Gerbes, Twisted K-theory and some Applications

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This Talk is based on the following papers and manuscripts:

- Bundle gerbes for Chern-Simons and Wess-Zumino-Witten theories. (with A. Carey, S. Johnson, M. Murray, D. Stevenson) Comm. in Math. Phys., 2005.
- Fusion of Symmetric D-branes and Verlinde ring. (with A. Carey) math.ph/0505040
- Thom isomorphism and Push-forward map in twisted K-theory. (with A. Carey) math.KT/0507414
- Groupoid interpretation of 2-gerbes. (with Chengchang Zhu). In preparation.
- Various recent works. In preparation.

1 Gerbes

Gerbe, a locally non-empty and locally connected stack in groupoids, has many new faces:

(1) Local line bundles: (Hitchin)

$$\{(L_{ij}, U_{ij}); \theta_{ijk}\}$$

- L_{ij} is a line bundle over $U_{ij} = U_i \cap U_j$;
- θ_{ijk} is an isomorphism: $L_{ij} \otimes L_{jk} \longrightarrow L_{ik}$ over U_{ijk};
- θ_{ijk} is associative over U_{ijkl}.

Gerbe connection and curving: L_{ij} can be equipped with a connection ∇_{ij} such that θ_{ijk} preserves the connection. There exists a local 2-form B_i (a curving, or B-field) such that

$$B_j - B_j = \text{ Curvature of } \nabla_{ij}.$$

(2) Bundle gerbes: (Murray)

- a surjective submersion $\pi : Y \to M$;
- a principal $U(1)$-bundle $\mathcal{G}$ over $Y^{[2]} = Y \times_M Y$, together with a groupoid structrue on $\mathcal{G}$,

$$
\begin{array}{ccc}
\mathcal{G} & & \\
\downarrow & & \\
Y^{[2]} & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & Y \\
 & & \downarrow \\
 & & M.
\end{array}
\qquad (1)
$$

Example 1. *Let G be a compact Lie group. A central extension of based loop group*

$$1 \to U(1) \to \widehat{\Omega G} \to \Omega G \to 1,$$

and the universal ΩG-bundle $\mathcal{A}_{S^1}$ over G (the space of G-connections on the trivial G-bundle over S^1) define a canonical (lifting) bundle gerbe over G.

(3) Groupoid picture:

A U(1) central extension of a groupoid.

Example 2. *1. The standard groupiod associated to a topological manifold M with a cover $\{U_i\}$:*

$$\coprod U_{ij} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \coprod U_i \longrightarrow M.$$

2. Action groupoid together with a group 2-cocycle. In particular, a disction group Γ action on X and a twisting $\sigma \in H^2(\Gamma, U(1))$ define a gerbe over X/Γ.

(4) Stack picture:

A principal stacky $\mathcal{B}S^1$-bundle over a differential stack, where the classifying stack $\mathcal{B}S^1$ is a stacky Lie group.

1.1 Deligne cohomology

For smooth manifolds, gerbes (with connection and curving) are geometric realizations of degree 3 Deligne cocycles

$$(g_{ijk}, A_{ij}, B_i).$$

In general, Deligne cohomology is characterized by the following diagram:

$$\begin{array}{ccccc} & & H^k(M, U(1)) & & \\ & & \downarrow & & \\ \dfrac{\Omega^k(M)}{\Omega^k_{\mathbb{Z}}(M)} & \to & H^{k+1}_{\mathcal{D}}(M, \mathbb{Z}) & \to & H^{k+1}(M, \mathbb{Z}) \\ & & \downarrow & & \downarrow \\ & & \Omega^{k+1}_{\mathbb{Z}}(M) & \to & H^{k+1}(M, \mathbb{R}) \end{array}$$

Here $H^k(M, U(1))$ classifies flat gerbes, and $\Omega^{k+1}_{\mathbb{Z}}(M)$ is the space of closed $k+1$-forms with integral period.

2 Application I

2.1 CS/WZW correspondence

Let G be a compact, connected, semi-simple Lie group. Dijkgraaf and Witten discussed a correspondence between Chern-Simons gauge theory and Wess-Zumino-Witten theory associated to G, which involves the transgression map

$$\tau : H^4(BG, \mathbb{Z}) \to H^3(G, \mathbb{Z}), \tag{2}$$

fixing a smooth infinite dimensional model of $\pi : EG \to BG$ by embedding G into $U(N)$.

