Course on Differential Geometry and Topology

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Introduction

Differential geometry is the language of modern physics as well as mathematics. Typically, one considers sets which are manifolds (that is, locally resemble Euclidean space) and which come equipped with a measure of distances. In particular, this includes classical studies of the curvature of curves and surfaces. Local questions both apply and help study differential equations; global questions often invoke algebraic topology and ideas from theoretical physics.

Motivated by practical surveying problems about mapping the surface of the earth, Gauss applied the powerful method of the calculus of Newton and Leibniz in his investigations on the curvature of surfaces. So one can say that Differential geometry begins with the study of curves and surfaces in three-dimensional Euclidean space. Using vector calculus and moving frames of reference on curves embedded in surfaces we can define quantities such as Gaussian curvature that allow us to distinguish among surfaces.

Though we live in 4-dimensional space and time, many occasions, people working at the frontier of science have to use the generalizations of our space-time, which leads to high dimensional manifolds.

It was Riemann who introduced the notion of a manifold as an appropriate form of space where one can study geometries. Euclidean geometry then became just one very special case among infinitely many geometries and the laws of Euclidean geometry were postulated to be true at a very small scale. These Euclidean measurements is now allowed to vary from one point to the next.

Einstein who realized that these new geometric ideas should be the basis for understanding not just the shape of the earth but that of the whole universe of space and time. His revolutionary General Theory of Relativity is a masterpiece of Geometric Physics explaining that mysterious fundamental force of Nature. Gravitation, which holds the universe together on a large scale, is a manifestation of the curvature of space-time itself. Later, Einstein dreamed of generalizing his theory to encompass all the other known forces of Nature: this is known as "Grand Unified Theory".

In recent decades, some spectacular new theoretical advances might ultimately be-come important stepping stones to realize Einstein's dream. Among those are Gauge Theory and String Theory. The idea is to treat particles and the forces between them not just through points and lines but by using higher dimensional object and to incorporate all the degrees of internal freedoms and symmetries that are needed to explain all the other forces of nature. The central concept is the Curvature, in its various manifestations, which is the fundamental invariant of Differential Geometry, and can be calculated by local Euclidean calculus.

This course is to present some basic concepts in differential geometry, such as differentiable manifolds, connections on bundles, curvatures and their characteristic classes.

Mathematical theory of connections and curvatures is the proper context for physicists' gauge theory, where the term "gauge potentials" are really the connection forms for a connection in a local form, and the "field strength" is the curvature form. Then the law of nature in physics is expressed by a set of differential equations for the curvature of the connection. Many discovery of physical laws in turn provide powerful tools for mathematicians to understand many intrinsic structures of low dimensional geometry and topology.

References of this course:

1. F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, 1983.

2. N. Kobayashi and K. Nomizu, *Foundations of Differential Geometry,* Wiley Classics Library Edition, New York. 1996.

3. D. Husemoller, *Fiber Bundles,* Graduate Texts in Math, Springer-Verlag, 1966

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1 Manifolds and de Rham Cohomology

1.1 Differentiable manifolds

A locally Euclidean space M of dimension d is a Hausdorff topological space for which each point has a neighborhood homeomorphic to an open subset of Euclidean space $\mathbb{R}^d$. That is, for any point $x \in M$, there is a neighborhood U and a homeomorphism ϕ from U to an open set in R^d.

Manifolds are locally Euclidean spaces with certain smooth structures so that the basic calculus can be carried over.

Definition 1.1. A differentiable manifold *M of dimension d is a Hausdorff topological space which is covered by countably many open set $\{U_\alpha\}_{\alpha\in I}$, and for each U_α, there exists a homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^d$ with $\phi_\alpha(U_\alpha)$ an open set in $\mathbb{R}^d$, such that*

$$\phi_\alpha \circ \phi_\beta^{-1} : \qquad \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta) \tag{1}$$

is a diffeomorphism. Such (U_α, ϕ_α) is called a coordinate chart, and $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ is an atlas on M which gives a **differential structure** *on differentiable manifold M. We call $(M, \mathcal{A})$ a smooth manifold.*

To understand this definition better, we recall some elementary calculus on Euclidean spaces.

