Seiberg-Witten-Floer Theory for Homology 3-Spheres

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Feb. 10, 1996

Abstract

We give the definition of the Seiberg-Witten-Floer homology group for a homology 3-sphere. Its Euler characteristic number is a Casson-type invariant. For a four-manifold with boundary a homology sphere, a relative Seiberg-Witten invariant is defined taking values in the Seiberg-Witten-Floer homology group, these relative Seiberg-Witten invariants are applied to certain homology spheres bounding Stein surfaces.

1 Introduction

Since Seiberg and Witten introduced a new gauge theory through the monopole equations [24], the power of these new invariants arising from the monopoles and the subtlety they capture through the scalar curvature has been used to prove the "Thom" conjecture and progress has been made on the "$\frac{11}{8}$" conjecture [15] [13]. Furthermore, as conjectured by Witten and supported by physicists' calculations, the Seiberg-Witten invariants appear to be equivalent to Donaldson's polynomial invariants.

In recent years progress on instanton theory has been rapid through the study of the relative Donaldson invariant for 4-manifolds with cylindrical ends, where Floer instanton homology enters very naturally. It is reasonable to expect that Seiberg-Witten theory for 3-manifolds could play a similar role. In this paper, we give a mathmatically rigorous definition of an analogue of the Floer homology group for a homology 3-sphere where in this case any line bundle has trivial first Chern class. For a 3-manifold with nontrivial first Betti number, Marcolli [18] constructs the Seiberg-Witten-Floer theory, which is independent of metric and perturbation, therefore, a topological invariant. In our case, since the reducible solution, whose spinor is vanishing, cannot be perturbed away, more care must be taken for the reducible solutions. For those metrics whose ordinary Dirac operator have trivial kernel, then the reducible solution as a critical point is isolated and unique, we can define the

Seiberg-Witten-Floer homology by removing this reducible solution. In this present paper, we analyse the asymptotic behaviour of the gradient flows connecting two critical pionts. Using the gluing arguments, we show that the Seiberg-Witten-Floer homology (removing the reducible solution) is well-defined. The subject of metric (perturbation) dependence is interesting. In concurrent joint work with M. Marcolli, we study equivariant Seiberg-Witten-Floer theory which is metric independent up to *index-shifting*. Interestingly, not only are there Seiberg-Witten invariants for two critical points with relative index 1, but also Seiberg-Witten invariants for two critical points with relative index 2, which essentially measure the interactions of the irreducible critical points with the unique reducible one (see [19] for details).

Our main strategy is to study the Seiberg-Witten equations near the critical points using hyperbolic, non-linear equations. We give a detailed analysis to derive exponential decay estimates for the solutions and use these decay estimates to get a useful gluing theorem analogous to *Taubes' constructions* in instanton theory. These gluing theorems reflect the fact that Floer's work for instanton homology can be adapted to define the Seiberg-Witten-Floer homology invariant. It is well-known now that the Floer homology group has a non-trivial cup-product (the so-called quantum cup-product), which has encoded the rich information from quantum string theory. One may expect these also from the Seiberg-Witten-Floer homology group. (In preparation is the case of a circle bundle over a Riemann surface.) This is likely to be a very interesting topic for the future in the field of topological quantum field theory and topological quantum gravity. The mathematical part of this is Taubes' astonishing result in [21] which asserts that the Seiberg-Witten invariant for a symplectic manifold X equals the Gromov invariant for a homology class in $H_2(X, \mathbb{Z})$ associated with the $Spin^c$ structure for the Seiberg-Witten invariant.

2 Seiberg-Witten equations on homology 3-sphere Y and its cylinder $Y \times \mathbb{R}$

2.1 Seiberg-Witten equations on $Y \times \mathbb{R}$

In this subsection, we review the Seiberg-Witten equations on a compact, connected, closed, oriented, homology 3-sphere Y and its cylinder $Y \times \mathbb{R}$, with a $Spin^c$ structure on $Y \times \mathbb{R}$ given by pulling back a $Spin^c$ structure on Y. The two $Spin^c$ bundle S^+ and S^- can be identified via the Clifford multiplication by dt, with the $Spin^c$ bundle S on Y. Since Y is a spin manifold, we can write the determinant bundle of S as L^2 for some line bundle L on Y.

The Seiberg-Witten equations for $Y \times \mathbb{R}$ are the equations for a unitary $U(1)$-connection A on L and a spinor section $\psi \in \Gamma(S)$:

$$
\begin{aligned}
F_A^+ &= \frac{1}{4} < e_i e_j \psi, \psi > e^i \wedge e^j \\
\not{D}_A(\psi) &= 0
\end{aligned} \tag{1}
$$

where F_A^+ is the self-dual part of the curvature of A, $\{e_i\}_{i=1}^4$ is the orthonormal frame for $T(Y \times \mathbb{R})$ acting on spinors by Clifford multiplication. We adopt the usual convention for the Clifford algebra: $e_i e_j + e_j e_i = -2\delta_{ij}$, with $\{e^i\}_{i=1}^4$ its dual basis elements for $T^*(Y \times \mathbb{R})$. We denote by $<,>$ the hermitian inner product and sometimes write $\overline{\psi}_1 \psi_2 =< \psi_1, \psi_2 >$. Here $\not{D}_A$ is the Dirac operator on $Y \times \mathbb{R}$ associated with the connection A.

We say that a pair (A, ψ) is in temporal gauge if the dt-component of A vanishes identically. In this temporal gauge, (A, ψ) on $Y \times \mathbb{R}$ can be written as a path $(A(t), \psi(t))$ on Y obtained by restricting (A, ψ) to the slices $Y \times \{t\}$. Then the Seiberg-Witten equations (1) read as follows ([15], [17], [4]).

$$
\begin{aligned}
\frac{\partial A}{\partial t} &= *F_A - q(\psi) \\
\frac{\partial \psi}{\partial t} &= \not{\partial}_A(\psi)
\end{aligned} \tag{2}
$$

where $q(\psi)$ is the quadratic function of ψ, in local coordinates,

$$
q(\psi) = \frac{1}{2} < e_i.\psi, \psi > e^i,
$$

and $*$ is the Hodge star operator on Y, $\not{\partial}_A$ is the Dirac operator on Y twisted with a time-depedent connection A.

As we work in temporal gauge, the permitted gauge transformations are constant with respect to the t-direction and we denote this gauge group by $\mathcal{G} = Map(Y, U(1))$. It is easy to see that the equations (2) are invariant under the gauge group $\mathcal{G}$

$$
(A, \psi) \mapsto (A - id\tau, e^{i\tau}\psi).
$$

As first noted in [15], the Seiberg-Witten equations (2) are a gradient flow equation on $\mathcal{A}$ (the space of pairs (A, ψ)) of a functional C. Before we introduce this functional, let us give an appropriate Sobolev norm on the configuration space $\mathcal{A}$. We choose to work with L^2_1-connections A and L^2_1-sections of the associated $Spin^c$ bundle, and the gauge transformations in $\mathcal{G}$ consist of L^2_2-maps from Y to $U(1)$, these gauge transformations are at least continuous by the Sobolev embedding theorem. With this metric on $\mathcal{A}$, the tangent space of $\mathcal{A}$ is the space of L^2_1-sections of $\Omega(Y, i\mathbb{R}) \oplus \Gamma(S)$. The inner product on this tangent space is the L^2-product on the one forms and twice the real part of the L^2-hermitian product, that is, for the tangent vectors $a, b \in \Omega(Y, i\mathbb{R})$, the L^2 inner product is given by

$$\langle a, b\rangle_{L^2} = -\int_Y a \wedge *b$$

where $*$ is the complex-linear Hodge-star operator on Y, for the tangent vectors $\psi_1, \psi_2 \in \Gamma(S)$, the L^2 inner product is given by

$$\langle \psi_1, \psi_2\rangle_{L^2} = \int_Y (\overline{\psi}_1\psi_2 + \overline{\psi}_2\psi_1)$$

We drop the subscript L^2 when there is no danger of confusion in the following definition of the functional.

Definition 2.1 *Fix a $U(1)$-connection B on L, define the functional C as*

$$C(A, \psi) = \frac{1}{2}\int_Y (A - B) \wedge F_A + \langle \psi, \partial\!\!\!/_A\psi\rangle dvol(Y) \tag{3}$$

For our $\mathbb{Z}$-homology sphere, this functional can be reduced to the quotient space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ (which is a Hausdorff space by the following lemma). Denote by $\mathcal{B}^*$ the space of the irreducible pairs (A, ψ) (where $\psi \neq 0$) modulo the action of the gauge group $\mathcal{G}$. The following lemma is a standard result of elliptic regularity.

