$\mathbb{Z}$-graded monopole homology and truncated relative Seiberg-Witten invariants

Matilde Marcolli and Bai-Ling Wang

Abstract

We define a $\mathbb{Z}$-graded monopole homology which gives an integer lift of the periodically graded monopole homology for any closed 3-manifold with a non-torsion Spin^c structure. We construct a spectral sequence, whose E^1-term coincides with this lift, converges to the periodically graded monopole homology. As an application, we define a trancated version of the relative Seiberg-Witten invariants that take values in the $E^k_{*,*}$ of the spectral sequence.

1 Introduction

For manifolds with $b_1(Y) > 0$ we know (cf. [3], [10], [11]) that there is a well defined $\mathbb{Z}_\ell$ graded Seiberg-Witten-Floer homology $HF^{SW}_*(Y, \mathfrak{s})$, for every choice of a non-trivial Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, where the integer $\ell = \ell(\mathfrak{s})$ satisfies

$$\ell(\mathfrak{s}) = g.c.d.\{\langle c_1(L), \sigma\rangle : \sigma \in H_2(Y, \mathbb{Z}))\}.$$

Here $L = \det W$ is the determinant line bundle of the spinor bundle W associated to the Spin^c structure $\mathfrak{s}$. In the following, we use equivalently the notation $L = \det \mathfrak{s}$ and $c_1(\mathfrak{s})$ to denote the corresponding Chern class. Here we are assuming that $c_1(\mathfrak{s})$ is non-torsion, hence $\ell \neq 0$.

The first issue regarding the Floer homology $HF^{SW}_*(Y, \mathfrak{s})$ that we discuss in this paper is the integer lift of the $\mathbb{Z}_\ell$ graded Floer homology. The arguments we develop in this section hold for any 3-manifold $(Y, \mathfrak{s})$ with non-trivial rational homology and with a Spin^c-structure satisfying $c_1(\mathfrak{s})(H_2(Y, \mathbb{Z})) = \ell\mathbb{Z} \neq 0$.

Analogous constructions of integer lifts of Floer homologies were derived by Fintushel and Stern [5], in the case of instanton homology, and by Weiping Li [7], in the case of symplectic Floer homology. We follow closely the construction of [5] and show that, in our case, there is a well defined integer lift of $HF^{SW}_{*,(\omega)}(Y,\mathfrak{s})$ of the $\mathbb{Z}_\ell$-graded Floer homology. Here $\omega \in \mathbb{R}$ is a regular value of the Chern-Simons-Dirac functional on the infinite cyclic cover space of the gauge equivalence classes of connections and spinor sections.

By studying the Chern-Simons-Dirac function on this infinite cyclic cover space, we will define an integer lift $\mathfrak{i}^{(\omega)}_Y$ of the indices of the critical points. We thus form a chain complex $C^{(\omega)}_*(Y,\mathfrak{s})$ depending on $\omega \in \mathbb{R}$. For any $n \in \mathbb{Z}_\ell$, the original $\mathbb{Z}_\ell$-graded Seiberg–Witten–Floer chain complex satisfies

$$C_n(Y,\mathfrak{s}) = \bigoplus_{k\in\mathbb{Z}} C^{(\omega)}_{j+k\ell}(Y,\mathfrak{s})$$

where $j = n(mod\ell)$.

In general, after defining a suitable boundary operator on $C^{(\omega)}_*(Y,\mathfrak{s})$, we observe that the resulting homology groups $HF^{SW}_{*,(\omega)}(Y,\mathfrak{s})$ do not satisfy the simple relation $\bigoplus_{k\in\mathbb{Z}} HF^{SW}_{*+k\ell,(\omega)}(Y,\mathfrak{s}) = HF^{SW}_*(Y,\mathfrak{s})$. However, the Floer homologies $HF^{SW}_{*+k\ell,(\omega)}(Y,\mathfrak{s})$ and $HF^{SW}_*(Y,\mathfrak{s})$ are related via a spectral sequence determined by a filtration of the chain complex $C_*(Y,\mathfrak{s})$. This spectral sequence converges to $HF^{SW}_*(Y,\mathfrak{s})$, and the E^1 term coincides with $HF^{SW}_{*,(\omega)}(Y,\mathfrak{s})$.

Theorem 1.1. *There exists a spectral sequence $E^k_{q,n}(Y,\mathfrak{s}), d^k)$ with $k > 0$, $n \in \mathbb{Z}_\ell$ and $q \in \mathbb{Z}$ with $q \equiv n(mod\ell)$, which is a topological invariant of $(Y,\mathfrak{s})$. Moreover, the spectral sequence $(E^k_{q,n}(Y,\mathfrak{s}), d^k)$ converges to the $\mathbb{Z}_\ell$-graded homology groups $HF^{SW}_*(Y,\mathfrak{s})$ and whose E^1-term gives the $\mathbb{Z}$-graded homology $HF^{SW}_{*,(\omega)}(Y,\mathfrak{s})$.*

We show that there are *truncated relative invariants* of 4-manifolds with boundary, that take values in the terms $E^k_{q,n}$ of the spectral sequence.

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2 Seiberg–Witten–Floer homology

In the following we present a construction of an integer lift of Seiberg–Witten–Floer homology for a compact 3-manifold Y with $b_1(Y) > 0$, and a non–torsion Spin^c-structure.

We recall some preliminary notions.

Let $\mathcal{M}_Y(\mathfrak{s})$ be the moduli space of gauge classes of solutions of suitably perturbed Seiberg–Witten equations on a 3-manifold $(Y, \mathfrak{s})$, with $b_1(Y) > 0$ and $c_1(\mathfrak{s})$ a non-torsion element. A perturbation of the 3-dimensional monopole equations by a co-closed 1-form ρ on Y can be chosen so that all the solutions in $\mathcal{M}_Y(\mathfrak{s})$ are irreducible critical points. Under a generic choice of such perturbation $\mathcal{M}_Y(\mathfrak{s})$ is a compact, oriented, 0-dimensional manifold, cut out transversely by the equations inside the configuration space $\mathcal{B} = \mathcal{A}/\mathcal{G}$, where $\mathcal{A}$ is the space of pairs formed by a $U(1)$-connection on $det(\mathfrak{s})$ and a spinor section of W, and $\mathcal{G}$ is the group of gauge transformations on $(Y, \mathfrak{s})$ (see [8] [10] [11]).