Recall that if $f : U \to \mathbb{R}$ is a continuous function defined on an open set U, we say that f is differentiable at $x \in U$ if the limit

$$\lim_{t\to 0} \frac{f(x+t) - f(x)}{t}$$

exists, is called the derivative of f at x, denoted by $f'(x)$. For a differentiable function f on U, its derivative is also a function on U.

For an open set $U \in \mathbb{R}^d$, a function $f : U \to \mathbb{R}$, we define its partial derivative along each direction $e_i = (0, \cdots, 1, \cdots, 0)$ which is 1 at the ith position and 0 elsewhere

$$\frac{\partial f}{\partial x^i}(x) = \lim_{t\to 0} \frac{f(x + te_i) - f(x)}{t},$$

which is called the ith partial derivative.

Let $U \subset \mathbb{R}^d$ be an open set, a function $f : U \to \mathbb{R}$ is differentiable of class C^k (where $k \in \mathbb{N}$) if the partial derivatives of f, up to order k, exist and continuous. f is smooth (or C^∞) if f is differentiable of class C^k for all $k > 0$. A map $f : U \to \mathbb{R}^n$ is differentiable of class C^k if each component of f is C^k.

Note that, in (1), both $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$ are open sets on $\mathbb{R}^d$. By definition, $\phi_\alpha \circ \phi_\beta^{-1}$ is a diffeomorphism if and only if $\phi_\alpha \circ \phi_\beta^{-1}$ is smooth, and has smooth inverse. This condition on $\phi_\beta(U_\alpha \cap U_\beta)$ is called the **compatibility** condition for two coordinate charts (U_α, ϕ_α) and (U_β, ϕ_β).

Example 1.2.

1. *$(\mathbb{R}^d, Id)$ is an atlas on $\mathbb{R}^d$, which makes $\mathbb{R}^d$ a manifold. Similarly, any open set U in $\mathbb{R}^d$ is a manifold.*

2. *The general linear group $GL(n, \mathbb{R})$ is the set of all $n \times n$ matrices with non-zero determinant, viewed as the open subset of $\mathbb{R}^{n^2}$, is a smooth manifold.*

3. *Let V be a vector space, any choice of basis gives rise to a coordinate chart of V. Two different bases give two compatible coordinate charts. In general, any linear isomorphism from V to $\mathbb{R}^d$ defines a manifold structure on V. In particular, complex d dimensional vector space $\mathbb{C}^d$ is a 2d dimensional manifold.*

4. *Let S^2 be the unit sphere in $\mathbb{R}^3$. Let $U_0 = S^2 - \{(0,0,1)\}$, Using the stereo-graphic projection from the point $\{(0,0,1)\}$ onto the xy plane, we can define a coordinate chart on U_0:*
$$\phi_0(x, y, z) = (\frac{x}{1-z}, \frac{y}{1-z}).$$
Let $U_1 = S^2 - \{(0,0,-1)\}$, Using the stereo-graphic projection from the point $\{(0,0,-1)\}$ onto the xy plane, we get a coordinate chart (U_1, ϕ_1) with
$$\phi_0(x, y, z) = (\frac{x}{1+z}, \frac{y}{1+z}).$$
Now we need to check that these two coordinate charts are compatible:

   (a) *$\phi_0(U_0 \cap U_1)$ and $\phi_1(U_0 \cap U_1)$ are equal to $\mathbb{R}^2 - \{(0,0)\}$, an open set in $\mathbb{R}^2$.*

(b) $\phi_0 \circ \phi_1^{-1} : \phi_1(U_0 \cap U_1) \to \phi_0(U_0 \cap U_1)$ *is a diffeomorphism, one can easily check that*

$$\phi_0 \circ \phi_1^{-1}(x, y) = (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}).$$

Similarly, any fixed length sphere in $\mathbb{R}^{d+1}$ *is a d-dimensional manifold.*