Lemma 2.2 *(1) The gauge group $\mathcal{G}$ acts smoothly on $\mathcal{B}$. The quotient $\mathcal{B}^*$ is a smooth Hilbert manifold with tangent spaces*

$$T_{[A,\psi]}(\mathcal{B}^*) = \{(\alpha, \phi) \in \Omega(Y, i\mathbb{R}) \oplus \Gamma(S) | d^*\psi = \overline{\psi}\phi - \overline{\phi}\psi\}.$$

(2) The charts for $\mathcal{B}^*$ near $[A, \psi]$ are given by

$$T_{[A,\psi]}(\mathcal{B}^*) \longrightarrow \mathcal{B}^*; \qquad (a, \phi) \mapsto [A + a, \psi + \phi]$$

Proposition 2.3 *([15][20]) Fix an interval $[0, 1]$, if $[A, \psi]$ is the solution of the Seiberg-Witten equations (2), then as a path in $\mathcal{B}$, $[A, \psi]$ satisfies the gradient flow equation*

$$\frac{\partial(A, \psi)}{\partial t} = \nabla C_{[A,\psi]}$$

where $\nabla C_{[A,\psi]} = (*F_A - q(\psi), \partial\!\!\!/_A(\psi))$ is the L^2-gradient vector field on $\mathcal{B}$ defined by the functional C.

2.2 Seiberg-Witten equations on Y and critical points for C

We have shown that the critical points of C are solutions of the Seiberg-Witten equations on Y,

$$
\begin{aligned}
*F_A &= q(\psi) \\
\partial\!\!\!/_A(\psi) &= 0
\end{aligned}
\tag{4}
$$

For the Seiberg-Witten equations (4) on a homology 3-sphere Y, we find an interesting phenomenon, that is the reducible solutions cannot be perturbed away by changing the metric or perturbing the curvature equations. However, for the metric whose oridinary Dirac operator has trivial kernel, the reducible solution as a critical point in $\mathcal{B}$ is non-degenerate, hence isolated. Note that for the homology sphere, there is only one reducible critical point.

Lemma 2.4 *For the metric g on Y, whose ordinary Dirac operator $\partial\!\!\!/$ has trivial kernel, then the only reducible critical point $[0,0]$, the trivial solution for (4), is isolated and nondegenerate.*

Proof. Since our Y is a $\mathbb{Z}$-homology 3-sphere, the reducible solution for (4) is $(A, 0)$ where $F_A = 0$, that means that A is abelian flat connection, determined by the $U(1)$-representation for the fundamental group $\pi_1(Y)$. The connection A must be in the orbit through the trivial connection on L. The isolated property can be proved using the Kuranishi model for the reducible solution $[0,0]$, here we use the perturbation theory by expanding the solution near $[0,0]$. First, note that if a solution $[A,\psi]$ of (4) is sufficiently close to $[0,0]$, then $[A,\psi]$ obeys the following equations

$$
\begin{aligned}
&d^*A = 0 \\
&*dA = \overline{\psi}\sigma\psi \\
&\partial\!\!\!/(\psi) + A.\psi = 0
\end{aligned}
$$

where $\sigma = c(e_i)e^i$, $c(e_i)$ is the Clifford multiplication of $\{e_i\}_{i=1}^3$. We can write (A,ψ) as

$$
\begin{aligned}
A &= \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 + \cdots \\
\psi &= \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \cdots
\end{aligned}
$$

Then from the Dirac equation and $ker\partial\!\!\!/ = 0$, we know that $\psi_1 = 0$, use this fact and the curvature equation, we know that a_i $(i = 1, 2, 3)$ are colsed and co-closed, for a homology 3-sphere, this leads to $a_i (i = 1, 2, 3)$ must be zero. Repeat this procedure, we get $[A, \psi]$ is zero. The non-degenerate property for $[0, 0]$ is equivalent to the isolated property in some sense. The Hessian operator at $[0, 0]$ is the map,

$$K : \qquad T_{[0,0]}(\mathcal{B}) \longrightarrow T_{[0,0]}(\mathcal{B})$$

$$(a, \phi) \mapsto (*da, \partial\!\!\!/(\phi))$$

where $d^* a = 0$, it is easy to see that $kerK = 0$. □

Let $\mathcal{M}_Y$ denote the set of the critical points of C on $\mathcal{B}$, i.e. the set of solutions of (4) modulo the gauge group $\mathcal{G}$ action.

For $[A_0, \psi_0] \in \mathcal{M}_Y, \psi_0 \neq 0$, the Hessian operator K is the following operator, it is the linearization of (4) on $\mathcal{B}^*$,

$$K_{[A_0,\psi_0]} : \qquad T_{[A_0,\psi_0]}(\mathcal{B}^*) \longrightarrow T_{[A_0,\psi_0]}(\mathcal{B}^*)$$

$$\begin{pmatrix} a \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} *d & -Dq_{\psi_0} \\ .\psi_0 & \partial\!\!\!/_{A_0} \end{pmatrix} \begin{pmatrix} a \\ \phi \end{pmatrix}$$

where Dq_{ψ_0} is the linearization of q at ψ_0, and ψ_0 acts on a by $a.\psi_0$.

In general, K may not be non-degenerate at $[A_0, \phi_0]$, but it has index zero, this can be seen from the fact that K is a compact perturbation of the Dirac operator and

$$\begin{pmatrix} 0 & d^* \\ d & *d \end{pmatrix} : \quad \Omega^0(Y, i\mathbb{R}) \oplus \Omega^1(Y, i\mathbb{R}) \to \Omega^0(Y, i\mathbb{R}) \oplus \Omega^1(Y, i\mathbb{R})$$

Both have index zero on three manifolds. This means that after a generic perturbation, K would have non-degenerate critical points set M_Y. Before introducing the perturbation, we study the geometry of the moduli space $\mathcal{M}_Y$ from the deformation complex at $[A_0, \phi_0]$:

$$0 \to \Omega^0(Y, i\mathbb{R}) \xrightarrow{G} \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(S) \xrightarrow{L} \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(S)/(G(\Omega^0(Y, i\mathbb{R})) \qquad (5)$$

where G is the infinitesimal action of the gauge group $\mathcal{G}$,

$$\tau \mapsto (-d\tau, \tau\psi)$$

and the map L is the linearization of (4) on $\mathcal{C}$ at (A_0, ψ_0),

$$\begin{pmatrix} a \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} *d & -Dq_{\psi_0} \\ .\psi_0 & \partial\!\!\!/_{A_0} \end{pmatrix} \begin{pmatrix} a \\ \phi \end{pmatrix}$$

The Zariski tangent space of $\mathcal{M}_Y$ at $[A_0, \phi_0]$ is given by

$$ker(L)/Im(G).$$

In order to see that $\mathcal{M}_Y^* = \mathcal{M}_Y \setminus \{[0,0]\}$ is a 0-dimensional smooth submanifold in $\mathcal{B}^*$, we need to consider a generic perturbation for (4) by proving $ker(L) = Im(G)$. In instanton theory, there is a generic metric theorem. It states that for a generic metric (that is lying in an open dense subset in the space of metrics) the instanton moduli space is a smooth manifold. There is no such generic metric theorem for us. But as in [15] [24], we can perturb the curvature equation by adding a suitable 1-form.

Now consider the perturbed functional C' as follows,

$$C'(A, \psi) = \frac{1}{2} \int_Y (A - B) \wedge (F_A - 2d\alpha) + \langle \psi, \partial\!\!\!/_A \psi \rangle dvol(Y)$$

where α is a purely imaginary 1-form on Y, as we remark below, this is the only possible perturbation for the curvature equation.

Then the perturbed Seiberg-Witten equations on Y (the solutions are the critical points for C') are given by

$$\begin{aligned} *F_A &= q(\psi) + *d\alpha \\ \partial\!\!\!/_A(\psi) &= 0 \end{aligned} \tag{6}$$

while the gradient flow equation (the perturbed Seiberg-Witten equations on the cylinder $Y \times \mathbb{R}$) is given by

$$\begin{aligned} \frac{\partial A}{\partial t} &= *F_A - q(\psi) - *d\alpha \\ \frac{\partial \psi}{\partial t} &= \partial\!\!\!/_A(\psi) \end{aligned} \tag{7}$$

Remark 2.5 *We point out that $q(\psi)$ is co-closed for ψ satisfying the Dirac equation $\partial\!\!\!/_A(\psi) = 0$, therefore the perturbing form in (6) (7) must be also co-closed by the Bianchi identity $d^*(*F_A) = 0$. For a homology 3-sphere, it is $*d\alpha$ where α is any imaginary-valued one form on Y.*

We denote the moduli space $\mathcal{M}_{Y,\alpha}$ for the solution space of (6) modulo $\mathcal{G}$.