Seiberg–Witten–Floer homology is the homology of a Morse–Smale–Witten complex associated to the Chern–Simons–Dirac functional on $\mathcal{A}$,

$$\begin{aligned} CSD(A,\psi) &= -\frac{1}{2}\int_Y (A - A_0)\wedge (F_A + F_{A_0} - 2\sqrt{-1} * \rho) \\ &\quad + \int_Y \langle \psi, \partial_A \psi\rangle dvol_Y. \end{aligned}$$

A suitable class of perturbations that achieves transversality of the moduli spaces of flow lines is introduced in §2 of [11]. For a different type of perturbations, more conveniently defined directly as perturbations of the Chern–Simons–Dirac functional, see [2] (cf. [4] [6]).

In particular, we work under the hypothesis that the additional perturbation of the Chern–Simons–Dirac functional is gauge invariant by construction (as in [2]), so that the perturbed CSD changes under gauge action by

$$CSD(\lambda.(A,\psi)) - CSD(A,\psi) = \langle (8\pi^2 c_1(L) + 4\pi[*\rho]) \cup [\lambda], [Y]\rangle,$$

where $[\lambda] \in H^1(Y,\mathbb{Z})$ determines the connected component of $\mathcal{G}$ that contains λ, and is represented by the closed 1-form $\frac{1}{2\pi\sqrt{-1}}\lambda^{-1} d\lambda$.

The Seiberg–Witten–Floer homology groups $HF_*^{SW}(Y,\mathfrak{s})$ (cf. [3], [10], [11]) are the homology groups of $(C_*(Y,\mathfrak{s}), \partial)$, where $C_*(Y,\mathfrak{s})$ is generated by the critical points of the perturbed CSD on $\mathcal{B}$. The entries of the boundary operator ∂ are defined by counting the points in the zero dimensional components of the moduli space of unparameterized flow lines of the perturbed functional CSD on $\mathcal{B}$. A considerable amount of technical work goes into checking that the moduli spaces of flow lines have all the desired properties that make this definition rigorous, and we refer the reader to [11] for a detailed account. The resulting Floer homology $HF_*^{SW}(Y,\mathfrak{s})$ is $\mathbb{Z}_\ell$-graded with $\ell\mathbb{Z} = c_1(\mathfrak{s})(H_2(Y,\mathbb{Z}))$. This is due to an ambiguity in the index formula computed in terms of the spectral flow of the Hessian operator for CSD around a loop in $\mathcal{B}$. To understand this ambiguity, let (A, ψ) represent a critical point a of CSD on $\mathcal{B}$, let λ be a gauge transformation whose class $[\lambda] \in H^1(Y, \mathbb{Z})$ is non-trivial. Then the spectral flow of the Hessian operator H from $\lambda.(A, \psi)$ to (A, ψ) can be calculated from the index formula for the linearization of the 4-dimensional Seiberg–Witten equations on $Y \times S^1$. This gives

$$SF(H)^{(A,\psi)}_{\lambda.(A,\psi)} = \langle [\lambda] \cup c_1(\mathfrak{s}), [Y] \rangle \in \ell\mathbb{Z}. \tag{1}$$

Notice that the periodicity ℓ is an even number. In fact, the map $\mathcal{S}(Y) \to H^2(Y, \mathbb{Z})$, given by $\mathfrak{s} \mapsto c_1(\det \mathfrak{s})$, is equivariant with respect to the action of $H^2(Y, \mathbb{Z})$, so that we have $\det(W \otimes H) = L \otimes H^2$, with $L = \det(W) = \det(\mathfrak{s})$ and $H \in H^2(Y, \mathbb{Z})$, cf. [9].

There is a natural and non-trivial way to lift these $\mathbb{Z}_\ell$-graded homology groups to $\mathbb{Z}$-graded homology groups, so that the Euler characteristic number agrees with the original one. Following the idea of [5], we will discuss this integer lift and construct a spectral sequence which has the integer lift as E^1 term and converges to the original $\mathbb{Z}_\ell$-graded Floer homology.

3 $\mathbb{Z}$-graded homology groups and the spectral sequence

The construction we present in this section holds for any 3-manifold Y with $b_1(Y) > 0$ and for any choice of a Spin^c-structure with non-torsion class $c_1(L)$.

As explained in [11], there is a cyclic covering of $\mathcal{B}$, obtained by taking the quotient of $\mathcal{A}$, the space of $U(1)$-connections and spinors, with respect to the subgroup $\mathcal{G}_\ell$ of the gauge group $\mathcal{G}$ given by

$$\mathcal{G}_\ell = \{\lambda \in \mathcal{G} \mid \langle c_1(L) \cup [\lambda], [Y] \rangle = 0\},$$

where ℓ satisfies $\ell\mathbb{Z} = c_1(L)(H_2(Y, \mathbb{Z}))$. This subgroup depends on $c_1(L)$.

The space $\mathcal{B}$ has the homotopy type of $\mathbb{C}P^\infty \times K(H^1(Y, \mathbb{Z}), 1)$, hence it has a universal covering obtained by taking the quotient of $\mathcal{A}$ by the identity component of the gauge group. The resulting space $\tilde{\mathcal{B}}$ covers $\mathcal{B}$ with fibers $H^1(Y, \mathbb{Z})$. Define H_ℓ to be

$$H_\ell = \{h \in H^1(Y, \mathbb{Z}) | \langle c_1(L) \cup h, [Y] \rangle = 0\}.$$

The group H_ℓ also depends on $c_1(L)$. Then the space $\mathcal{B}_\ell = \mathcal{A}/\mathcal{G}_\ell$ is a covering of $\mathcal{B}$ with fiber $H^1(Y,\mathbb{Z})/H_\ell \cong \mathbb{Z}$. Hence $\mathcal{B}_\ell$ is an infinite cyclic covering space of $\mathcal{B}$.