Remark 3. *The CS/WZW correspondence is not surjective for non-simply connected Lie groups, which explains why chiral algebras only exist at certain particular values of the level.*

To understand the WZW model in the image of CS/WZW correspondence, we need to refine the transgression map (2) using Deligne cohomology:

$$\begin{array}{ccc} H^4_{\mathcal{D}}(BG, \mathbb{Z}) & \longrightarrow & H^3_{\mathcal{D}}(G, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^4(BG, \mathbb{Z}) & \rightarrow & H^3(G, \mathbb{Z}), \end{array} \tag{3}$$

This transgression map is obtained from the canonical prinicpal G-bundle over $S^1 \times G$ with connection, which is

$$\frac{\mathcal{A}_{S^1} \times S^1 \times G}{\Omega G},$$

where $\mathcal{A}_{S^1}$ is the space of G-connections on $S^1 \times G$, $\mathcal{A}_{S^1} \to G$ is a universal ΩG-bundle.

Let $\phi \in H^4(BG, \mathbb{Z})$ and Φ be the corresponding G-invariant polynomial on the Lie algebra of G.

Then a connection $\mathbb{A}$ on the universal bundle EG over BG uniquely determines a Deligne class

$$c_\phi(EG, \mathbb{A}) \in H^4_{\mathcal{D}}(BG, \mathbb{Z}),$$

via $\big(\phi, \Phi(F_{\mathbb{A}})\big) \in H^4(BG, \mathbb{Z}) \times \Omega^4_{\mathbb{Z}}(BG)$. Pull $c_\phi(EG, \mathbb{A})$ back to $S^1 \times G$, then integrate along S^1,

$$\int_{S^1} : \qquad H^4_{\mathcal{D}}(S^1 \times G, \mathbb{Z}) \to H^3_Z(G, \mathbb{Z}),$$

we get a degree 3 Deligne class $H^3_{\mathcal{D}}(G, \mathbb{Z})$.

The degree 4 Deligne class $c_\phi(EG, \mathbb{A})$ defines a 2-gerbe $\mathcal{Q}_\phi$ over BG, called the universal Chern-Simons bundle 2-gerbe with curvature $\Phi(F_{\mathbb{A}})$ and C-field given by Chern-Simons forms.

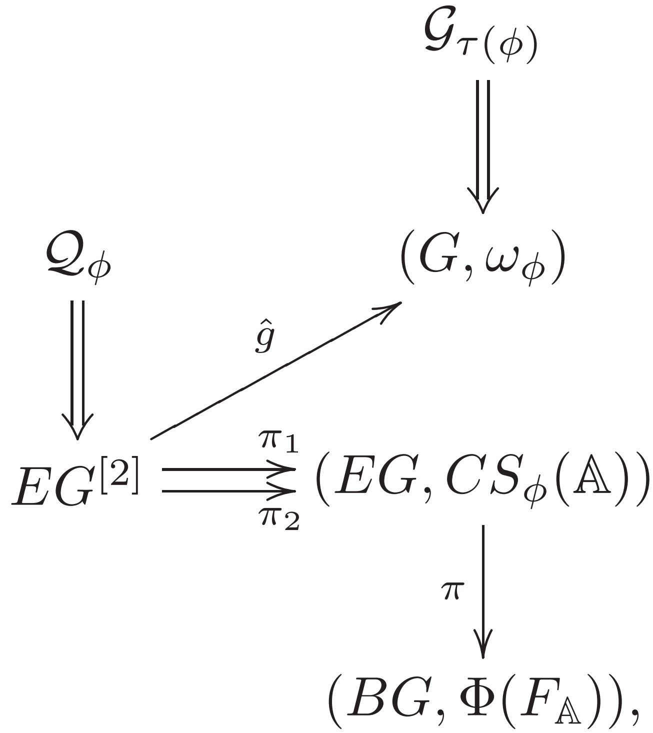
Denote by $\mathbf{BGrb}_M$ the bi-category of bundle gerbes over M and stable isomorphisms between bundle gerbes. The space of 2-morphisms between two stable isomorphisms is one-to-one corresponding to the space of line bundles over M.

For a smooth submersion $\pi : X \to M$, there is a natural associated simplicial manifold $\{X_n = X^{[n+1]}\}$ with $X_0 = M$, $X_1 = X$ and $X^{[n+1]}$ is the $(n+1)$-fold fiber product of π.