5. *(***Real Projective space***) Consider the space of all lines through the origin in* R^{d+1}*, denoted by* $\mathbb{RP}^d$*. Note that a line through the origin is determined by an equivalent class of non-zero vectors in* $\mathbb{R}^{d+1}$ *where two non-zero vectors* x_1 *and* x_2 *are equivalent if and only if* $x_1 = tx_2$ *for a non-zero* $t \in \mathbb{R}$*. Hence.* $\mathbb{RP}^d$ *can be identified as the quotient space of* $\mathbb{R}^{d+1} - 0$ *by this equivalent relation. Denote by* $[x]$ *be the line through the non-zero vector* $x = (x^0, x^1, \cdots, x^d)$*. For each* $i = 0, \cdots, d$*, consider the open subset of* $\mathbb{RP}^d$*:*

$$U_i = \{[x] | x = (x^0, x^1, \cdots, x^d) \text{ with } x^i \neq 0, \}$$

which covers all of $\mathbb{RP}^d$*. Define a map from* U_i *to* $\mathbb{R}^d$ *by*

$$\phi_i([x^0, x^1, \cdots, x^d]) = (\frac{x^0}{x^i}, \cdots, \frac{x^d}{x^i}).$$

We can check that $\{(U_i, \phi_i)\}$ *are compatible coordinate charts, which make* $\mathbb{RP}^d$ *a d-dimensional manifold.*

Similarly, the **Complex Projective Space** $\mathbb{CP}^d$*, the space of all complex lines through the original in* $\mathbb{C}^{d+1}$*, is a* $2d$*-dimensional manifold.*

6. *Let* M *and* N *be manifolds of dimension* m *and* n*, then their product* $M \times N$ *is a manifold of dimension* $m + n$*.*

A collection of $\{U_\alpha\}$ of open subsets of M is called a cover of M. A subset of the collection $\{U_\alpha\}$ which still covers M is called a subcover. A refinement $\{V_\beta\}$ of the cover $\{U_\alpha\}$ is a cover such that for each β there is an α with $V_\beta \subset U_\alpha$. A locally finite cover $\{U_\alpha\}$ is a cover such that for any point $x \in M$ there exists a neighborhood U of x such that $U_\alpha \cap U$ non-empty for only finitely many α.

Remark 1.3. *1. Any atlas on a manifold has a locally finite refinement, as any locally Euclidean space has this property.*

2. Two atlas $\mathcal{A}$ and $\mathcal{B}$ are equivalent if $\mathcal{A} \cup \mathcal{B}$ is also atlas. Therefore, a differentiable structure on M is really a choice of a equivalent class of atlas.

3. For any atlas $\mathcal{A}$ on M, take all the coordinate charts compatible with every chart in $\mathcal{A}$, this defines a maximal atlas containing $\mathcal{A}$, actually, this is the unique maximal atlas containing $\mathcal{A}$.

In summary, a **differential or smooth manifold** is a Hausdorff topological space M with an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$. We call the choice of an atlas on M is a choice of differential structure on M.

For a differential manifold M, we often just say that (U, ϕ) is a coordinate chart on M rather than (U, ϕ) is a member of a specific atlas $\mathcal{A}$, in any sense, it is a member of the unique maximal atlas of the differential structure. In practical situation, we can choose any convenient atlas as we want, as seen in the above examples.

Now we can carry over many concepts of calculus on Euclidean spaces to manifolds using coordinate charts.

Definition 1.4. *1. f is a smooth function on M if $f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbb{R}$ is a C^∞-function for any coordinate chart (U_α, ϕ_α) on M.*

2. A continuous map f between two manifolds M and N is smooth if and only if for every point $x \in M$ there are coordinate charts (U, ϕ) on M and (V, ψ) on N such that $x \in U$, $f(U) \subset V$ and

$$\psi \circ f \circ \phi^{-1} : \qquad \phi(U) \to \psi(V)$$

is smooth.

Example 1.5. *1. The determinant function on $GL(n, \mathbb{R})$ is a smooth function.*

2. Let S^2 be the unit sphere in $\mathbb{R}^3$, for any point $(x, y, z) \in S^2$,

$$f(x, y, z) = z$$

is a smooth function, this is called the height function of S^2.