The perturbed Chern–Simons–Dirac functional is a real valued functional $CSD : \mathcal{B}_\ell \to \mathbb{R}$ on the covering space $\mathcal{B}_\ell$. The critical manifold $\tilde{\mathcal{M}}_Y(\mathfrak{s})$ is a $\mathbb{Z}$-covering of $\mathcal{M}_Y(\mathfrak{s})$. The critical values form a discrete set in $\mathbb{R}$, which is a finite set mod $\mathbb{Z}$. Let $\Omega \subset \mathbb{R}$ denote the set of regular values. Let $\omega \in \Omega$ be a regular value. Given any point $a \in \mathcal{M}_Y(\mathfrak{s})$, there is a unique element a^ω in the fiber $\pi^{-1}(a)$ in $\tilde{\mathcal{M}}_Y(\mathfrak{s})$ that satisfies $CSD(a^\omega) \in (\omega, \omega + 8\pi^2\ell)$.

We have the following Lemma which shows that the relative indices on $\tilde{\mathcal{M}}_Y(\mathfrak{s})$, defined by the spectral flow of the Hessian operator, take values in $\mathbb{Z}$.

Lemma 3.1. *1. The spectral flow of the Hessian operator $H_{A(t),\psi(t)}$ of the perturbed CSD functional around a loop in $\mathcal{B}_\ell$ is zero, hence the relative index of any two points $\tilde{a}$ and $\tilde{b}$ in $\tilde{\mathcal{M}}_Y(\mathfrak{s})$ is well defined in $\mathbb{Z}$.*

2. Let $\tilde{a}$ be a critical point in $\tilde{\mathcal{M}}_Y(\mathfrak{s}) \in \mathcal{B}_\ell$, and let $\lambda \in \mathcal{G}/\mathcal{G}_\ell$ be a gauge transformation. We have the following identities:

$$CSD(\lambda(\tilde{a})) - CSD(\tilde{a}) = 8\pi^2\langle[\lambda] \cup c_1(\mathfrak{s}), [Y]\rangle;$$
$$SF(H_{(A(t),\psi(t))})|^{\tilde{a}}_{\lambda(\tilde{a})} = \langle[\lambda] \cup c_1(\mathfrak{s}), [Y]\rangle,$$

with $[\lambda] = [\frac{1}{2\pi\sqrt{-1}}\lambda^{-1}d\lambda]$. Here $SF(H_{(A(t),\psi(t))})|^{\tilde{a}}_{\lambda(\tilde{a})}$ denotes the spectral flow of the Hessian operator $H_{(A(t),\psi(t))}$ along a path from $\lambda(\tilde{a})$ to $\tilde{a}$ in $\mathcal{B}_\ell$.

Proof. By the Atiyah-Patodi-Singer index theorem, we have

$$SF(H_{(A(t),\psi(t))})|^{\tilde{a}}_{\lambda(\tilde{a})} = Index(\frac{\partial}{\partial t} + H_{(A(t),\psi(t))}).$$

This index calculates the virtual dimension of the 4-dimensional Seiberg–Witten monopole moduli space on $Y \times S^1$, with the Spin^c structure obtained by gluing the Spin^c structure $\mathfrak{s}$ on $Y \times \mathbb{R}$ along the two ends with the gauge transformation λ. By the index formula for the Seiberg–Witten monopoles, we obtain

$$\begin{aligned} & Index(\frac{\partial}{\partial t} + H_{(A(t),\psi(t))}) \\ = & \frac{1}{2\pi\sqrt{-1}}\int_Y c_1(L) \wedge \lambda^{-1}d\lambda \\ = & \langle[\lambda] \cup c_1(\mathfrak{s}), [Y]\rangle. \end{aligned}$$

The remaining claims are direct consequence of this index formula. □

Upon fixing a base point $\tilde{a}_0$ in $\tilde{\mathcal{M}}_Y(\mathfrak{s})$, we can define the following $\mathbb{Z}$-lifting of the $\mathbb{Z}_\ell$-grading of the elements of $\mathcal{M}_Y(\mathfrak{s})$.

Definition 3.2. *We define the grading of elements in $\mathcal{M}_Y(\mathfrak{s})$ as the relative index of the ω-lifting in $\tilde{\mathcal{M}}_Y(\mathfrak{s})$,*

$$\mathfrak{i}_Y^{(\omega)}(a) = \mathfrak{i}_Y(a^\omega) = SF(H_{(A(t),\psi(t))})|_{a^\omega}^{\tilde{a}_0}.$$

This definition of the grading depends on the choice of the base point $\tilde{a}_0$ and on the choice of a regular value ω. Notice that we can reduce the choice of $\tilde{a}_0$ to the choice of a base point a_0 in $\mathcal{M}_Y(\mathfrak{s})$.

Since now on, we fix a base point a_0 in $\mathcal{M}_Y(\mathfrak{s})$. Then $\tilde{a}_0$ is chosen to be the unique critical point in $\pi^{-1}(a_0)$ with $CSD(\tilde{a}_0) \in (\omega, \omega + 8\pi^2\ell)$. It is easy to see, from the definition, that we have $\mathfrak{i}_Y^{(\omega)} = \mathfrak{i}_Y^{(\omega')}$ whenever ω and ω' are connected by a path in the set Ω of regular values. Moreover, we have $\mathfrak{i}_Y^{(\omega+8\pi^2 k\ell)} = \mathfrak{i}_Y^{(\omega)}$, for any $k \in \mathbb{Z}$.