Proposition 2.6 *For any metric g on Y, there is an open dense set in $*d(\Omega^1_{L^2_2}(Y, i\mathbb{R}))$, such that $\mathcal{M}^*_{Y,\alpha}$ is a smooth 0-dimensional manifold and the only reducible solution $[\alpha, 0]$ is isolated for any $*d\alpha$ in this set.*

Proof. This is an application of the Sard-Smale theorem as [15] [18]. First we construct the parametrized moduli space,

$$\begin{aligned}\mathcal{M} &= \bigcup_{\delta = *d\alpha} \mathcal{M}_{Y,\alpha} \\ &\subset (\mathcal{A} \times *d(\Omega^1_{L^2_2}(Y, i\mathbb{R})))/\mathcal{G}\end{aligned}$$

We write $\mathcal{M}^*, \mathcal{M}^*_{Y,\alpha}$ for the moduli spaces of irreducible solutions. Let (A_0, ψ_0, δ) be the solution of (6) with $\psi \neq 0$. The linearization of the equations at this point is the following operator:

$$P: \qquad \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(S) \oplus *d(\Omega^1(Y, i\mathbb{R})) \to \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(S)/(G(\Omega^0(Y, i\mathbb{R}))$$

$$\begin{pmatrix} a \\ \phi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} *d & -Dq_{\psi_0} & -Id \\ .\psi_0 & \partial\!\!\!/_{A_0} & 0 \end{pmatrix} \begin{pmatrix} a \\ \phi \\ \eta \end{pmatrix}$$

To estabilish surjectivity for P, it is sufficient to show that no element (b, ρ) in the range can be L^2-orthogonal to the image of P. If such (b, ρ) is orthogonal, then by varying η alone, one can get that b is 0. Varying ϕ alone, ρ must be in the kernel of $\partial\!\!\!/_{A_0}$, then varying a alone, one can see ρ is also 0, otherwise since being a solution of the Dirac equation $\partial\!\!\!/_{A_0} = 0$, ψ and ρ cannot vanish on an open set, note that, modulo the gauge action in the range, we can construct an a such that $(0, \rho)$ is not orthogonal to $P(a, 0, 0)$.

Hence by the implicit function theorem, we know that $\mathcal{M}^*$ is a smooth manifold, now apply the Sard-Smale theorem to the projection map:

$$\pi: \quad \mathcal{M}^* \longrightarrow *d(\Omega^1(Y, i\mathbb{R})$$

we get there is a generic $\delta = *d\alpha$ such that $\mathcal{M}^*_{Y\alpha}$ is a smooth manifold, by the index calculation, it is 0-dimensional.

Consider the isolated property of the only reducible point $[\alpha, 0]$. By an argument similar to the proof of Lemma 2.4, the perturbing form $*d\alpha$ should satisfy $ker\partial\!\!\!/_\alpha = 0$, this is a co-dimension one condition for α. The existence of an open dense set of such α still holds. $\square$

3 Properties of various moduli spaces

3.1 The compactness of the moduli space

There is a great simplicity in Seiberg-Witten gauge theory, that is, the moduli space of Seiberg-Witten equations is always compact. In this subsection, we will show that $\mathcal{M}_{Y,\alpha}$ is compact and, since the reducible point in $\mathcal{M}_{Y,\alpha}$ is isolated, for a generic sufficiently small perturbation, $\mathcal{M}^*_{Y,\alpha}$ is a set with finitely many points, consisting of the irreducible, non-degenerate critical points of C'. Moreover, we can also prove the moduli space of the Seiberg-Witten equations (7) connecting the two critical points in $\mathcal{M}_{Y,\alpha}$ is also compact.

Proposition 3.1 *$\mathcal{M}_{Y,\alpha}$ is sequentially compact, that is, for any sequence $\{(A_i, \psi_i)\}$ of the solution (4) or (6), there is a subsequence (up to L^2_2 gauge tranformations) converging to a solution in C^∞-topology. Therefore, $\mathcal{M}_{Y,\alpha}$ contains only finite points in $\mathcal{B}$.*

This follows from a priori bounds for any solution of (4). There are several proofs on the compactness of the Seiberg-Witten moduli space in the literature such as [15] [23] [17]. We omit the proof of this proposition.

Now we are in a position to discuss the moduli space of the connecting orbits. The functional C or C' has a nice property, since it satisfies the Palais-Smale condition.

Lemma 3.2 *For any $\epsilon > 0$, there is $\lambda > 0$ such that if $[A, \psi] \in \mathcal{B}$ has the L^2_1-distance at least ϵ from all the critical points in $\mathcal{M}_{Y,\alpha}$, then*

$$\|\nabla C'_{[A,\psi]}\|_{L^2} > \lambda$$

A similar result was obtained in [20] for three manifolds $S^1 \times \Sigma$, where Σ is a Riemann surface of genus $g > 1$ whose determinant line bundle for the $Spin^c$ bundle is pulled back from a line bundle on the Riemann surface which has degree $2 - 2g$.

Proof. Suppose there is a sequence (A_i, ψ_i) in $\mathcal{B}$ whose L^2_1-distance are at least ϵ from all the critical points in $\mathcal{M}_{Y,\alpha}$ for which

$$\|\nabla C'_{[A_i,\psi_i]}\|_{L^2} \to 0 \qquad \text{as } i \to \infty$$

Then as $i \to \infty$, we have

$$\| * F_{A_i} - q(\psi_i) - *d\alpha\|_{L^2} \to 0$$
$$\|\partial\!\!\!/_{A_i}(\psi_i)\|_{L^2} \to 0$$

where $*d\alpha$ is sufficiently small. This means that there is a contant $C > 0$ such that

$$\int_Y |*F_{A_i} - q(\psi_i) - *d\alpha|^2 + |\partial\!\!\!/_{A_i}(\psi_i)|^2 < C$$

Resorting to the Weitzenbock formula for $\partial\!\!\!/^2_{A_i}$ and for $*d\alpha$ sufficiently samll, the above inequality reads as

$$\int_Y |F_{A_i}|^2 + |q(\psi_i)|^2 + \frac{s}{2}|\psi_i|^2 + 2|\nabla_{A_i}\psi_i|^2 < 2C$$

It follows that $\|\psi_i\|_{L^4}, \|F_{A_i}\|_{L^2}, \|\nabla_{A_i}\psi_i\|_{L^2}$ are all bounded independent of i. Now use the standard elliptic argument as in the proof of the compactness, we know that there is a subsequence converging in L^2_1-topology to a solution of (6), this contradicts with the assumption (A_i, ψ_i) in $\mathcal{B}$ whose L^2_1-distance are at least ϵ from all the critical points in $\mathcal{M}_{Y,\alpha}$. □

Definition 3.3 *A finite energy solution of (7) on $Y \times \mathbb{R}$ is the solution $[A(t), \psi(t)]$ whose square of the energy*

$$\int_{-\infty}^{+\infty} \|\nabla C'([A(t), \psi(t)])\|^2_{L^2(Y)}$$

is finite.

Choose a sufficiently small ϵ for all the finite points in $\mathcal{M}_{Y,\alpha}$ such that Lemma 3.2 holds, we label the nondegenerate, irreducible critical points of C' as $x_1, \cdots, x_N$. Denote by $U(x_i, \epsilon)$ the L^2_1-open ball of radius ϵ. Let $\gamma(t) = [A(t), \psi(t)]$ is the gradient flow for C' with finite energy. From Lemma 3.2, one can see

$$\{t \in \mathbb{R} | [A(t), \psi(t)] \in U(x_i, \epsilon) \text{ for any } i\}$$

has total length finite and has only finite intervals running between different $U(x_i, \epsilon)$.

Therefore, for sufficiently large $T >> 1$, $\gamma|_{[T,\infty)} \subset U(x_i, \epsilon)$ for some x_i.

Similarly, $\gamma|_{[-T,-\infty)} \subset U(x_j, \epsilon)$ for some x_j. Actually, one can use the finite energy to get the limits of $\gamma(t)$ which are x_i, x_j respectively. In the next subsection, we will prove that $\gamma(t)$ exponentially decays to the limits.

We denote the moduli space of finite energy solutions for (7) which connect the two critical points a, b by

$$\mathcal{M}(a,b) = \left\{ x(t) \;\middle|\; \begin{array}{l} (1)\ x(t) \text{ is the finite energy solution of} \\ \quad\ (7) \text{ modulo the gauge group } \mathcal{G}, \\ (2)\ \lim_{t\to+\infty} x(t) = a, \\ (3)\ \lim_{t\to-\infty} x(t) = b. \end{array} \right\}$$

Remark 3.4 *Since any Seiberg-Witten monopole on $S^1 \times Y$ is invariant under the rotation action of S^1, there is no closed non-constant gradient flow connecting the same critical point, that is, $\mathcal{M}(a,a)$ is just one piont $\{a\}$. One can also see this from the following useful length equality:*

$$\begin{aligned} & C'([A(t_1),\psi(t_1)]) - C'([A(t_2),\psi(t_2)]) \\ = & \int_{t_1}^{t_2}\int_Y (\|\frac{\partial A(t)}{\partial t}\|^2 + \|\frac{\partial \psi(t)}{\partial t}\|^2) dvol_Y \\ = & \int_{t_1}^{t_2} \|\nabla C'([A(t),\psi(t)])\|^2_{L^2(Y)} \end{aligned}$$

Note that in this case, the finite energy condition for a connecting orbit is the same as the finite length condition for the corresponding gradient flow.

3.2 Transversality for $\mathcal{M}(a,b)$

In this subsection, we will prove that $\mathcal{M}(a,b)$ is a smooth manifold after a generic perturbation. As we need $\mathbb{R}$-translation action on the moduli space, one may expect to construct a time-invariant perturbation of the gradient flow equation to achieve the smoothness for $\mathcal{M}(a,b)$. Unfortunately, this perturbation can't achieve the transversality for $\mathcal{M}(a,b)$. In [12], Froyshov construct an explicit perturbation, here, we show that the transversality for $\mathcal{M}(a,b)$ is actually generic.