1.2 Directional derivatives and the chain rule

Let U be an open set in $\mathbb{R}^d$, $f : U \to \mathbb{R}^m$ is a smooth map, for any $v \in \mathbb{R}^d$, we can define the directional derivative of f along v at $x \in U$ as

$$d_v f(x) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} \in \mathbb{R}^m.$$

Lemma 1.6. *Define $df(x)(v) = d_v f(x)$, then $df(x)$ is a linear operator $\mathbb{R}^d \to \mathbb{R}^m$.*

The proof is left to readers as an exercise.

Theorem 1.7. *Let $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^m$ be open sets, $f : U \to \mathbb{R}^m$ and $g : V \to \mathbb{R}^n$ be smooth functions with $f(U) \subset V$. Then $g \circ f : U \to \mathbb{R}^n$ is also smooth and*

$$d(g \circ f)(x) = dg(f(x)) \circ df(x),$$

where $\circ$ on the right hand side means the composition of linear operators.

Proof. Use the standard basis of Euclidean space, we write the coordinates on $\mathbb{R}^d$, $\mathbb{R}^m$ and $\mathbb{R}^n$ to be $(x^1, \cdots, x^d)$, $(y^1, \cdots, y^m)$ and $(z^1, \cdots, z^n)$ respectively. Then the directional derivative of $f = (f^1, \cdots, f^m)$, as a linear operator from $\mathbb{R}^d \to \mathbb{R}^m$, can be written as an $m \times d$ matrix:

$$\Big[\frac{\partial f^i}{\partial x^j}(x)\Big].$$

Similarly, for $g = (g^1, \cdots, g^n)$, the linear operator $dg(f(x)) : \mathbb{R}^m \to \mathbb{R}^n$ can be expressed by an $n \times m$ matrix of form

$$\Big[\frac{\partial g^k}{\partial y^i}(f(x))\Big],$$

and the linear operator $d(g \circ f)(x) : \mathbb{R}^d \to \mathbb{R}^n$ can be written as

$$\Big[\frac{\partial (g^k \circ f)}{\partial x^j}(x)\Big].$$

By the chain rule for partial derivatives, that is,

$$\frac{\partial (g^k \circ f)}{\partial x^j} = \sum_{i=1}^{d} \frac{\partial g^k}{\partial y^i} \frac{\partial f^i}{\partial x^j},$$

we get the chain rule as matrix form, hence, as linear operators, $d(g \circ f)(x) = dg(f(x)) \circ df(x)$. □

1.3 Tangent space and cotangent space

For a smooth path in $\mathbb{R}^d$, that is a smooth function $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^d$, the tangent vector to γ at $\gamma(0)$ is give by the limit definition:

$$\gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t} \in \mathbb{R}^d.$$

For a point x in a smooth manifold M, how to define a tangent vector at x? We know that if M sits inside a Euclidean space $\mathbb{R}^N$, we may take all possible smooth path inside M, and use the limit definition to define the tangent vector to these paths as in $\mathbb{R}^N$.

For example, take the 2-dimensional unit sphere $S^2 \subset \mathbb{R}^3$, we get the tangent space of S^2 at $x \in S^2$, denoted by $T_x S^2$:

$$T_x S^2 = \{v \in \mathbb{R}^3 | \langle x, v \rangle = 0.\}$$

To show this claim, we take $x \in S^2$ and a vector $v \in \mathbb{R}^3$ with $\langle x, v \rangle = 0$, then $\gamma(t) = \dfrac{x + tv}{\|x + tv\|}$ for $t \in (-\epsilon, \epsilon)$ is a smooth path in S^2 through x.

Sometime, it is much convenient to have an intrinsic definition of tangent space without resorting to some embedding in an Euclidean space. For this reason, we need smooth paths inside a smooth manifold.

A smooth path $\gamma : (-\epsilon, \epsilon) \to M$ through $x = \gamma(0)$, as a smooth map between these two manifolds, we choose a coordinate chart (U, ϕ) with $\gamma(-\epsilon, \epsilon) \subset U$, then $\phi \circ \gamma$ is a smooth path in $\mathbb{R}^d$, hence,

$$(\phi \circ \gamma)'(0) \in \mathbb{R}^d$$

is well-defined.