Definition 3.3. *For any $q \in \mathbb{Z}$, we define*

$$\mathcal{M}_{Y,q}^{(\omega)}(\mathfrak{s}) = \{a \in \mathcal{M}_Y(\mathfrak{s}) \ |\mathfrak{i}_Y^{(\omega)}(a) = q\}.$$

The q-chains in the $\mathbb{Z}$-graded Floer complex are the elements of the abelian group $C_q^{(\omega)}(Y,\mathfrak{s})$ generated by the monopoles in $\mathcal{M}_{Y,q}^{(\omega)}(\mathfrak{s})$. The boundary operator

$$\partial_q^{(\omega)} : C_q^{(\omega)}(Y,\mathfrak{s}) \to C_{q-1}^{(\omega)}(Y,\mathfrak{s})$$

is defined as

$$\partial_q^{(\omega)}(a) = \sum_{b \in \mathcal{M}_{Y,q-1}^{(\omega)}(\mathfrak{s})} \#\big(\hat{\mathcal{M}}^0(a,b)\big)b,$$

where $\hat{\mathcal{M}}^0(a,b)$ is the zero dimensional components of the moduli space of unparameterized flow lines in $\hat{\mathcal{M}}(a,b)$. The compactness theorem (cf. §4 of [11]) tells us that $\hat{\mathcal{M}}^0(a,b)$ is an oriented, compact, 0-dimensional manifold. Thus, the coefficient $\#(\hat{\mathcal{M}}^0(a,b))$ is well-defined as the algebraic sum of points in $\hat{\mathcal{M}}^0(a,b)$.

The following Lemma shows that we have $\partial_{q-1}^{(\omega)} \circ \partial_q^{(\omega)} = 0$. The resulting homology groups are denoted as $HF_{*,(\omega)}^{SW}(Y,\mathfrak{s})$, $* \in \mathbb{Z}$.

Lemma 3.4. *1. For $n \in \mathbb{Z}_\ell$ and $q \in \mathbb{Z}$ with $q = n (mod\ \ell)$, we have*

$$C_n(Y, \mathfrak{s}) = \bigoplus_{k\in\mathbb{Z}} C^{(\omega)}_{q+k\ell}(Y, \mathfrak{s}). \tag{2}$$

2. Under the decomposition as above, the boundary operator ∂_n on $C_n(Y, \mathfrak{s})$ can be expressed as follows. Assume that a is a generator in $C^{(\omega)}_q(Y, \mathfrak{s})$. Upon regarding a as a generator of $C_n(Y, \mathfrak{s})$ for $n = q (mod\ \ \ell)$, we obtain

$$\partial_n(a) = \partial^{(\omega)}_q(a) + \sum_{k>0} \partial^{(\omega)}_{q,k}(a),$$

with $\partial^{(\omega)}_{q,k} : C^{(\omega)}_q(Y, \mathfrak{s}) \to C^{(\omega)}_{q-1+k\ell}(Y, \mathfrak{s})$ for $k > 0$. In particular, the relation $\partial_{n-1} \circ \partial_n = 0$ implies that $\partial^{(\omega)}_{n-1} \circ \partial^{(\omega)}_n = 0$ is also satisfied.

Proof. The first statement about the decomposition of the chain complex follows directly from the definition. Now we study the boundary operator under this decomposition.

For any $k < 0$, $a \in \mathcal{M}^{(\omega)}_{Y,q}(\mathfrak{s})$, and $b \in \mathcal{M}^{(\omega)}_{Y,q-1+k\ell}(\mathfrak{s})$, we shall prove that the 0-dimensional components $\hat{\mathcal{M}}^0(a, b)$ in $\hat{\mathcal{M}}(a, b)$ is empty, hence the entry of the boundary operator is trivial, $\langle b, \partial_n(a)\rangle = 0$.

Notice that we have

$$\mathfrak{i}^{(\omega)}_Y(a) = \mathfrak{i}_Y(a^\omega) = q \quad \text{and} \quad \mathfrak{i}^{(\omega)}_Y(b) = \mathfrak{i}_Y(b^\omega) = q - 1 + k\ell.$$

Thus, the moduli space of flow lines on $\mathcal{B}_\ell$, $\hat{\mathcal{M}}(a^\omega, b^\omega)$ has virtual dimension $-kl > 0$. There exists a unique element $[\lambda] \in \mathcal{G}/\mathcal{G}_\ell$, such that

$$\langle [\lambda] \cup c_1(L), [Y]\rangle = -kl.$$

This implies that we have $\mathfrak{i}_Y(\lambda(b^\omega)) = SF(H_{(A(t),\psi(t))})^{\tilde{a}_0}_{\lambda(b^\omega)} = q - 1$, and $CSD(\lambda(b^\omega)) = CSD(b^\omega) - 8\pi^2 k\ell$, see Lemma 3.1.

When non-empty, $\hat{\mathcal{M}}^0(a, b)$ is isomorphic to $\hat{\mathcal{M}}(a^\omega, \lambda(b^\omega))$. We can prove that $\hat{\mathcal{M}}(a^\omega, \lambda(b^\omega))$ is empty, since the CSD functional is non-increasing along the gradient flow lines and the difference of CSD between a^ω and $\lambda(b^\omega)$ is negative:

$$\begin{aligned} & CSD(a^\omega) - CSD(\lambda(b^\omega)) \\ = \; & CSD(a^\omega) - CSD(b^\omega) - (CSD(\lambda(b^\omega)) - CSD(b^\omega)) \\ = \; & CSD(a^\omega) - CSD(b^\omega) + 8\pi^2 k\ell < 0, \end{aligned}$$

as $k < 0$ and $|CSD(a^\omega) - CSD(b^\omega)| < 8\pi^2\ell$. This proves that the entries below the diagonal are always zero.