Suppose a, b are irreducible critical points for a fixed perturbation α, choose any smooth representations (A_1,ψ_1), (A_2,ψ_2) in $\mathcal{A}_{L^2_1}$ for a, b respectively, note that $\mathcal{A}_{L^2_1}$ is the L^2_1-configuration space, then (A_i,ψ_i) satisfies equation (6)

$$*F_A = q(\psi) + *d\alpha$$

$$\not{\partial}_A(\psi) = 0$$

Denote by $\mathcal{A}_{L^2_1}(a,b)$ the set

$$\left\{ (A,\psi):\ \mathbb{R} \to \mathcal{A}_{L^2_1} \ \middle|\ \begin{array}{l} (1)\ \lim_{t\to-\infty}(A,\psi) \text{ lies in the gauge orbit of } (A_1,\psi_1), \\ (2)\ \lim_{t\to+\infty}(A,\psi) \text{ lies in the gauge orbit of } (A_2,\psi_2), \\ (3)\ \int_{-\infty}^{+\infty}(\|\frac{\partial A(t)}{\partial t}\|^2_{L^2_1} + \|\frac{\partial \psi(t))}{\partial t}\|^2_{L^2_1})dt < \infty. \end{array} \right\}$$

It is easy to see that the tangent space of $\mathcal{A}_{L^2_1}(a,b)$ at $(A(t),\psi(t))$ is

$$T_{(A(t),\psi(t))}(\mathcal{A}_{L^2_1}(a,b)) = L^2_{1,0}(\mathbb{R}, \Omega^1(Y,i\mathbb{R}) \oplus \Gamma(S)) \oplus \Omega^0(Y,i\mathbb{R})$$

Then $\mathcal{M}(a,b)$ is the moduli space of the gradient flow eqation (7) defined on $\mathcal{A}_{L_1^2}(a,b)$. The perturbation we choose is to add a function of $A(t)$ in $L_{2,0}^2(\mathbb{R},\Omega^1(Y,i\mathbb{R}))$ to the curvature equation in (7).

$$\begin{aligned} \frac{\partial A}{\partial t} &= *F_A - q(\psi) - *d\alpha_{A(t)} \\ \frac{\partial \psi}{\partial t} &= \partial\!\!\!/_A(\psi) \end{aligned} \tag{8}$$

where $\alpha_{A(t)} \in \alpha + L_{2,0}^2(\mathbb{R},\Omega^1(Y,i\mathbb{R}))$. Note that this perturbed gradient flow equation preserves $\mathbb{R}$-translation, that is, for a fixed $\alpha_t = \alpha_{A(t)}$, if $(A(t),\psi(t))$ is a solution for (8), then $(A(t+s),\psi(t+s)$ (for $s \in \mathbb{R}$) is also a solution for (8).

Define the parametrized moduli space $\mathcal{M}^{\mathcal{P}}$ as

$$\mathcal{M}^P = \{(A,\psi,\alpha_t)\ |(A,\psi,\alpha_t) \text{ is a solution of } (8)\}/\mathcal{G}$$

For any solution $[A_0(t),\psi_0(t),\alpha_t] \in \mathcal{M}^P$, the linearisation of (8) is given by

$$L_{1,0}^2(\mathbb{R},\Omega^1(Y,i\mathbb{R}) \oplus \Gamma(S)) \oplus L_{2,0}^2(\mathbb{R},\Omega^1(Y,i\mathbb{R})) \to L_{1,0}^2(\mathbb{R},\Omega^1(Y,i\mathbb{R}) \oplus \Gamma(S))$$

$$D : \begin{pmatrix} A \\ \phi \\ \beta_t \end{pmatrix} \mapsto \frac{\partial}{\partial t}\begin{pmatrix} A \\ \phi \end{pmatrix} - \begin{pmatrix} *d & -Dq_{\psi_0} \\ .\psi_0 & \partial\!\!\!/_{A_0} \end{pmatrix}\begin{pmatrix} A \\ \phi \end{pmatrix} + \begin{pmatrix} \beta_t \\ 0 \end{pmatrix}$$

where $\beta_t \in L_{2,0}^2(\mathbb{R},\Omega^1(Y,i\mathbb{R}))$, we will choose α_t in (8) very small such that the index and the spectral flow defined in the next subsection are tha same as the perturbed operators.

From the surjectivity of D, we know that $\mathcal{M}^P$ is a smooth manifold. Now apply the Sard-Smale theorem to the projection map from $\mathcal{M}^P$ to $\alpha + L_{2,0}^2(\mathbb{R},\Omega^1(Y,i\mathbb{R}))$, for a generic perturbation (consisting an open dense set in $\alpha + L_{2,0}^2(\mathbb{R},\Omega^1(Y,i\mathbb{R}))$), to see that $\mathcal{M}(a,b)$ is a smooth manifold whose dimension is given by the relative Morse index defined in the next subsection.

3.3 Spectral flow and relative Morse index for critical pionts

As in subsection 3.2, the moduli space of the connecting orbits $\mathcal{M}(a,b)$ is a smooth, compact manifold with the correct dimension given by the index of the deformation complex. In this subsection, we describe a relative Morse index for the critical points in $\mathcal{M}_{Y,\alpha}$. In general, as in Floer instanton homology, the Morse index defined by the dimension of the negative space for the Hessian operator at such critical points is not well-defined, since the Hessian has infinite dimensional negative space. It is Floer's insight to introduce a relative index to overcome this diffculty. The dimension of the moduli space of the connecting orbits is given by this relative index. We adopt the same idea to give the index for our critical points.

In [1], Atiyah, Patodi and Singer proved that

$$Index(\frac{d}{dt} + T(t)) = \text{ spectral flow of } T(t) \tag{9}$$

where $T(t)$ is a path of Fredholm operators with invertible limits as $t \to \pm\infty$.

For any connecting $[A(t), \psi(t)] \in \mathcal{M}(a, b)$, here $a, b \in \mathcal{M}^*_{Y,\alpha}$. The linearization on $\mathcal{A}$ is given by

$$\frac{\partial}{\partial t} - \begin{pmatrix} *d & -Dq_{\psi(t)} \\ .\psi & \partial\!\!\!/_{A(t)} \end{pmatrix} \tag{10}$$

with the limits (as $t \to \pm\infty$):

$$\begin{pmatrix} *d & -Dq_{\psi_\pm} \\ .\psi_\pm & \partial\!\!\!/_{A_\pm} \end{pmatrix} \tag{11}$$

where $(A_\pm, \psi_\pm)$ are smooth representatives for a, b. These are precisely the Hessian operators at a, b and are invertible (note that Hessian at the only reducible critical point is also invertible see Proposition 2.6). Here we think of (11) as an operator on the L^2 tangent bundle, i.e., fixing the unique gauge $G(\tau)$, $\tau \in \Omega^0(Y, i\mathbb{R})$ given by (5), such that $K(t)+G(\tau)$ is well defined as a L^2-tangent vector of $\mathcal{B}^*$. Denote this gauged operator as $\hat{K}(t)$. At the critical point, such a τ is zero. We abuse the notation $K(t)$ and $\hat{K}(t)$ when it doesn't cause any confusion.

By the Atiyah-Patodi-Singer index theorem (9), the dimension of $\mathcal{M}(a, b)$ is determined by the spectral flow of $\hat{K}(t)$, the number of the eigenvalues crossing 0 from negative to positive minus the number of the eigenvalues crossing 0 from positive to negative. In [4], we use the results of this paper to give a spectral flow version of the definition of Casson type invariant and its $\mathbb{Z}_2$-version. In [18], M. Marcolli also gave a description of this invariant for other 3-manifolds.

To give a $\mathbb{Z}$-valued index for $a \in \mathcal{M}_{Y,\alpha}$, we need to consider a path $\gamma(t)$ from the trivial point $[0, 0]$ to a, then define the index at a as follows,

$$\mu(a) = \text{ spectral flow of } \hat{K} \text{ along } \gamma(t)$$

There is no ambiguity in the definition of the index $\mu(a)$, since for our homology 3-sphere Y, $H^1(\mathcal{A}, \mathbb{Z}) = 0$. One can see this by a simple observation $\dim\mathcal{M}(a, a)$ is zero, actually,

$\mathcal{M}(a,a)$ is just one point, the constant solution, by the identity

$$\int_{-\infty}^{+\infty} \|\nabla C'([A(t),\psi(t)])\|^2_{L^2(Y)}$$
$$= \|\frac{\partial A(t)}{\partial t}\|^2_{Y\times\mathbb{R}} + \|\frac{\partial \psi(t))}{\partial t}\|^2_{Y\times\mathbb{R}}$$
$$= \lim_{t\to\infty} C'(A(t),\psi(t)) - \lim_{t\to-\infty} C'(A(t),\psi(t))$$

which is zero for the solution $[A(t),\psi(t)]$ in $\mathcal{M}(a,a)$.

This will make the Seiberg-Witten-Floer homology graded by $\mathbb{Z}$. From the Atiyah-Patodi-Singer index theorem, we have the follow lemma.