Definition 4. *A bundle 2-gerbe on M consists of a smooth surjective submersion $\pi\colon X \to M$ together with*

1. *An object $(\mathcal{Q}, X^{[2]})$ in $\mathbf{BGrb}_{X^{[2]}}$.*
2. *A stable isomorphism m: $\pi_1^*\mathcal{Q} \otimes \pi_3^*\mathcal{Q} \to \pi_2^*\mathcal{Q}$ in $\mathbf{BGrb}_{X^{[3]}}$ defining the bundle 2-gerbe product which is associative up to a 2-morphism in $\mathbf{BGrb}_{X^{[4]}}$.*
3. *The 2-morphism satisfies a natural coherency condition in $\mathbf{BGrb}_{X^{[5]}}$.*

Remark 5. *Geometrically, $\mathcal{Q}_\phi$ is a bundle gerbe over $EG^{[2]}$, obtained from the pull-back of a bundle gerbe $\mathcal{G}_{\tau(\phi)}$ over G by the canonical map $\hat{g} : EG^{[2]} \to G$.*



where $\pi^*\Phi(F_\mathbb{A}) = dCS_\phi(\mathbb{A}),$ *and*

$$(\pi_2^* - \pi_1^*)CS_\phi(\mathbb{A}) = \hat{g}^*\omega_\phi - d\Phi(\mathbb{A}, d\hat{g} \cdot \hat{g}^{-1}).$$

Here ω_ϕ is the corresponding left-invariant closed 3-form ω_ϕ on G. The bundle gerbe $\mathcal{G}_{\tau(\phi)}$ can be equipped with a connection and curving whose curvature is ω_ϕ.

Theorem 6. *Given a bundle gerbe $\mathcal{G}$ over G, the pull back bundle gerbe $\hat{g}^*\mathcal{G}$ over $EG^{[2]}$ is a bundle 2-gerbe on BG if and only if the Dixmier-Douady class of $\mathcal{G}$ is in the image of the transgression map τ.*

Definition 7. *A bundle gerbe $\mathcal{G}$ over G, whose Dixmier-Douady class is transgressive, is called a multiplicative bundle gerbe.*

For a multiplicative bundle gerbe $\mathcal{G}$ over G, the corresponding WZW action satisfies the following property:

$$e^{S_{WZW}(g_1 \cdot g_2)} = e^{S_{WZW}(g_1)} \cdot e^{S_{WZW}(g_2)}$$

for any pair of smooth maps $g_i : \Sigma \to G$. Here the WZW action is given by the bundle gerbe holonomy of the Deligne class.

Example 8. *The Wess-Zumino-Witten model on $SO(3)$ is multiplicative if and only if the level is an even class in $H^3(SO(3), \mathbb{Z})$, or a multiple of 4 in $H^3(SU(2), \mathbb{Z})$ under the pull-back homomorphism.*

2.2 Loop group representations

Using geometry of gerbes, we now understand more about positive energy representations of loop groups, in particular loop groups of non-simply connected, compact simple Lie groups.

Simply connected case

Positive energy representations of loop group LG at level $k > 0$ can be obtained from the geometric quantization of affine coadjoint orbits at level k. .

Given a Riemann surface Σ with one boundary component which is pointed by fixing a base point on the boundary, denote by $\mathcal{M}_\Sigma$ the based moduli space of flat G-connections on Σ.

The boundary holonomy map defines a group valued moment map

$$\mu_\Sigma : \mathcal{M}_\Sigma \longrightarrow G,$$

in the sense of Alekseev-Malkin-Meinrenken, $\mathcal{M}_\Sigma$ is called a quasi-Hamiltonian G-space.