For $a \in \mathcal{M}^{(\omega)}_{Y,q}(\mathfrak{s})$ and $b \in \mathcal{M}^{(\omega)}_{Y,q-1}(\mathfrak{s})$, it is easy to see that we have

$$\langle \partial_n(a), b\rangle = \langle \partial^{(\omega)}_q(a), b\rangle.$$

For $a \in \mathcal{M}^{(\omega)}_{Y,q}(\mathfrak{s})$ and $b \in \mathcal{M}^{(\omega)}_{Y,q-1+k\ell}(\mathfrak{s})$ with $(k > 0)$, we can define

$$\langle \partial^{(\omega)}_{q,k}(a), b\rangle = \langle \partial_n(a), b\rangle = \#\big(\hat{\mathcal{M}}^0(a,b)\big).$$

This counts the points in the zero dimensional components $\hat{\mathcal{M}}^0(a,b)$ in the moduli space of trajectories on $\mathcal{B}$ from a to b. Equivalently, we have

$$\langle \partial^{(\omega)}_{q,k}(a), b\rangle = \#\big(\hat{\mathcal{M}}(a^\omega, \lambda(b^\omega))\big),$$

where $\hat{\mathcal{M}}(a^\omega, \lambda(b^\omega))$ is the moduli space of trajectories on $\mathcal{B}_\ell$ from a^ω to $\lambda(b^\omega)$ and λ represents the unique element $[\lambda]$ in $\mathcal{G}/\mathcal{G}_\ell$ such that

$$\langle [\lambda] \cup c_1(L), [Y]\rangle = -kl.$$

Thus, we have $\mathfrak{i}_Y(\lambda(b^\omega)) = q-1$. Therefore, if non-empty, the moduli space $\hat{\mathcal{M}}(a^\omega, \lambda(b^\omega))$ is an oriented, compact 0-dimensional manifold with energy given by

$$8\pi^2 k\ell + CSD(a^\omega) - CSD(b^\omega) > 8\pi^2(k-1)\ell > 0.$$

This completes the proof of the Lemma. □

Example 3.5. *Let $(S^1 \times \Sigma_g, \mathfrak{s}_n)$ be a surface of genus g times a circle with a* Spin^c *structure $\mathfrak{s}_k$ such that $c_1(\mathfrak{s}_n) = 2nPD([S^1])$ where $n \neq 0$ and $|n| \leq g-1$, then the results in [13] show that as $\mathbb{Z}_{2n}$-graded homology, $HF^{SW}_*(S^1 \times \Sigma_g, \mathfrak{s}_n) \cong H^*(Sym^d(\Sigma_g))$ where $d = g-1-|n|$ and $Sym^d(\Sigma_g)$ is the d-fold symmetric product of Σ_g. For a proper choice of $\omega \in \mathbb{R}$, we know that*

$$HF^{SW}_{*,(\omega)}(S^1 \times \Sigma_g, \mathfrak{s}_n) \cong H^*(Sym^d(\Sigma_g))$$

as $\mathbb{Z}$-graded homology groups. This is the case that

$$HF^{SW}_k(S^1 \times \Sigma_g, \mathfrak{s}_n) = \bigoplus_{q\in\mathbb{Z}} HF^{SW}_{m+qn,(\omega)}(S^1 \times \Sigma_g, \mathfrak{s}_n)$$

for $k \in \mathbb{Z}_n$ and $m \in \mathbb{Z}$ with $k = k(mod\ n)$.

The expression of ∂_n and the appearance of $\partial^{(\omega)}_{q,k}$ in Lemma 3.4 lead us naturally to introduce a filtration of the $\mathbb{Z}_\ell$ graded complex $C_*(Y,\mathfrak{s})$. The filtration is given by

$$F^{(\omega)}_q C_n = \bigoplus_{k\geq 0} C^{(\omega)}_{q+k\ell}(Y,\mathfrak{s}).$$

for $n \in \mathbb{Z}_\ell$, and $q \in \mathbb{Z}$ with $q \equiv n (mod\ \ell)$. Thus, we have

$$\cdots \subset F^{(\omega)}_{q+\ell} C_n \subset F^{(\omega)}_q C_n \subset F^{(\omega)}_{q-\ell} C_n \subset \cdots \subset C_n(Y,\mathfrak{s}), \tag{3}$$

a finite length decreasing filtration of the $\mathbb{Z}$-graded Seiberg–Witten–Floer chain complex. From Lemma 3.4, we see that the boundary operator

$$\partial_n : \qquad F^{(\omega)}_q C_n \longrightarrow F^{(\omega)}_{q-1} C_{n-1}$$

preserves the filtration. Let $F^{(\omega)}_q H_n$ denote the homology of the complex

$$\cdots \xrightarrow{\partial} F^{(\omega)}_q C_n \xrightarrow{\partial} F^{(\omega)}_{q-1} C_{n-1} \xrightarrow{\partial} \cdots .$$

We define

$$F^{(\omega)}_q HF^{SW}_n(Y,\mathfrak{s}) = Im\big(F^{(\omega)}_q H_n \to HF^{SW}_n(Y,\mathfrak{s})\big).$$

We thus obtain a bounded filtration on $HF^{SW}_n(Y,\mathfrak{s})$,

$$\begin{aligned} \cdots \subset F^{(\omega)}_{q+\ell} HF^{SW}_n(Y,\mathfrak{s}) &\subset F^{(\omega)}_q HF^{SW}_n(Y,\mathfrak{s}) \\ \subset F^{(\omega)}_{q-\ell} HF^{SW}_n(Y,\mathfrak{s}) &\subset \cdots \subset HF^{SW}_n(Y,\mathfrak{s}). \end{aligned}$$

The standard procedure of constructing the spectral sequence for a filtration [14] gives the following theorem on the relation between the $\mathbb{Z}$-graded and the $\mathbb{Z}_\ell$-graded homology groups.

