the homogeneous components of all the elements in $I_1^g$ have degree strictly bigger than $d$, except for $R_1^g$, which has degree $d$. So $K \otimes K \to \mathbb{Q}$ has rank 1. Hence, for $z_1 \in A(\Sigma)$ and $z_2 \in A(\Sigma)$, we have that

$$\sum_{h \in R_{im}} \SW_{\mathcal{X}_1, \delta_i + h(z_1 z_2)} = \sum_{n,m \in \mathbb{Z}} c \SW_{\mathcal{X}_1, \delta_1 + n[\Sigma_1]} (z_1 x^{d/2}) \SW_{\mathcal{X}_2, \delta_2 + m[\Sigma_2]} (z_2 x^{d/2}),$$

where we have used $\SW_{\mathcal{X}_1, \delta_1 + n[\Sigma_1]} (z_1 R_1^g) = \SW_{\mathcal{X}_1, \delta_1 + n[\Sigma_1]} (z_i x^{d/2})$, and with $c = \langle R_1^g, R_1^g \rangle^{-1}$. To compute $c$, note that $\theta R_1^g = 0$ so

$$\langle R_1^g, R_1^g \rangle = \langle R_1^g, \eta^{d/2} \rangle = \sum_{i=0}^{\alpha} \frac{\binom{d/2}{i}}{i!(g-1)!} (-1)^i \frac{g^i}{(g-i)!} = \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} \frac{g^i}{(g-i)!} = (-1)^\alpha \binom{g-1}{\alpha}^{-1},$$

with $\alpha = d/2$. Finally Corollary 4.3 implies that $\SW_{\mathcal{X}_1, \delta_1 + n[\Sigma_1]} (z_1 x^{d/2}) = 0$ for any $z_1$ with $\deg(z_1) > 0$. \hfill $\square$

The following corollary is analogue to the result in [15, Corollary 15] regarding the Kronheimer-Mrowka basic classes.

**Corollary 5.8** Suppose $\mathcal{X}_1$ is of strong simple type and $\mathcal{X}_2$ has $H_1$-simple type. Suppose that the connected sum $X = \mathcal{X}_1 \# \Sigma \mathcal{X}_2$ is admissible. Then there are no basic classes $s$ with $0 < |c_1(s), [\Sigma]| < 2g - 2$. The basic classes for $X$ with $|c_1(s), [\Sigma]| = \pm (2g - 2)$ are indexed by pairs of basic classes $(\delta_1, \delta_2)$ for $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively, such that $(\delta_1, [\Sigma_1]) = (\delta_2, [\Sigma_2]) = \pm (2g - 2)$. 

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6 Higher type adjunction inequalities

In this section we shall reprove the higher type adjunction inequalities for non-simple type 4-manifolds obtained by Osszváth and Szabó in [21]. Our method of proof is considerably simpler and parallels the proof of the higher type adjunction inequalities in the context of Donaldson invariants given in [19]. On the weak side, we cannot deal with the case \( \langle c_1(\mathcal{L}), [\Sigma] \rangle = 0 \) due to the fact that we have to restrict to non-torsion Spin\(^c\) structures for our study of the Seiberg–Witten–Floer homology.

Proof of Theorem 1.4. Without loss of generality, by reversing the orientation of \( \Sigma \) in the case \( b^+ > 1 \), we can suppose that \( \langle c_1(\mathcal{L}), [\Sigma] \rangle \leq 0 \). We reduce to the case of self-intersection zero by blowing-up. Let \( n = [\Sigma]^2 \) and consider the blow-up \( \tilde{X} = X \# nCP^2 \) with exceptional divisors \( E_1, \ldots, E_n \). Let \( \Sigma = E_1 - \cdots - E_n \) be the proper transform of \( \Sigma \), which is an embedded surface of self-intersection zero and genus \( g \), with \( b \in \mathbb{A}(\Sigma) \cong \mathbb{A}(\Sigma) \). Consider the Spin\(^c\) structure \( \mathcal{L} \) on \( \tilde{X} \) with \( c_1(\mathcal{L}) = c_1(\mathcal{L}) - E_1 - \cdots - E_n \). Then \( d(\mathcal{L}) = d(\mathcal{L}) \),

\[
-\langle c_1(\mathcal{L}), [\Sigma] \rangle + [\Sigma]^2 = -\langle c_1(\mathcal{L}), [\Sigma] \rangle + [\Sigma]^2,
\]

and \( SW_{\mathcal{L}}(a,b) = SW_{\mathcal{L}}(a,b) \neq 0 \).