Remark 9. *Note that* $(\mathcal{M}_\Sigma, \mu_\Sigma)$ *and the universal* ΩG*-bundle* $\mathcal{A}_{S^1}$ *define a canonical Hamiltonian* LG*-manifold*

$$\hat{\mathcal{M}}_\Sigma = \mathcal{M}_\Sigma \times_G \mathcal{A}_{S^1}$$

with a proper moment map

$$\hat{\mu}_\Sigma : \hat{\mathcal{M}}_\Sigma \longrightarrow \mathcal{A}_{S^1} \cong L\mathfrak{g}^* \times \{k\}.$$

Moreover, the pull-back of the canonical lifting G*-equivariant bundle gerbe* $\mathcal{G}_k$ *to* $\mathcal{M}_\Sigma$ *admits a trivialization given by a* $\widehat{LG}$*-equivariant pre-quantization line bundle over* $\hat{\mathcal{M}}_\Sigma$*. The geometric quantization of* $\hat{\mathcal{M}}_\Sigma$ *realizes all the positive energy representions of* LG *at level* k*.*

Denote by $\mathcal{Q}_{G,k}$ the category of quasi-Hamiltonian G-spaces admitting a $\widehat{LG}$-equivariant pre-quantization line bundle. There exists a well-defined fusion product $\boxtimes$ on $\mathcal{Q}_{G,k}$ induced from the group multiplication $G \times G \to G$. The category $(\mathcal{Q}_{G,k}, \boxtimes)$ is called the fusion category of generalized symmetric D-branes.

Theorem 10. *There exists a quantization functor defined by the Spin^c quantization along various coadjoint orbits*

$$\mathcal{Z}_{k,G} : (\mathcal{Q}_{G,k}, \boxtimes) \longrightarrow (R_k(LG), *)$$

satisfying

$$\boxed{\mathcal{Z}_{k,G}(M_1 \boxtimes M_2) = \mathcal{Z}_{k,G}(M_1) * \mathcal{Z}_{k,G}(M_2),}$$

where the product $*$ on the right hand side denotes the fusion ring structure on the Verlinde ring $R_k(LG)$.

Remark 11. *The non-simply connected case is more subtle and interesting. We explained certain modular invariants from the WZW theory on non-simply connected compact Lie groups using transgressive WZW model. For those Deligne classes not in the image of the transgression map, we need higher spin CS/WZW correspondence to understand the fusion ring structure on the representation rings of LG at level k.*

Non-simply connected case

For a non-simply connected, connected, compact, simple Lie group G. Equivalence class of central extensions of its loop group is classified by

$$(\chi, \kappa) \in Hom(\pi_1(G), U(1)) \oplus H^3(G, \mathbb{Z}),$$

or an element in $H^3_G(G, \mathbb{Z})$. Note that $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$ for all such G except $PSO(4n)$

$$H^3(PSO(4n), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

Applying the quantization functor

$$\mathcal{Z}_{G,(\chi,\kappa)} : (\mathcal{Q}_{G,(\chi,\kappa)}, \boxtimes) \longrightarrow R_{(\chi,\kappa)}(LG),$$

we know that $R_{(\chi,\kappa)}(LG)$ admits a fusion ring structure if and only if $\chi = +1$ and κ is transgressive.

Example 12. *For $G = PSU(3)$, $(\chi, \kappa) = (+1, 6)$, we get the simple current modular invariant of $PSU(3)$ at level 6 in terms of the Kac-Peterson character of $LSU(3)$:*

$$\begin{gathered} |\chi_{6,(0,0)} + \chi_{6,(6,0)} + \chi_{6,(0,6)}|^2 \\ +|\chi_{6,(1,1)} + \chi_{6,(4,1)} + \chi_{6,(1,4)}|^2 \\ +|\chi_{6,(3,3)} + \chi_{6,(0,3)} + \chi_{6,(3,0)}|^2 + 3|\chi_{6,(2,2)}|^2. \end{gathered}$$

3 Twisted K-theory

There are four equivalent definitions of twisted K-groups. For simplicity, we assume that M is a compact manifold. Given

$$\sigma \in H^3(M, \mathbb{Z}) \cong H^2(M, \underline{U(1)}) \cong H^1(M, \underline{PU(\mathcal{H})}),$$

where $PU(\mathcal{H})$ is the projective unitary group (with norm topology) of an infinite dimensional, complex, separable Hilbert space $\mathcal{H}$, there is a principal $PU(\mathcal{H})$-bundle $\mathcal{P}_\sigma$ over M, unique up to an isomorphism.