Theorem 3.6. *There exists a spectral sequence $(E^k_{q,n}(Y,\mathfrak{s}), d^k)$ with*

$$E^1_{q,n}(Y,\mathfrak{s}) \cong HF^{SW}_{q,(\omega)}(Y,\mathfrak{s})$$

for $n \in \mathbb{Z}_\ell$ and $q \in \mathbb{Z}$ with $q \equiv n (mod\ \ell)$. The higher differentials

$$d^k : \qquad E^k_{q,n}(Y,\mathfrak{s}) \longrightarrow E^k_{q-1+k\ell,n-1}(Y,\mathfrak{s})$$

are induced by the maps $\partial^{(\omega)}_{q,k}$ defined in Lemma 3.4. Furthermore, the spectral sequence $(E^k_{q,n}(Y,\mathfrak{s}), d^k)$ converges to the $\mathbb{Z}_\ell$-graded homology groups $HF^{SW}_n(Y,\mathfrak{s})$.

Proof. By construction, the $\mathbb{Z}_\ell$-graded chain complex $C_n(Y,\mathfrak{s})$ has a bounded filtration (3) with the associated graded complex given by

$$F_q^{(\omega)} C_n / F_{q+\ell}^{(\omega)} C_n = C_q^{(\omega)}(Y,\mathfrak{s}).$$

Then by the standard technique of [14], we derive the existence of a spectral sequence

$$(E^k_{q,n}(Y,\mathfrak{s}), d^k)$$

with $E^1_{q,n}(Y,\mathfrak{s}) \cong HF^{SW}_{q,(\omega)}(Y,\mathfrak{s})$, and

$$\begin{aligned} Z^k_{q,n}(Y,\mathfrak{s}) &= \{a \in F_q^{(\omega)} C_n |\ \partial_n(a) \in F^{(\omega)}_{q-1+k\ell} C_{n-1}\} \\ E^k_{q,n}(Y,\mathfrak{s}) &= Z^k_{q,n}(Y,\mathfrak{s}) / \big(Z^{k-1}_{q+\ell,n}(Y,\mathfrak{s}) + \partial_{n+1} Z^{k-1}_{q+1-(k-1)\ell,n+1}(Y,\mathfrak{s}) \big). \end{aligned}$$

The higher differentials are induced by ∂_n. The expression of ∂_n, as discussed in Lemma 3.4, tells us that the higher differentials acting on $E^k_{q,n}(Y,\mathfrak{s})$ are defined by $\partial^{(\omega)}_{q,k}$. □

3.1 Poincaré series and Euler characteristics

For a graded vector space V_*, the Poincaré series is defined as

$$P(V_*, t) := \sum_{N \geq 0} \dim V_N \, t^N,$$

and the Euler characteristic $\chi(V_*) := P(V_*, -1)$, whenever this expression makes sense. The Poincaré series of a bigraded vector space $E_{*,*}$ is defined as

$$P(E_{*,*}, t) := \sum_{N \geq 0} \dim \left(\oplus_{p+q=N} E_{p,q} \right) t^N, \tag{4}$$

and the Euler characteristic of $E_{*,*}$ is again defined as $\chi(E_{*,*}) := P(E_{*,*}, -1)$ whenever the expression makes sense. If V_* is a graded vector space with a filtration $F^0 V_* \subset F^1 V_* \subset \cdots F^N V_* \subset \cdots V_*$, then $P(V_*, t) = P(E^0_{*,*}(V_*), t)$. Given two graded vector spaces V_* and W_*, we say that $P(V_*, t) \geq P(W_*, t)$ if the power series $P(V_*, t) - P(W_*, t)$ has non-negative coefficients.

In the following, we denote by $SW^{(\omega)}(Y,\mathfrak{s})$ the invariant of 3-manifolds obtained as the Euler characteristic of the $\mathbb{Z}$–graded Floer homology, $SW^{(\omega)}(Y,\mathfrak{s}) := \chi(HF^{SW}_{*,(\omega)}(Y,\mathfrak{s}))$. We denote by $SW(Y,\mathfrak{s}) := \chi(HF^{SW}_*(Y,\mathfrak{s}))$, the invariant of 3-manifolds obtained as Euler characteristic of the $\mathbb{Z}_\ell$ graded Floer homology.

Proposition 3.7. *The Poincaré series of $E^k_{q,n}(Y,\mathfrak{s})$ satisfy the following properties.*

1. $P(E^1_{q,n}(Y,\mathfrak{s}),t) \geq P(E^2_{q,n}(Y,\mathfrak{s}),t) \geq \cdots P(HF^{SW}_*(Y,\mathfrak{s}),t)$
2. $\chi(E^k_{q,n}(Y,\mathfrak{s})) = SW(Y,\mathfrak{s})$, *for all* $k \geq 1$.

Proof. The first property follows from standard results on spectral sequences (cf. [12]). The last statement is obtained by computing the Euler characteristics at the chain level. In fact, using the identification (2) of chain complexes, we see that

$$SW(Y,\mathfrak{s}) = \chi(C_*(Y,\mathfrak{s})) = \chi(C^\omega_*(Y,\mathfrak{s})) = SW^{(\omega)}(Y,\mathfrak{s}).$$

This implies that $P(E^1_{q,n}(Y,\mathfrak{s}),-1) = P(HF^{SW}_*(Y,\mathfrak{s}),-1)$, then the statement follows from the first property. $\square$

We see, this way, that the Poincaré series of the terms $E^k_{q,n}(Y,\mathfrak{s})$, $k \geq 2$ in the spectral sequence give refined invariants of the 3-manifold Y, all of which correspond to the same Euler characteristic $SW(Y,\mathfrak{s})$.