Therefore we can suppose that \( [\Sigma]^2 = 0 \) and \( \langle c_1(\mathcal{L}), [\Sigma] \rangle = -2r, \) with \( 0 < r \leq g - 1 \). Let \( \{ \gamma_i \} \) be a symplectic basis of \( H_1(\Sigma) \) with \( \gamma_1 + \cdots + \gamma_{i+1} = 1, 1 \leq i \leq g \). Without loss of generality we may also suppose that \( b = x^p \gamma_1 \cdots \gamma_{i+1}, \) with \( \deg(b) = 2p + m \). Now let \( \tilde{Z} = \Sigma \times D^2 \) be a small tubular neighbourhood of \( \Sigma \subset X \) and consider the splitting \( X = X_1 \cup_{\eta} \tilde{Z} \), where \( X_1 \) is the closure of the complement of \( \tilde{Z} \) and \( \partial X_1 = \partial \tilde{Z} = \Sigma = \Sigma \times S^1 \). In this case \( \mathcal{L} \) is the only Spin\(^c\) structure appearing in the gluing formula in Theorem 2.2, so

\[
0 \neq SW_{X_1}(a,b) = \langle \phi^{SW}_{X_1}(a,a), \phi^{SW}(\mathcal{L},\mathcal{L}) \rangle.
\]

(In the case \( b^+ = 1 \), the metric giving a long neck for \( X \) has period point close to \( [\Sigma] \). Therefore we are calculating the invariants in the component \( K(X) \) of the positive cone containing PD, \( [\Sigma] \in H^2(X; \mathbb{Z}) \), since \( \langle c_1(\mathcal{L}), [\Sigma] \rangle < 0 \).) Then \( \phi^{SW}_{X_1}(a,a) \in HFSW^*(\tilde{Z},\mathcal{L}) = V_{-r} \cong V_r \) is non-zero and therefore \( \eta \psi_1 \cdots \psi_{i+1} \neq 0 \) \( \forall V_r \). By Corollary 4.3 this implies \( 2p + m \leq 2d = 2(g - 1 - \deg(b)) \). Therefore \( 2r + \deg(b) \leq 2g - 2 \).

Proof of Theorem 1.6. Again we may suppose that \( [\Sigma]^2 = 0 \) and \( \langle c_1(\mathcal{L}), [\Sigma] \rangle = -2r, \) with \( 0 < r \leq g - 1 \). Suppose also that \( b = x^p \gamma_1 \cdots \gamma_{i+1}, \) with \( \deg(b) = 2p + m \). Now let \( \tilde{Z} = \Sigma \times D^2 \) be a small tubular neighbourhood of \( \Sigma \subset X \) and consider the splitting \( X = X_1 \cup_{\eta} \tilde{Z} \). Then

\[
0 \neq SW_{X_1}(a,b) = \langle \phi^{SW}_{X_1}(a,a), \phi^{SW}(\mathcal{L},\mathcal{L}) \rangle.
\]

Here \( \psi_1 \cdots \psi_{i+1} \) lives in the kernels of \( \psi_1, \ldots, \psi_{i+1} \), since as \( \eta \psi_1 \cdots \psi_{i+1} \) \( \in H_1(X) \),

\[
\psi_j, \phi^{SW}_{X_1}(a,a) = \phi^{SW}_{X_1}(a,\gamma_j a) = 0, \quad j = 1, \ldots, l.
\]

Therefore it must be \( \phi^{SW}_{X_1}(a,a) = \eta \psi_1 \cdots \psi_{i+1} \notin (\psi_1, \ldots, \psi_l) \) in \( V_{-r} \). The argument in [19, Proposition 4.5] (using Proposition 4.5) shows that any element of degree strictly bigger than \( g - 1 - \deg(b) \) must lie in the ideal \((\psi_1, \ldots, \psi_{i+1}) \) of \( V_{-r} \). So if \( l \geq g - 1 - \deg(b) \), then \( 2p + m \leq g - 1 - \deg(b) \), i.e., \( \deg(b) \leq g - 1 - |r| \) and \( 2r + 2 \deg(b) \leq 2(g - 1) \). On the other hand, if \( l + 1 \leq g - 1 - |r| \) then obviously \( 2r + 2 \deg(b) \leq 2(g - 1) \) as \( \deg(b) \leq 1 \) by hypothesis.

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References


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[16] V. Muñoz, Ring structure of the Floer cohomology of \( \Sigma \times S^1 \), Topology 38, 517–528 (1999).