For two smooth paths γ_1 and γ_2 through the same point x in M, we call that γ_1 and γ_2 are tangent at x if there is a coordinate chart (U, ϕ) such that

$$(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0).$$

Exercise 1.8. *Show that the tangency relation between two paths is an equivalence relation on the set of all paths through x, and doesn't depend on the choice of coordinate charts on M.*

Definition 1.9. *The tangent space of a smooth manifold M at x, denoted by T_xM, is defined to be the space of equivalent classes of smooth paths through x in M.*

Proposition 1.10. *T_xM is a linear vector space of dimension d.*

Proof. Choose a coordinate chart (U, ϕ) at x and define a map $d\phi(x)$ between T_xM and $\mathbb{R}^d$ as:

$$d\phi(x)([\gamma]) = (\phi \circ \gamma)'(0).$$

By definition, $d\phi(x)$ is injective. For any $v \in \mathbb{R}^d$, choose ϵ such that for any $|t| < \epsilon$, $\phi(x) + tv$ is a path through $\phi(x)$ in $\phi(U)$, and $\phi^{-1}(\phi(x) + tv)$ is a smooth path through x then we can check that

$$d\phi(x)([\phi^{-1}(\phi(x) + tv)]) = v.$$

Hence, $d\phi(x)$ is a bijection. We can equip T_xM with a unique vector space structure such that the map $d\phi(x)$ is a linear isomorphism from T_xM to $\mathbb{R}^d$. □

Though $\gamma'(0)$ doesn't make sense for a smooth path in a general manifold M from the limit definition, sometimes, we still denote $[\gamma]$ by $\gamma'(0)$. This agrees with the natural definition when M sits inside some Euclidean space.

Let $f : M \to N$ be a smooth map. To know the rate of change of f at x along a smooth path, we need the differential of f at $x \in M$, which is defined to be

$$\begin{array}{rccc} df(x): & T_xM & \to & T_{f(x)}N. \\ & [\gamma] & \mapsto & [f \circ \gamma]. \end{array}$$

Lemma 1.11. *With the natural linear structures on tangent spaces T_xM and $T_{f(x)}N$, $df(x)$ is a linear map. Moreover, for two smooth maps $f : M \to N$ and $g : N \to K$, then*

$$d(g \circ f)(x) = dg(f(x)) \circ df(x).$$

Proof. Choose coordinate charts (U, ϕ) and (V, ψ) at x and $f(x)$ respectively such that $f(U) \subset V$. The directional derivative of $\psi \circ f \circ \phi^{-1}$, $d(\psi \circ f \circ \phi^{-1})(\phi(x))$, is linear.

The linear structures on T_xM and $T_{f(x)}N$ are obtained from the bijective linear maps $d\phi(x)$ and $d\psi(f(x))$. Hence, $df(x)$ is a linear map. The second property follows from the chain rule properties of the directional derivatives. □

Let M and N be two smooth manifolds. For $(x, y) \in M \times N$, we have

$$T_{(x,y)}(M \times N) \cong T_xM \oplus T_yN. \tag{2}$$

The proof of this claim is straight forward by choosing smooth paths in $M \times N$ such that the projection to M or N is just the point x or y, together with dimension counting.

For a smooth function $f : M \to \mathbb{R}$, note that for any tangent vector $[\gamma] \in T_xM$ with a representing path $\gamma : [-e, \epsilon] \to M$, we know that

$$df(x)([\gamma]) = [f \circ \gamma] = (f \circ \gamma)'(0) \in \mathbb{R}.$$

The differential $df(x)$ at x is a linear map: $T_xM \to \mathbb{R}$. This motivates the definition of cotangent space.

Definition 1.12. *The cotangent space of M at x, denoted by T_x^*M, is the space of all linear maps from T_xM to $\mathbb{R}$. Elements of T_x^*M are called cotangent vectors.*

For any smooth function on M, $df(x)$ is a cotangent vector at x. For a smooth map $f : M \to N$, the differential $df(x) : T_xM \to T_{f(x)}N$ induces a dual map

$$df^*(x) : \qquad T^*_{f(x)}N \to T^*M.$$

If g is a smooth function on N, $dg(f(x))$ is a cotangent vector at $f(x) \in N$, then

$$df^*(x)\big(dg(f(x))\big) = d(f \circ g)(x),$$

as $f \circ g$ is a smooth function on M.