3.1 First Definition

Denote by

$$\mathcal{A}_\sigma = \Gamma(M, \mathcal{P}_\sigma \times_{PU(\mathcal{H})} \mathcal{K}(\mathcal{H}))$$

the associated stable, continuous trace C*-algebra with spectrum M. Then the twisted K-group is defined by

$$K^i_\sigma(\mathcal{A}_\sigma) := K_i(\mathcal{A}_\sigma) = KK(\mathbb{C}, \mathcal{A}_\sigma).$$

3.2 Second Definition (Rosenberg)

Note that $PU(\mathcal{H})$ acts continuously on $Fred(\mathcal{H})$, the space of Fredholm operators with norm or compact-open topology, via the conjugagtion action of $U(\mathcal{H})$, also acts on

$$\mathcal{U}_1 = \{1 + u \in U(\mathcal{H}) | u \text{ is of trace class}\}$$

continuously. Rosenberg proved that

$$K^0_\sigma(M) \cong \pi_0(\Gamma(M, \mathcal{P}_\sigma \times_{PU(\mathcal{H})} Fred(\mathcal{H}))$$

$$K^1_\sigma(M) \cong \pi_0(\Gamma(M, \mathcal{P}_\sigma \times_{PU(\mathcal{H})} \mathcal{U}_1(\mathcal{H})).$$

So a twisted K^0-class is represented by families of locally defined Fredholm operators

$$\phi_\alpha : U_\alpha \longrightarrow Fred(\mathcal{H})$$

satisfying $\phi_\beta = Ad_{\hat{g}_{\alpha\beta}}(\phi_\alpha)$.

Remark 13. *Mickelson and Freed-Hopkins-Teleman constructed G-equivariant twisted K-classes on G using families of cubic Dirac operators for WZW models.*

3.3 Third Definition (Adelaide School)

Consider the lifting bundle gerbe $\mathcal{G}_\sigma$ associated to the principal bundle $\mathcal{P}_\sigma$ and the central extension

$$1 \to U(1) \to U(\mathcal{H}) \to PU(\mathcal{H}) \to 1.$$

Bouwknegt-Carey-Mathai-Murray-Stevenson studied Hiblert bundles over $\mathcal{P}_\sigma$ with structure group

$$\mathcal{U}_{\mathcal{K}} = \{1 + \text{compact operator}\} \subset U(\mathcal{H}),$$

admitting a bundle gerbe action of $\mathcal{G}_\sigma$. Let $\mathcal{E}_{bg}^{U_K}(\mathcal{P}_\sigma)$ be the category of bundle gerbe modules of $\mathcal{G}_\sigma$. They showed that

$$K^0_\sigma(M) \cong K(\mathcal{E}_{bg}^{U_K}(\mathcal{P}_\sigma)).$$

3.4 Fourth Definition (Canberra Branch)

Carey-Wang introduced the bundle of Hilbert-Schmidt operators associated to $\mathcal{P}_\sigma$,

$$\mathbb{A}_\sigma = \mathcal{P}_\sigma \times_{PU(\mathcal{H})} End_{HS}(\mathcal{H}).$$

Denote

$$\mathcal{U}_2 = \{1 + \text{Hilbert-Schmidt operator}\} \subset U(\mathcal{H}).$$

A Hilbert bundle E over M with structure group $\mathcal{U}_2$ is called a $\mathbb{A}_\sigma$-module if there is a bundle homomorphism:

$$\mathbb{A}_\sigma \otimes E \longrightarrow E.$$

Let $\mathcal{E}_{\mathcal{U}_2}^{\mathbb{A}_\sigma}(M)$ be the category of $\mathbb{A}_\sigma$-modules and $\mathbb{A}_\sigma$-morphisms. We showed that

$$K(\mathcal{E}_{bg}^{U_K}(\mathcal{P}_\sigma)) = K(\mathcal{E}_{\mathcal{U}_2}^{\mathbb{A}_\sigma}(M)).$$

Remark 14.

- *The above definitions can be generalized to locally compact manifolds, then* $K_\sigma^i(M) \cong K_\sigma^{i+1}(M \times \mathbb{R})$.
- *Equipped the underlying gerbe with a bundle gerbe connection and curving, a variant twisted K-theory with twisting given by a degree 3 Deligne class can be* similarly *defined as in Fourth definition. This version is crucial for the definition of twisted Chern characters.*

3.5 Main properties of twisted K-theory

1. Functorial property for proper maps:

2. Mayer-Vietoris sequence:

3. Thom isomorphism: Let $\pi : V \to M$ be a real oriented vector bundle of rank k, then

$$K^i_{\pi^*\sigma}(V) \cong K^{i+k \ mod(2)}_{\sigma+W_3(V)}(M)$$

for any $\sigma \in H^3(M, \mathbb{Z})$ and $W_3(V)$ is the third integral Stiefel-Whitney class.