Lemma 3.5 *For $a,b \in \mathcal{M}^*_{Y\ \alpha}$ with $\mu(a) > \mu(b)$, the moduli space of the "connecting orbits" $\mathcal{M}(a,b)$ is a $\mu(a)-\mu(b)$ dimensional, smooth, compact manifold. For $a \neq b$ with $\mu(a) \leq \mu(b)$, $\mathcal{M}(a,b)$ is empty.*

Proof. It is sufficient to prove that for $a \neq b$ with $\mu(a) = \mu(b)$ then $\mathcal{M}(a,b)$ is empty, this follows from the equations (8) being invariant under the action of the $\mathbb{R}$-translation, if $\mathcal{M}(a,b)$ is non-empty, its dimension has to be at least 1, this is impossible from the index calculation. □

Remark 3.6 *We didn't discuss the dimension of $\mathcal{M}(a,b)$ where either a or b is the only reducible critical point (which is also non-degenerate according to Lemma 2.4). By the excision principle,*

$$dim\mathcal{M}(a,b) = \mu(a) - \mu(b) - 1$$
$$dim\mathcal{M}(c,a) = \mu(c) - \mu(a)$$

where a is the reducible critical point. We delay this proof until we introduce our gluing theorem.

3.4 Decay estimates for the gradient flow near the critical points

In this subsection, we will study the gradient flow near the critical point $[A_0,\psi_0]$ of C', $\psi_0 \neq 0$, where $[A_0,\psi_0]$ obeys

$$*F_A = q(\psi) + *d\alpha$$
$$\not{\partial}_A(\psi) = 0 \tag{12}$$

By Lemma 2.2 (2), we know that $U([A_0,\psi_0],\epsilon)$ is diffeomorphic to a neighbourhood $U(0,\epsilon)$ of $T_{[A_0,\psi_0]}(\mathcal{B}^*)$. Using this relation, we can rewrite the Seiberg-Witten equations (7) on $T_{[A_0,\psi_0]}(\mathcal{B}^*)$ as follows,

$$\frac{\partial}{\partial t}\begin{pmatrix} A_0 + A \\ \psi_0 + \phi \end{pmatrix} = \begin{pmatrix} *F_{(A_0+A)} - q(\psi_0+\phi) - *d\alpha \\ \partial\!\!\!/_{(A_0+A)}(\psi_0+\phi) \end{pmatrix}$$

Applying (12), we simplify the gradient flow equation on $T_{[A_0,\psi_0]}(\mathcal{B}^*)$ as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ \phi \end{pmatrix} = \begin{pmatrix} *d & -Dq_{\psi_0} \\ .\psi_0 & \partial\!\!\!/_{A_0} \end{pmatrix}\begin{pmatrix} A \\ \phi \end{pmatrix} + \begin{pmatrix} -q(\phi) \\ A.\phi \end{pmatrix} \tag{13}$$

Write the quadratic term in (13) as $Q(A,\phi)$. Denote

$$\begin{pmatrix} *d & -Dq_{\psi_0} \\ .\psi_0 & \partial\!\!\!/_{A_0} \end{pmatrix}$$

as $K_{[A_0,\psi_0]}$, it is the Hessian operator of C' at $[A_0,\psi_0]$ the linearization of the perturbed Seiberg-Witten equations on Y. $K_{[A_0,\psi_0]}$ is a closed, self-adjoint, Fredholm operator from the L^2_1-completion of $T_{[A,M]}(\mathcal{B})$ to the L^2-completion of $T_{[A,M]}(\mathcal{B})$, and has only discrete spectrum without accumulation points.

From the preceding sections, we know that $K_{[A_0,\psi_0]}$ is a invertible operator acting on $T_{[A_0,\psi_0]}(\mathcal{B}^*)$, we can decompose $T_{[A_0,\psi_0]}(\mathcal{B}^*)$ as the direct sum of the positive eigenvalue spaces and the negetive eigenvalue space.

$$T_{[A_0,\psi_0]}(\mathcal{B}^*) = (\oplus_{\lambda>0}\mathcal{H}_\lambda)\oplus(\oplus_{\lambda<0}\mathcal{H}_\lambda) \tag{14}$$

where $\mathcal{H}_\lambda$ is the eigenspace with eigenvalue λ.

Fix $\delta < \min\{|\lambda|\}$, let the semi-group generated by $K_{[A_0,\psi_0]}$ be $\Phi(t_0,t)$, then $\Phi(t_0,t)$ preserves the decomposition (14) We write $x \in T_{[A_0,\psi_0]}(\mathcal{B}^*)$ as

$$x = x^+ + x^-$$

where $x^+ \in \mathcal{H}^+ = \oplus_{\lambda>0}\mathcal{H}_\lambda$ is called the stable part, $x^- \in \mathcal{H}^- = \oplus_{\lambda<0}\mathcal{H}_\lambda$ is called the unstable part. Denote $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.

We have the following hyperbolicity formula for $\Phi(t_0,t)$:

- For $x^+ \in \mathcal{H}^+, t \geq t_0$, we have

$$\|\Phi(t,t_0)x^+\| \leq \exp^{-\delta(t-t_0)}\|x^+\|.$$

- For $x^- \in \mathcal{H}^-, t \leq t_0$, we have

$$\|\Phi(t, t_0)x^-\| \leq \exp^{-\delta(t_0 - t)} \|x^-\|.$$

We can solve the equation (13) near 0 in $\mathcal{H}^+ \oplus \mathcal{H}^-$, let

$$x = \begin{pmatrix} A \\ \phi \end{pmatrix}$$

satisfy the following equation with some boundary data:

$$\begin{cases} \dfrac{\partial x}{\partial t} = K_{[A_0, \psi_0]}x + Q(x) \\ \|Q(x)\|_{L^2} \leq C_1 \|x\|^2_{L^2_1} \\ \|D_x Q(x)\|_{L^2} \leq \epsilon \\ Q(0) = 0. \end{cases} \tag{15}$$

Generally, the solution is described by the following integral equation,

$$x(t) = \Phi(t_0, t)x(t_0) + \int_{t_0}^{t} \Phi(s, t)Q(x(s))ds.$$

The following two lemmas can be obtained by applying the theory developed in [2], which is a nice model for decay estimates.

Lemma 3.7 *For sufficiently small p, q in $\mathcal{H}^+, \mathcal{H}^-$ respectively, $T_1 < T_2$, there is a unique solution $x(t) : [T_1, T_2] \to \mathcal{H}$ satisfying (13) such that*

$$\begin{cases} x^+(T_1) = p \\ x^-(T_2) = q \\ \|x(t)\|_{L^2_1} < \epsilon. \end{cases}$$

This solution depends smoothly on p, q, T_1, T_2. Denote $x(t) = x(t, p, q, T_1, T_2)$.

Proof. The solution must have the following form

$$\begin{aligned} x(t) &= \Phi(T_1, t)p + \int_{T_1}^{t} \Phi(s, t)Q^+(x(s))ds \\ &\quad + \Phi(T_2, t)q - \int_{t}^{T_2} \Phi(s, t)Q^-(x(s))ds. \end{aligned} \tag{16}$$

Write (16) as a fixed point equation:

$$x = F(x)$$

where x lies in the Banach space $C^0([T_1, T_2], \mathcal{H})$ with the supremum norm. The existence and uniqueness can be obtained by showing that the above map is a strong contraction on a sufficiently small ball $\|x\| < \eta$. This can be verified by (15) as follows,

$$\begin{aligned} \|F(x)\| &< C(\epsilon + |\int_{T_1}^{T_2} \exp^{\delta s} ds| \sup_{\|x\|<\eta} \|Q(x)\|) \\ &< C(\epsilon + \frac{C_1}{\delta}\eta^2) < \frac{1}{2}\eta \end{aligned}$$

provided that ϵ is sufficiently small. □

Lemma 3.8 *Suppose $x(t)$ is the solution in Lemma 3.7, then there are a priori estimates for $x(t)$ as follows where C_3 is a positive constant,*

$$\|x^+(t)\|_{L_1^2} \le C_3 \exp^{-\delta(t-T_1)}$$

$$\|x^-(t)\|_{L_1^2} \le C_3 \exp^{-\delta(T_2-t)}$$

$$\|\frac{\partial x^+(t)}{\partial p}\|_{L_1^2} \le C_3 \exp^{-\delta(t-T_1)}$$

$$\|\frac{\partial x^+(t)}{\partial q}\|_{L_1^2} \le C_3 \exp^{-\delta(t-T_1)}$$

$$\|\frac{\partial x^-(t)}{\partial p}\|_{L_1^2} \le C_3 \exp^{-\delta(T_2-t)}$$

$$\|\frac{\partial x^-(t)}{\partial q}\|_{L_1^2} \le C_3 \exp^{-\delta(T_2-t)}$$

Note that from the first two estimates, it is easy to see that the solution of (13) satisfies the following property,

$$\|x(t)\|_{L_1^2} \le C_3 \exp^{-\delta d(t)}$$

where $d(t) = \min\{t - T_1, T_2 - t\}$.