4 Truncated relative invariants

Let X be an oriented Riemannian 4-manifold with a cylindrical end modeled on $Y \times [0,\infty)$. Let $\mathfrak{s}_X$ be a Spin^c structure on X and $\mathfrak{s}$ the induced Spin^c structure on Y, with $\ell\mathbb{Z} = c_1(\mathfrak{s})(H_2(Y,\mathbb{Z}))$. We denote by $\mathcal{M}_Y(\mathfrak{s})$ the moduli space of Seiberg–Witten monopoles on $(Y,\mathfrak{s})$ and by $\tilde{\mathcal{M}}_Y(\mathfrak{s})$ its $\mathbb{Z}$-covering as in the previous section. If $\mathcal{M}_{X,Y}(\mathfrak{s})$ denotes the covering of $\mathcal{M}_Y(\mathfrak{s})$ with fiber $H^1(Y,\mathbb{Z})/Im(i^*)$, with $i^* : H^1(X,Z) \to H^1(Y,\mathbb{Z})$, we have a diagram of covering maps

$$\begin{array}{ccc} \mathcal{M}_{X,Y}(\mathfrak{s}) & & \\ \downarrow{\scriptstyle \tilde{\pi}_X} & \searrow^{\pi_X} & \\ \tilde{\mathcal{M}}_Y(\mathfrak{s}) & \xrightarrow{\pi} & \mathcal{M}_Y(\mathfrak{s}) \end{array} \qquad (5)$$

For each $a \in \mathcal{M}_{X,Y}(\mathfrak{s})$ it is possible to define a moduli space $\mathcal{M}_X(\mathfrak{s}_X, a)$ of gauge classes of solutions of the the Seiberg–Witten equations on X, whose asymptotic limit lies in the class of a. The analysis required to set up the theory of these moduli spaces of relative invariants and the main results about their properties are presented in [3]. We simply recall here that there is a suitable perturbation theory, such that, for generic choice of the perturbation $\mathfrak{p}$, the moduli space $\mathcal{M}_{X,\mathfrak{p}}(\mathfrak{s}_X, a)$ is smooth of the expected dimension, computed via the index $i_X(a)$ of the deformation complex of the perturbed Seiberg–Witten equations on X. Components of a given dimension $i_X(a) = d$ admit a compactification to a smooth manifold with corners, where the boundary strata are identified with moduli spaces of Seiberg–Witten monopoles on X and on the cylinder $Y \times \mathbb{R}$. Thus, it is possible to define relative invariants of the 4–manifold with boundary X, of the form

$$SW_X(\mathfrak{s}_X, \cdot) : \mathbb{A}(X) \to HF^{SW}_{*,[Im(i^*)]}(Y, \mathfrak{s}), \tag{6}$$

where $\mathbb{A}(X) = \mathrm{Sym}^*(H_0(X)) \otimes \Lambda^*(H_1(X)/\mathrm{Torsion})$. These are obtained by constructing, for each monomial $z = U^k \gamma_1 \wedge \gamma_2 \cdots \wedge \gamma_\ell$ with $2k + \ell = d \geq 0$, compact, zero-dimensional, oriented, smooth submanifolds

$$\mathcal{M}^{\Lambda,\Xi}_X(\mathfrak{s}_X, a) \subset \mathcal{M}_{X,\mathfrak{p}}(\mathfrak{s}_X, a),$$

and setting

$$SW_X(\mathfrak{s}_X, z, a) := \#\mathcal{M}^{\Lambda,\Xi}_X(\mathfrak{s}_X, a).$$

The independence of this definition on the additional choices that determine the data (Λ, Ξ), in the construction of these submanifolds, is discussed in [3]. The Floer homology in (6) is a natural lift of the $\mathbb{Z}_\ell$ graded Floer homology $HF^{SW}_*(Y, \mathfrak{s})$ obtained by considering moduli spaces with respect to the action of the group of gauge transformations on Y that extend to X. This is again discussed in detail in [3]. For $d = \deg(z)$, the chain element in $C_{*,[Im(i^*)]}(Y, \mathfrak{s})$ is given by

$$SW_X(\mathfrak{s}_X, z) := \sum_{a \in \mathcal{M}_{X,Y}(\mathfrak{s}), i_X(a) = d} SW_X(\mathfrak{s}_X, z, a)\langle a \rangle. \tag{7}$$

The properties of the compactification of the moduli spaces $\mathcal{M}_{X,\mathfrak{p}}(\mathfrak{s}_X, a)$ show that this is a cycle, hence it defines a class in $HF^{SW}_{*,[Im(i^*)]}(Y, \mathfrak{s})$.

This definition of relative invariant is the one best suited in order to have gluing formulae of relative invariants that correspond to splitting a compact 4–manifold along a separating compact 3-manifold, into two 4–manifolds with boundary, with the invariants of the compact 4–manifolds obtained via a pairing of the Floer groups of the 3–manifold, along the lines of Atiyah's formulation of a TQFT. However, if we are interested in relative invariants just as invariants of 4–manifolds with boundary, then it is possible to introduce other versions, which involve smaller moduli spaces, and are therefore, in principle, more easily computable.

First we notice that the coverings of moduli spaces (5) allow us to define a relative invariant

$$\widetilde{SW}_X(\mathfrak{s}_X, z) := \sum_{\tilde{a}\in\tilde{\mathcal{M}}_Y(\mathfrak{s}), i_X(\tilde{a})=d} \widetilde{SW}_X(\mathfrak{s}_X, z, \tilde{a})\langle\tilde{a}\rangle, \tag{8}$$

as a formal linear combination of elements in $\tilde{\mathcal{M}}_Y(\mathfrak{s})$, with $\widetilde{SW}_X(\mathfrak{s}_X, z, \tilde{a}) = \sum_{a\in\tilde{\pi}_X^{-1}(\tilde{a})} SW_X(\mathfrak{s}_X, z, a)$.