Wrong way functoriality: Let $f : X \to Y$ be a differential map, there exists a canonical homomorphism:

$$f_! : K^*_{f^*\sigma + W_3(f)}(X) \longrightarrow K^{*+d(f)}_{\sigma}(Y)$$

for $\sigma \in H^3(Y, \mathbb{Z})$, $W_3(f) = W_3(X) + f^* W_3(Y)$ and $d(f) = dim(X) - dim(Y)(mod\ (2))$.

Moreover,

$$(f \circ g)_! = f_! \circ g_!$$

for smooth maps $f : X \to Y$ and $g : Y \to Z$.

Remark 15.

- *Note that* $W_3(f) = 0$ *if and only if* f *is K-oriented. The above result generalizes the wrong way functoriality of Connes-Skandalis for the K-oriented push-forward map in ordinary K-theory, and the result of Bouwknegt-Evslin-Mathai for the K-oriented push-forward map in twisted K-theory.*
- *Equivariant version for a compact Lie group action can be established with due attention to various topologies and third integral equivariant Stiefel-Whitney classes.*

4 Application II

4.1 D-brane charge

Freed-Witten defined a Type IIB D-brane on a 10-dimensional spin manifold X with a B-field, whose characteristic class is $\sigma \in H^3(X, \mathbb{Z})$, as a cycle

$$\iota : M \longrightarrow (X, \sigma)$$

satisfying

$$\iota^*\sigma + W_3(M) = 0,$$

together with a complex vector bundle E over M.

With the push-forward map at hand, we can define the charge of D-branes taking values in twisted K-theory:

$$\iota_! : K^0(M) \cong K^0_{\iota^*\sigma + W_3(M)}(M) \longrightarrow K^*_\sigma(X).$$

4.2 Index theory (Even case)

Recall the Atiyah-Singer index theorem for an even dimensional closed manifold X

$$
\begin{array}{ccccc}
 & & K^0(T^*X) & & \\
 & \swarrow & & \searrow & \\
K_0^t(X) & & \xrightarrow{\quad\cong\quad} & & K_0^a(X), \\
 & \searrow^{\epsilon_*} & & \swarrow_{\epsilon_*} & \\
 & & \mathbb{Z} & &
\end{array}
$$

where $K_0^t(X)$ and $K_0^a(X)$ are the topological and analytical K-homologies respectively, $\epsilon : X \to pt$.

Analytical D-branes: $\mathbb{Z}_2$-graded Fredholm module over $C(X)$, modulo operator homotopy equivalence.

Topological D-brane: a closed $Spin^c$ manifold M with a continuous map $\iota : M \to X$, a complex vector bundle E over M; modulo cobordism, disjoint & direct sum and vector bundle modifications (K-oriented maps).

Apply the Thom isomorphism $K^0(T^*X) \cong K^0_{W_3(X)}(X)$ and the push-forward map $\epsilon_!$, we have another interpretation of the Atiyah-Singer index theorem:

$$
\begin{array}{ccc}
K^0(T^*X) & \xrightarrow{\cong} & K^0_{W_3(X)}(X). \\
 & {\scriptstyle a-Index} \searrow \quad \swarrow {\scriptstyle \epsilon_!} & \\
 & \mathbb{Z} &
\end{array}
$$

Similarly, the equivariant Atiyah-Singer index theorem for compact group G action:

$$
\begin{array}{ccc}
K^0_G(T^*X) & \xrightarrow{\cong} & K^0_{W_3^G(X)}(X). \\
 & {\scriptstyle a-Index} \searrow \quad \swarrow {\scriptstyle \epsilon_!} & \\
 & R(G) &
\end{array}
$$