Proof. These estimates can be obtained by the standard open and closed argument and continuity. We only prove the first estimate. Define the non-empty open set

$$S = \{t \in [T_1, T_2] | \|x^+(t)\|_{L_1^2} \le C_3 \exp^{-\delta(t-T_1)}$$

The aim is to prove S is also closed. Since $T_1 \in S$, suppose $[T_1, t) \subset S$. Note that $x^+(t)$ can be write as

$$x^+(t) = \Phi(T_1, t)p + \int_{T_1}^{t} \Phi(s,t)Q^+(x)ds$$

Estimating the norm $x^+(t)$ shows that $t \in S$, hence $S = [T_1, T_2]$. $\square$

Proposition 3.9 *There are positive constants ϵ, δ, C_4, such that for any $T_2 >> T_1$, if $[A(t), \psi(t)]$ is the solution for the perturbed Seiberg-Witten equations in a temporal gauge on $[T_1, T_2] \times Y$ near the critical point $[A_0, \psi_0]$, that is, if for each $t \in [T_1, T_2]$*

$$dist_{L_1^2}([A(t), \psi(t)], [A_0, \psi_0]) < \epsilon$$

then there is a exponential decay estimate as follows,

$$dist_{L_1^2}([A(t), \psi(t)], [A_0, \psi_0]) \leq C_4 \exp^{-\delta d(t)}$$

where $d(t) = \min\{t - T_1, T_2 - t\}$. Therefore, if the gradient flow $[A(t), \psi(t)]|_{[T,\infty)}$ is sufficiently close to the critical point $[A_0, \psi_0]$, then as $t \to \infty$, $[A(t), \psi(t)]$ decays to $[A_0, \psi_0]$ exponentially. There is a similar exponential decay for the finite energy solution $[A(t), \psi(t)]$ as $t \to -\infty$.

Proof. We have identifed the perturbed Seiberg-Witten equations in temporal gauge and (13) on $\mathcal{H}$, then this proposition is a direct consequence of this identification and the above lemmas for the small solutions. $\square$

The following lemma gives a nice picture for the limits of the solution near the critical point, the broken trajectories appear in the limit. This will play an important role in the definitions of the boundary operators for the Floer complex and various chain-maps for metric-independence.

Lemma 3.10 *Let $t \mapsto x(t, p, q, -T, T)$ be the solution as in Lemma 3.7, As $T \to \infty$,*
(1) The trajectories: $\gamma_1(t): [0, T] \to \mathcal{H}$

$$\gamma_1(t) = x(t - T, p, q, -T, T)$$

approach the limit which lies in $\mathcal{H}^+$ (locally stable manifold).
(2) The trajectories: $\gamma_2(t): [0, T] \to \mathcal{H}$

$$\gamma_1(t) = x(T - t, p, q, -T, T)$$

approach the limit which lies in $\mathcal{H}^-$ (locally unstable manifold).

Proof. From Lemma 3.8, we know that

$$\|\gamma_1(t)^-\|_{L^2_1} = \|x^-(t-T,p,q,-T,T)\|_{L^2_1} \le C_3 \exp^{-\delta(T-(t-T))}$$

$$\|\gamma_2(t)^+\|_{L^2_1} = \|x^+(T-t,p,q,-T,T)\|_{L^2_1} \le C_3 \exp^{-\delta(T-t-(-t))}$$

then this lemma follows. □

3.5 Gluing arguments

In this subsection, we construct the gluing map which will enable us to build a Floer complex and various chain homomorphisms. For simplicity, we denote $\mathcal{M}_{Y,\alpha}$ as $\mathcal{R}$.

Since there is a natural $\mathbb{R}$-action on $\mathcal{M}(a,b)$ (time translation), we define the $\mathbb{R}$-quotient space by,

$$\hat{\mathcal{M}}(a,b) = \mathcal{M}(a,b)/\mathbb{R}.$$

Proposition 3.11 *For T sufficiently large, let $a,b,c \in \mathcal{R}$ with $\mu(a) > \mu(b) > \mu(c)$. Suppose b is irreducible, then there is an embedding map, for T sufficiently large,*

$$g: \qquad \hat{\mathcal{M}}(a,b) \times \hat{\mathcal{M}}(b,c) \times [T,\infty) \to \hat{\mathcal{M}}(a,c).$$

If b is the reducible critical point, then there is a gluing group $U(1)$ arising from the stabiliser of b in the gluing map; this means that there is a local diffeomorphism,

$$\hat{g}: \qquad \hat{\mathcal{M}}(a,b) \times \hat{\mathcal{M}}(b,c) \times U(1) \times [T,\infty) \to \hat{\mathcal{M}}(a,c).$$

Proof. Choose $x(t) \in \hat{\mathcal{M}}(a,b)$ and $y(t) \in \hat{\mathcal{M}}(b,c)$, by Proposition 3.9, assume $b = [A_0,\psi_0]$, we can write (for T sufficiently large)

$$x(t) = b + \begin{pmatrix} A_1 \\ \phi_1 \end{pmatrix}$$

$$y(-t) = b + \begin{pmatrix} A_2 \\ \phi_2 \end{pmatrix}$$

for $t > 2T$, then $x_i(t) = (A_i,\phi_i)$ satisfies (13), that is,

$$\frac{\partial}{\partial t}\begin{pmatrix} A_i \\ \phi_i \end{pmatrix} = \begin{pmatrix} *d & -Dq_{\psi_0} \\ .\psi_0 & \not{\partial}_{A_0} \end{pmatrix}\begin{pmatrix} A_i \\ \phi_i \end{pmatrix} + \begin{pmatrix} -q(\phi_i) \\ A_i.\phi_i \end{pmatrix}.$$

We simplify this equation as

$$\frac{\partial x_i}{\partial t} = Kx_i + Q(x_i) \tag{17}$$

with the exponential decay estimates,

$$\|(A_i, \phi_i)\|_{L^2_1} < C_5 \exp^{-\delta T}.$$

We construct the following gluing path on $\mathcal{B}$,

$$x\sharp_T y(t) = \begin{cases} x(t+2T) & \text{for} \quad t \le -1 \\ b + \rho(t)x_1(t+2T) + (1-\rho(t))x_2(t-2T) & \text{for} \quad -1 \le t \le 1 \\ y(t-2T) & \text{for} \quad t \ge 1 \end{cases}$$

where the calculation is done near b under the identification $U(b, \epsilon)$ with a domain in $\mathcal{H}$, $\rho(t)$ is the cut-off function on $[-1, 1]$, whose derivative has support in $[-\frac{1}{2}, \frac{1}{2}]$, is 1 near -1 and 0 near 1. Assume $|\partial\rho| < C_6$. We will find a solution for (7) uniquely determined by the gluing data $x(t), y(t), T$.

For $t \in [-1, 1]$, write

$$x'(t) = x\sharp_T y(t) - b = \begin{pmatrix} A' \\ \psi' \end{pmatrix}$$

then for T sufficiently large, $x\sharp_T y(t) - b$ is an approximate solution for (17), in the sense that,

$$\eta = \partial_t x' - Kx' - Q(x')$$

is sufficiently small as long as T is sufficiently large.

We aim to find a $\xi'(t) \in L^2_{1,0}([-1, 1], \mathcal{H})$ (very small) such that $x' + \xi'$ solves the flow equation. This means that ξ' obeys the following equation,

$$-\frac{\partial \xi'}{\partial t} + K\xi' + Q(\xi') + \begin{pmatrix} 0 & -Dq_{x'} \\ .\psi' & A' \end{pmatrix} \xi' = \eta.$$

Note that, for T sufficiently large:

- $S = \begin{pmatrix} 0 & -Dq_{x'} \\ .\psi' & A' \end{pmatrix}$ has arbitrarily small operator norm on $L^2_{1,0}([-1, 1], \mathcal{H})$.
- $-\dfrac{\partial}{\partial t} + K$ is a bounded, invertible operator on the following space,

$$L^2_{1,0}([-1, 1], \mathcal{H}) \longrightarrow L^2_0([-1, 1], \mathcal{H}).$$

Only the second assertion needs some explanation. Since K is invertible, if

$$z(t) \in L^2_{1,0}([-1,1], \mathcal{H})$$

in $ker(-\frac{\partial}{\partial t} + K)$, then $z(t)$ satisfies

$$\begin{cases} \dfrac{\partial z(t)}{\partial t} = Kz(t) \\ z(0) = 0 \\ z(1) = 0 \end{cases}$$

From the flow $\Phi(0,t)$ generated by K, we see that $z(t)$ must be zero. Then $-\frac{\partial}{\partial t} + K$ is an invertible operator. The boundness is obvious. So is $-\frac{\partial}{\partial t} + K + S$. Therefore, there is a right inverse P, P is a bounded operator acting on $L^2_0([-1,1], \mathcal{H})$ with range $L^2_{1,0}([-1,1], \mathcal{H})$.

We write $\xi' = P(\xi)$, then ξ satisfies

$$\xi + Q(P(\xi)) = \eta$$

where η can be made arbitrarily small by making T larger.