Then we choose a regular value ω of the Chern–Simons–Dirac functional CSD on $\tilde{\mathcal{M}}_Y(\mathfrak{s})$, with the property that

$$\omega > \max\{CSD(\tilde{a}) : \widetilde{SW}_X(\mathfrak{s}_X, z, \tilde{a}) \neq 0\}. \tag{9}$$

We can then write (8), separating the generators $\langle\tilde{a}\rangle$ according to the values of $CSD(\tilde{a})$:

$$\widetilde{SW}_X(\mathfrak{s}_X, z) = \sum_{k\geq 0} \left(\sum_{CSD(\tilde{a})\in I_k, i_X(\tilde{a})=d} \widetilde{SW}_X(\mathfrak{s}_X, z, \tilde{a}) \right) \langle\tilde{a}\rangle, \tag{10}$$

where we use the notation $I_k = (\omega - 8\pi^2(k+1)\ell, \omega - 8\pi^2 k\ell)$.

We define a *truncation* of (10) at level N by setting

$$\tau_N \widetilde{SW}_X(\mathfrak{s}_X, z) := \sum_{0\leq k\leq N} \left(\sum_{CSD(\tilde{a})\in I_k, i_X(\tilde{a})=d} \widetilde{SW}_X(\mathfrak{s}_X, z, \tilde{a}) \right) \langle\tilde{a}\rangle. \tag{11}$$

Notice then that, if $\tilde{a} \in \tilde{\mathcal{M}}_Y(\mathfrak{s})$ satisfies $CSD(\tilde{a}) \in I_k$, the corresponding element a^ω, in the same fiber of the map $\pi : \tilde{\mathcal{M}}_Y(\mathfrak{s}) \to \mathcal{M}_Y(\mathfrak{s})$, is related to $\tilde{a}$ by a gauge transformation $a^\omega = \lambda_k \cdot \tilde{a}$, with $\langle[\lambda_k] \cup c_1(\mathfrak{s}), [Y]\rangle = k\ell$. Thus, if $\langle\pi(\tilde{a})\rangle$ is a generator in $C_n(Y, \mathfrak{s})$, then the element a^ω defines a generator $\langle a^\omega\rangle \in C^{(\omega)}_{q+k\ell}(Y, \mathfrak{s})$, with $q \equiv n \mod \ell$. Thus, we can apply to (11) a *shift* given by the gauge transformations λ_k.

We obtain an element

$$SW_X(\mathfrak{s}_X, z, N) := \sum_{0\leq k\leq N} \left(\sum_{a^\omega,\, i_X(\lambda_k^{-1}a^\omega)=d} \widetilde{SW}_X(\mathfrak{s}_X, z, \lambda_k^{-1}a^\omega) \right) \langle a^\omega\rangle, \tag{12}$$

where each term $\sum_{a^\omega,\, i_X(\lambda_k^{-1}a^\omega)=d} \widetilde{SW}_X(\mathfrak{s}_X, z, \lambda_k^{-1}a^\omega)\langle a^\omega\rangle$ defines a chain in $C^{(\omega)}_{q+k\ell}(Y,\mathfrak{s})$.

Notice that this shift by the gauge transformations λ_k also induces the $\mathbb{Z}$-action:

$$k : HF^{(\omega)}_*(Y,\mathfrak{s}) \stackrel{\simeq}{\to} HF^{(\omega-8\pi^2 k\ell)}_{*+k\ell}(Y,\mathfrak{s}), \tag{13}$$

where $H^1(Y,\mathbb{Z})/Ker(c_1(\mathfrak{s})) \cong \mathbb{Z}$.

Theorem 4.1. *The truncated relative invariants $SW_X(\mathfrak{s}_X, z, N)$ of* (12) *define classes in $E^N_{q,n}(Y,\mathfrak{s})$.*

Proof. The fact that (7) is a cycle with respect to the boundary operator of the Floer complex of $HF^{SW}_{*,[Im(i^*)]}(Y,\mathfrak{s})$ implies that, when we apply ∂_n to the truncation $SW_X(\mathfrak{s}_X, z, N)$ we obtain

$$\left(\partial^{(\omega)}_q + \sum_{0<k\leq N-1} \partial^{(\omega)}_{q,k}\right) SW_X(\mathfrak{s}_X, z, N) = 0.$$

In fact, we know that $\partial^{(\omega)}_{q,k}: \; C^{(\omega)}_q(Y,\mathfrak{s}) \to C^{(\omega)}_{q-1+k\ell}(Y,\mathfrak{s})$ for $k > 0$, and ω is chosen satisfying condition (9). This implies that

$$\partial_n(SW_X(\mathfrak{s}_X, z, N)) \in F^{(\omega)}_{q-1+N\ell}(Y,\mathfrak{s}),$$

hence $SW_X(\mathfrak{s}_X, z, N) \in Z^N_{q,n}(Y,\mathfrak{s})$ defines a class in $E^N_{q,n}(Y,\mathfrak{s})$.

□

The construction presented here is analogous to the relative invariants defined by Fintushel and Stern in [5], with values in the $E^r_{q,n}$ terms of their spectral sequence. Using their formulation, we should consider the invariant $SW_X(\mathfrak{s}_X, z, N)$ as obtained by counting oriented points in zero-dimensional moduli spaces $\mathcal{M}^{\Lambda,\Xi,(\omega),N}_X(\mathfrak{s}_X, \tilde{a})$ inside the $d = \deg(z)$-dimensional moduli spaces of $[A,\Psi] \in \mathcal{M}_X(\mathfrak{s}_X, \tilde{a})$ satisfying

$$-\int_X F_A \wedge F_A < 8\pi^2 N\ell - \omega. \tag{14}$$

Notice, however, that $CSD(A,\psi) = CS(A)$ at a solution $\tilde{a} = [A,\psi]$ of the 3-dimensional SW equations, since the quadratic term in the spinor vanishes at a 3d monopole solution. Thus, the condition (14) can be rephrased as $CS(\tilde{a}) = CSD(\tilde{a}) > \omega - 8\pi^2 N\ell$, hence it agrees with the condition used in the truncation on (11).

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Matilde Marcolli and **Bai-Ling Wang**,
Max–Planck–Institut für Mathematik,
Vivatsgasse 7, D-53111 Bonn, Germany.

marcolli@mpim-bonn.mpg.de
bwang@mpim-bonn.mpg.de