4.3 Index theory for discrete torsion

Let Γ be a discrete group, acting properly on a manifold X. A $U(1)$-valued group 2-cocycle σ determines a central extension

$$1 \to U(1) \to \hat{\Gamma}_\sigma \to \Gamma \to 1.$$

The $\hat{\Gamma}_\sigma$ equivariant K-group $K^0_{\Gamma,\sigma}(X)$ is defined by Baum-Connes using Γ-compactly supported complexes of $\hat{\Gamma}_\sigma$ equivariant vector bundles over X. They showed that

$$K^i_{\Gamma,\sigma}(X) \cong K_i(C_0(X) \rtimes_\sigma \Gamma).$$

There is a canonical (flat) gerbe associated to this action groupoid and σ. If Γ acts on X freely, then we have

$$K^i_{\Gamma,\sigma}(X) \cong K^i_{f^*\sigma}(X/\Gamma)$$

where $f : X/\Gamma \to B\Gamma$ is the classifying map of the Γ-bundle $\pi : X \to X/\Gamma$, and

$$\sigma \in H^2(\Gamma, U(1)) \cong H^2(B\Gamma, U(1)).$$

For simplicity, we assume that $w_2^\Gamma(X) = 0$, that is, X admits a Γ-invariant spin structure. We can think $f^*\sigma$ as a degree 3 Deligne cohomology class whose curvature is zero. Then there is a well-defined Chern character

$$ch_\sigma : K^0_{\Gamma,\sigma}(X) \longrightarrow H^*(X/\Gamma, \mathbb{R}).$$

As $w_2^\Gamma(X) = 0$, we have

$$K^0_{\Gamma,\sigma}(T^*X) \cong K^0_{\Gamma,\sigma}(X).$$

Let E represent a twisted K-class in $K^0_{\Gamma,\sigma}(X)$, the corresponding twisted K-class $[\not{D}^\sigma_E]$ in $K^0_{\Gamma,\sigma}(T^*X)$ has a well-defined analytical index (see Baum-Connes)

$$a - Index(\not{D}^\sigma_E) \in K_0(C^*_r(\Gamma, \sigma)).$$

Applying the Γ-trace, we have the following index theorem for discrete torsion:

$$Trace_\Gamma(a - Index(\not{D}^\sigma_E)) = \int_{X/\Gamma} \hat{A}(X/\Gamma) ch_\sigma(E).$$

Assume that $\sigma = e^{2\pi i\rho}$ for a $\mathbb{R}$-valued group 2-cocyle ρ. Baum-Connes showed that there exists an isomorphism $r : K^0_\Gamma(X) \to K^0_{\Gamma,\sigma}(X)$ making the following diagram commutative:

$$\begin{array}{ccc} K^0_\Gamma(X) & \longrightarrow & K^0_{\Gamma,\sigma}(X) \\ \Big\downarrow{\scriptstyle ch} & & \Big\downarrow{\scriptstyle ch} \\ H^*(X/\Gamma,\mathbb{R}) & \xrightarrow{e^{2\pi i f^*\rho}} & H^*(X/\Gamma,\mathbb{R}). \end{array}$$

Now applying the index theorem for discrete torsion, we get

$$Trace_\Gamma(a - Index(\not{D}^\rho_E) = \int_{X/\Gamma} \hat{A}(X/\Gamma) e^{2\pi i f^*\rho} ch(E).$$

This is the index theorem of Baum-Connes (1982), restated by Gromov (1991), and proved by Mathai (1998) using heat kernel method.

Potential application to problems in geometry and topology is under investigation.

4.4 Freed-Hopkins-Teleman Theorem

$$\boxed{K^{dimG}_{G,(\kappa+h^v,\chi)}(G) \cong R_{\kappa,\chi}(LG)},$$

for $(\kappa, \chi) \in H^3_G(G, \mathbb{Z})$ and $\kappa + h^v > 0$, where h^v is the dual Coxeter number of G.

In the light of our work, we can interpret Freed-Hopkins-Teleman Theorem using the quantization functor and the push-forward map along ΩG:

$$\begin{array}{ccc} & K(\mathcal{E}_{G,(\chi,\kappa)}) & \\ {\scriptstyle \mathcal{Z}_{G,(\chi,\kappa)}} \swarrow & & \searrow {\scriptstyle \text{Push-forward}} \\ R_{\kappa,\chi}(LG) & \xrightarrow{\;\cong\;} & K^{dimG}_{G,(\kappa+h^{v},\chi)}(G) \end{array}$$

where the dual Coxeter shift comes from the level of spin representation along ΩG.

Remark 16. $K^{dimG}_{G,(\kappa+h^{v},\chi)}(G)$ *admits a ring structure if and only if the twisting* (κ,χ) *is transgressive. For those* (κ,χ) *not transgressive, we need further twisting from* $H^1(G,U(1)) = Hom(\pi_1(G),\mathbb{Z})$.