Since Q is a quadratic function, then we have

$$\|Q(P(\xi_1)) - Q(P(\xi_2))\| \leq C_7 \|\xi_1 - \xi_2\| (\|\xi_1\| + \|\xi_2\|)$$

for some constant C_7. Then the existence and uniqueness of the solution determined by $x(t), y(t), T$ follows from the map

$$\xi \mapsto \eta - P(Q(\xi))$$

is a strong contraction. The fixed point of this map, ξ, is our $g(x(t), y(t), T)$. g is an injection by our construction. It is easy to see that the image of g is one end of $\hat{\mathcal{M}}$.

When gluing at the reducible solution, basically the same procedure applies. We need to work with the framed configuration space near the reducible critical point, the stabiliser of the reducible solution then enters the gluing map by a free action. We only construct the $\sharp$-map in this case, gluing two gradient flows $x(t), y(t)$ along a reducible point b, suppose T

is sufficeintly large, $\exp(i\theta) \in U(1)$, then as before

$$x\sharp_T^\theta y(t) = \begin{cases} x(t+2T) & \text{for} \quad t \le -2 \\ b + \exp^{i\theta(t+2)}.x_1(t+2T) & \text{for} \quad -2 \le t \le -1 \\ b + \exp^{i\theta}.(\rho(t)x_1(t+2T) + (1-\rho(t))x_2(t-2T)) & \text{for} \quad -1 \le t \le 1 \\ b + exp^{i\theta(t-2)}.x_2(t-2T) & \text{for} \quad 1 \le t \le 2 \\ y(t-2T) & \text{for} \quad t \ge 2 \end{cases}$$

It is easy to see that $x\sharp_T^\theta y(t)$ is an approximate solution, to find the unique correct solution defined by x, y, T, θ, one follows the steps we did for the irreducible gluing map. The details we leave to the reader. □

This proposition gives an explicit picture for the boundary of various moduli spaces, we give here just one example of these arguments, we will meet several other interesting moduli spaces when we discuss relative Seiberg-Witten invariants. Proposition 3.11 also tells us how a trajectory connecting two critical points a, b ($\mu(a) > \mu(b)$) breaks into two pieces, which become two trajectories satisfying (7) and breaking at another critical point c with $\mu(a) < \mu(c) < \mu(b)$.

From this Proposition, we can prove

$$dim\mathcal{M}(a,b) = \mu(a) - \mu(b) - 1$$

and

$$dim\mathcal{M}(c,a) = \mu(c) - \mu(a)$$

where a reducible. Suppose $\mathcal{M}(c,a), \mathcal{M}(a,b)$ are nonempty moduli spaces, then the gluing map $\hat{g}$ in Proposition 3.11 tells us the identity,

$$dim\mathcal{M}(c,a) - 1 + dim\mathcal{M}(a,b) - 1 + 2 = dim\mathcal{M}(c,b) - 1$$

since c, b must be irreducible, we have $dim\mathcal{M}(c,b) = \mu(c) - \mu(b)$, the assertion follows. In particular, when $\mu(a) - \mu(b) = 1$ and a is reducible, $\mathcal{M}(a,b)$ is empty.

Corollary 3.12 *Suppose a, c are two irreducible critical points with the index given by $\mu(a) = \mu(c) + 2$, the boundary of $\hat{\mathcal{M}}(a,c)$ consists of the union*

$$\bigcup_{\{b|\mu(b)=\mu(a)-1\}} \hat{\mathcal{M}}(a,b) \times \hat{\mathcal{M}}(b,c)$$

where b runs over the set of the irreducible critical point.

Proof. From the gluing map (see Proposition 3.11)and the above arguments, we know that the limit of a sequence of trajectories doesn't break through the reducible critical point, and only breaks at the irreducible critical point b with $\mu(b) = \mu(a) - 1$. $\square$

4 Floer Homology

We use the simplified notations as before, such as $\mathcal{R}, \mathcal{R}^*, \mathcal{M}(a,b), \hat{\mathcal{M}}(a,b)$, where $\mathcal{R}^*$ is a finite set of irreducible, non-degenerate critical points, indexed by the spectral flow of a path starting from the trivial point. For $a, b \in \mathcal{R}^*$ with $\mu(a) - \mu(b) = 1$, then $\hat{\mathcal{M}}(a,b)$ is also a finite set of points, specifically, we recall $\hat{\mathcal{M}}(a,b)$ defined by

$$\hat{\mathcal{M}}(a,b) = \left\{ x(t) \;\middle|\; \begin{array}{l} (1) x(t) \text{ is the finite energy solution of (6)} \\ \quad \text{modulo the gauge group and } \mathbb{R}\text{-translation,} \\ (2) \lim_{t\to+\infty} x(t) = a \in \mathcal{R}^*, \\ (3) \lim_{t\to-\infty} x(t) = b \in \mathcal{R}^*. \end{array} \right\} \tag{18}$$

Theorem 4.1 Seiberg-Witten-Floer homology: *Let $C_k(Y)$ be the free abelian group over $\mathbb{Z}$ with generators consisting of points in $\mathcal{R}^*$ whose index is k. Let $n_{a\ b}$ be the sum of ± 1 over $\hat{\mathcal{M}}(a,b)$ whenever $\mu(a) - \mu(b) = 1$, where the sign is determined by comparing the orientation on $\mathcal{M}(a,b)$ with the $\mathbb{R}$-translation on each isolated 1-dimensional component, the orientation on $\mathcal{M}(a,b)$ is determined by the orientation of Y. Then the boundary operator*

$$\partial : \qquad C_k(Y) \longrightarrow C_{k-1}(Y)$$

$$\partial(a) = \sum_{b\in\mathcal{R}^* \;\; \mu(b)=k-1} n_{a\ b} b$$

satisfies $\partial\partial = 0$. The homology group

$$HF_k^{SW}(Y,g) = ker\partial_k / Im\partial_{k+1}$$

is a topological invariant (up to a canonical isomorphism). Moreover, HF_*^{SW} is a functor on the category of homology 3-spheres and their cobordisms.

Remark 4.2 *The Casson-type invariant for a homology 3-sphere defined in [4] is the Euler characteristic of the Seiberg-Witten-Floer homology* HF_*^{SW}*:*

$$\begin{aligned} \chi(HF_*^{SW}(Y,g)) &= \textstyle\sum_k (-1)^k \dim HF_k^{SW}(Y,g) \\ &= \textstyle\sum_k (-1)^k \dim C_k(Y) \\ &= \textstyle\sum_{a\in\mathcal{R}^*} (-1)^{\mu(a)} \\ &= \lambda(Y,g) \end{aligned} \tag{19}$$

A similar formula was found by M. Marcolli [18] for a 3-manifold with $Spin^c$ *structure whose determinant line bundle has non-zero first Chern class.*

Remark 4.3 *Obviously, for a diffeomorphism* $f: Y \to Y$*, there is an isomorphism*

$$HF_*^{SW}(Y,g) \cong HF_*^{SW}(Y, f^*g)$$

Proof. We only prove $\partial\partial = 0$, the remaining assertions will be proved in the following subsection. By definition,

$$\begin{aligned} \partial^2(a) &= \sum_{b\in\mathcal{R}^* \ \mu(b)=\mu(a)-1} n_{a\ b}\,\partial(b) \\ &= \sum_{b\in\mathcal{R}^* \ \mu(b)=\mu(a)-1} \ \sum_{c\in\mathcal{R}^* \ \mu(c)=\mu(b)-1} n_{a\ b} n_{b\ c}\, c \end{aligned}$$

To see

$$\sum_{b\in\mathcal{R}^* \ \mu(b)=\mu(a)-1} n_{a\ b} n_{b\ c} = 0$$

for any $a, c \in \mathcal{R}^*$ whenever $\mu(a) = \mu(c) + 2$. We know that the number

$$\sum_{b\in\mathcal{R}^* \ \mu(b)=\mu(a)-1} n_{a\ b} n_{b\ c}$$

is the number of oriented boundary points of $\hat{\mathcal{M}}(a,c)$ (Proposition 3.12), hence is zero. Now the Seiberg-Witten-Floer homology group is well-defined. □

To understand the metric (perturbation) dependence of $HF_k^{SW}(Y,g)$, we should consider the equivariant Seiberg-Witten-Floer homology $HF_{k,U(1)}^{SW}(Y,g)$ (see [19] for the definition), which is independent of metric and perturbation up to index-shifting. Here we apply the results in [19] to illustrate how the irreducible critical points interact with the reducible one (denoted by θ).

Theorem 4.4 *(Proposition 7.8, Proposition 8.1 in [19])*
For $k < 0$,

$$HF_k^{SW}(Y, g_0) \cong HF_{k,U(1)}^{SW}(Y, g_0).$$

For $k \geq 0$, we have the following exact sequences

$$0 \to HF_{2k+1,U(1)}^{SW}(Y, g_0) \stackrel{i_{2k+1}}{\to} HF_{2k+1}^{SW}(Y, g_0) \stackrel{\triangle_k}{\to} \mathbb{R}\Omega^k \to$$

$$\to HF_{2k,U(1)}^{SW}(Y, g_0) \to HF_{2k}^{SW}(Y, g_0) \to 0.$$

where $\triangle_k$ is given by

$$\triangle_k(\sum_a x_a a) = x_a m_{ac} m_{ce} \cdots m_{\alpha'\alpha} n_{\alpha\theta} \Omega^k$$

Here a sum over repeated indices is over all critical points with indices $\mu(a) = 2k+1$, $\mu(c) = 2k-1$, $\mu(\alpha') = 3$, $\mu(\alpha) = 1$ and $m_{\alpha\beta}$ for critical points α, β with relative index 2 is the Seiberg-Witten invariant on $Y \times \mathbb{R}$ with boundary conditions α, β respectively.

Let α be a critical point with index $2k+1$, donote

$$m(\alpha, \theta) = m_{\alpha a} m_{ac} m_{ce} \cdots m_{\beta'\beta} n_{\beta,\theta}$$

the number presented in $\triangle_k$, sum over $\mu(a) = 2k-1$, $\mu(c) = 2k-3$, $\cdots$, $\mu(\beta') = 3$, $\mu(\beta) = 1$. Then we see that

$$\begin{aligned} Im(i_{2k+1}) &= Ker(\triangle_k) \\ &= \left\{ \sum_\alpha x_\alpha \alpha \;\middle|\; \begin{array}{ll} (1) & \sum_\alpha x_\alpha n_{\alpha\ \beta} = 0, \text{ for } \beta, \mu(\beta) = 2k-1, \\ (2) & \sum_\alpha x_\alpha m(\alpha, \theta) = 0 \end{array} \right\} \end{aligned}$$

Therefore, $Ker(\triangle_k)$ measure the interaction of $HF_*^{SW}(Y, g)$ with the reducible critical points. There is a similar analogue for Seiberg-Witten-Floer cohomology.

5 Relative Seiberg-Witten invariants

In this section, we define the relative Seiberg-Witten invariant for a 4-manifold with a cylindrical end (a homology 3-sphere), this invariant takes its values in the Seiberg-Witten-Floer homology group we defined in Theorem 4.1. In instanton theory, this is true for a general manifold see [6] [3]. From the result of Marcolli, there is also a well-defined primary relative Seiberg-Witten invariant valued in the Seiberg-Witten-Floer homology group for 3-manifold with $Spin^c$ structure whose determinant line bundle has non-zero first Chern class.

First we give the definition of the finite energy solution on a 4-manifold with a cylindrical end which is isometric to $[0,\infty)\times Y$ for a homology 3-sphere Y. Where the boundary is $S^1\times\Sigma_g$ $(g>1)$ for a Riemnanian surface Σ_g, this definition appears in [20].

Definition 5.1 *Let X be a 4-manifold with end isometric to $[0,\infty)\times Y$ for a homology 3-sphere Y. Fix a Spin^c structure on X whose restriction to the cylindrical end $[0,\infty)\times Y$ is the pull-back of the Spin^c structure on Y with the line bundle we introduced earlier. For any solution (A,ψ) to the Seiberg-Witten equations on X with respect to this Spin^c structure, in a temporal gauge on the cylindrical end, there is a unique gradient flow $x(t):[1,\infty)\to\mathcal{B}^*$, determined by the solution (A,ψ). A finite energy solution to the Seiberg-Witten equations on X is a solution for which the associated gradient flow line $x(t):[0,\infty)\to\mathcal{B}^*$ satisfies the condition that*

$$\lim_{t\to\infty}(C'(x(t))-C'(x(1)))$$

is finite.

In the above definition, the (perturbed) Seiberg-Witten equations on X are

$$\begin{aligned} F_A^+ &= \frac{1}{4} < e_ie_j\psi,\psi > e^i\wedge e^j+\rho(d\alpha+dt\wedge *d\alpha) \\ \not{D}_A(\psi) &= 0 \end{aligned} \tag{20}$$

where $*$ is the complex Hodge star operator on Y, the cut-off function ρ has support in $(0,\infty)$, and equals 1 over $[1,\infty)$.

Define $\mathcal{M}(X,a)$ as the moduli space with asymptotic limit $a\in\mathcal{R}$

$$\mathcal{M}(X,a)=\left\{ [A,\psi] \;\middle|\; \begin{array}{l} (1)\ [A,\psi]\ \text{ denotes the gauge orbit of} \\ \qquad \text{the solution } (A,\psi) \text{ of } (20) \\ (2)\ x(t) \text{ is the finite energy solution on } Y\times[1,\infty) \\ \qquad \text{associated with } (A,\psi) \text{ as in Definition 5.1} \\ (3)\ \lim_{t\to\infty}x(t)=a \end{array} \right\} \tag{21}$$

Similar to the exponential decay in Proposition 3.9, any finite energy solution to the Seiberg-Witten equations (20) decays exponentially to a critical point in $\mathcal{R}$, from this fact we can get a similar gluing theorem from which we draw the following conclusion.

Proposition 5.2 *After a small compact support perturbation of (20), $\mathcal{M}(X,a)$ is a smooth manifold, whose boundary consists of*

$$\mathcal{M}(X,b)\times\hat{\mathcal{M}}(b,a)$$

where b runs over $\mathcal{R}$ with $\mu(b) \geq \mu(a) + 1$ and $\hat{\mathcal{M}}(b, a)$ is defined by (18).

For any $a \in \mathcal{R}^*$, we can attach an invariant for the 0-dimensional component of $\mathcal{M}(X, a)$, called $n_{X,\ a}$. Now we define the relative Seiberg-Witten invariant with values in the Floer complex, by

$$SW_{X,L} = \sum_{a \in \mathcal{R}^*} n_{X,\ a} a \tag{22}$$

then from Proposition 5.2 and Remark 3.6, $SW_{X,L}$ is closed under the Floer boundary operator, that means

$$SW_{X,L} \in HF_*^{SW}(Y).$$

For a closed, oriented four manifold $X = X_1 \cup_Y X_2$, which splits along a homology 3-sphere Y, we put a $Spin^c$ structure on X whose restriction to Y is L and then define the Seiberg-Witten-Floer homology $HF_*^{SW}(Y)$. Then the relevant invariant of X is given by the following gluing formula

$$SW_X = \sum_{a \in \mathcal{R}^*} n_{X,\ a} n_{X_2,\ a}.$$

Example 5.3 *As an example, we apply these relative Seiberg-Witten invariants to study certain homology spheres which bound Stein surfaces. For the background of contact structures and Stein surfaces, see [9] [10] [14].*

Suppose Y is a homology sphere, which is the boundary of a Stein surface (X, J) with J the associated complex structure on X , then Y has a natural induced holomorphically fillable contact structure ξ (the J-invariant tangent plane distributions). Define $\mu(Y, \xi) = SW_{X,J}$, this definition can be extended to the symplectically fillable contact structure ξ on Y.

Proposition 5.4 *Let J_1, J_2 be two complex structures on a Stein surface X, if the induced contact structure ξ_1, ξ_2 on Y is isotopic, that is, there is a diffeomorphism $f : Y \to Y$ which is homotopic to identity on Y and sends ξ_1 to ξ_2, then $\mu(Y, \xi_1) = \mu(Y, \xi_2)$.*

Proof. Under these assumptions, Lisca and Matic prove that $c_1(J_1) = c_1(J_2)$ hence J_1, J_2 induce the same $Spin^C$-structure. From the definition of the relative Seiberg-Witten invariant for (Y, ξ_i), one can obtain $\mu(Y, \xi_1) = \mu(Y, \xi_2)$. $\square$

Since any homology 3-sphere Y is obtained by surgery along a link $L = \bigcup_{i=1}^{n} K_i$ in S^3, whose link matrix $Q = (a_{ij})$ (where a_{ij} is the link number for K_i and K_j when $j \neq j$, and a_{ii} is the framing on K_i) is unimodular and symmetric. One can get a four-manifold X bounded by Y by attaching 2-handles $h_i = D^2 \times D^2$ $(i = 1, \cdots, n)$ along each knot K_i in $S^3 = \partial(D^4)$ with framing given by a_{ii}, that is, $X = D^4 \bigcup_L (h_L)$. Note that $H_2(X, \mathbb{Z})$ has a basis $\{\alpha_1, \cdots, \alpha_n\}$, determined by $\{K_1, \cdots, K_n\}$, such that the intersection matrix is Q. From the works of Eliashberg and Gompf, we know that if the link L can be realized by a Legendrian link in S^3 for the canonical contact structure, such that the Thurston-Bennequin invariant for each knot K_i is $a_{ii} + 1$, then X has a Stein structure J. Moreover, the first Chern class $c_1(J) \in H^2(X, \mathbb{Z})$ of such a Stein structure J is represented by a cocycle whose value on each such basis element α_i is the rotation number of the corrresponding oriented Legendrian link component. We can apply the relative Seiberg-Witten invariants to study these induced holomorphically fillable contact structures ξ_J on Y, simple examples show that these relative Seiberg-Witten invariants can distinguish certain induced contact structures. Full understanding of these relative invariants will be interesting for further investigations.
Acknowledgement: I wish to acknowledge my gratitude to my supervisor Alan Carey for his advice, encouragement and support throughout this work. I am grateful to Matilde Marcolli, Tom Mrowka, Ruibin Zhang for many invaluable discussions.

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