Sometimes, it is instructive to do local calculations in coordinates of a coordinate chart, here we express tangent vector and cotangent vector in coordinates.

Let (U, ϕ) be a coordinate chart at $x \in M$, with

$$\phi = (\phi^1, \cdots, \phi^d) : U \to \mathbb{R}^d.$$

Each component ϕ^i is a smooth function, this implies that $d\phi^i(x) \in T_x^*M$.

Recall that $d\phi(x) : T_xM \to \mathbb{R}^d$ is a linear isomorphism. Take the standard basis $\{e_1, \cdots, e_d\}$ for $\mathbb{R}^d$, we denote by

$$\frac{\partial}{\partial \phi^i}(x) = (d\phi(x))^{-1}(e_i).$$

We get a basis

$$\{\frac{\partial}{\partial \phi^1}(x), \cdots, \frac{\partial}{\partial \phi^d}(x)\}$$

for the tangent space T_xM.

Proposition 1.13. *$\{d\phi^1(x), \cdots, d\phi^d(x)\}$ is a dual basis of T_x^*M to the basis $\{\frac{\partial}{\partial \phi^1}(x), \cdots, \frac{\partial}{\partial \phi^d}(x)\}$.*

Proof. Apply $d\phi^i(x)$ to element $\frac{\partial}{\partial \phi^i}(x)$, we get

$$\begin{aligned} d\phi^i(x)\big(\frac{\partial}{\partial \phi^j}(x)\big) &= d\phi^i(x)\big((d\phi(x))^{-1}(e_j)\big) \\ &= \big(d\phi^i(x) \circ d(\phi^{-1})(\phi(x))\big)(e_j) \\ &= d(\phi^i \circ \phi^{-1})(\phi(x))(e_j) \\ &= \lim_{t\to 0} \frac{\phi^i \circ \phi^{-1}(\phi(x) + te_j) - \phi^i(x)}{t} \\ &= \delta_{ij}. \end{aligned}$$

□

Exercise 1.14. 1. *Assume that (U_α, ϕ_α) and (U_β, ϕ_β) are two compatible coordinate charts at x. Show that the basis change is given by the Jacobian matrix of $\phi_\alpha \circ \phi_\beta^{-1}$.*

2. *Express the tangent vector $[\gamma]$ in T_xM in terms of a basis from a coordinate chart.*

1.4 Submanifolds

As our manifolds are locally Euclidean space, we know that, in Euclidean space $\mathbb{R}^d$, a Euclidean subspace can be written as a linear subspace like

$$\{(x^1, \cdots, x^k, 0, \cdots, 0) | (x^1, \cdots, x^k) \in \mathbb{R}^k\}.$$

A submanifold is a subset which is locally of Euclidean subspace.

We begin with a submanifold with a Euclidean space.

Definition 1.15. *Let Z be a subset of $\mathbb{R}^n$. We call Z a smooth submanifold of dimension d if we can cover Z with domains of coordinate charts on $\mathbb{R}^n$ such that for each chart $\big(U, \phi = (\phi^1, \cdots, \phi^n)\big)$*

$$U \cap Z = \{x \in U | \phi^{d+1}(x) = \cdots = \phi^n(x) = 0.\}.$$

We have to check that a submanifold as defined above is indeed a manifold. Let $\{U_\alpha, \phi_\alpha)\}$ be an atlas of $\mathbb{R}^n$ whose domains cover Z. Then $\mathcal{A} = \{(U_\alpha \cap Z, \psi_\alpha)\}$ with $\psi_\alpha = (\phi_\alpha^1|_{U_\alpha \cap Z}, \cdots, \phi_\alpha^d|_{U_\alpha \cap Z})$ is an atlas of Z which makes Z a differentiable manifold with differential structure given by $\mathcal{A}$.

Theorem 1.16. *Let $f : \mathbb{R}^n \to \mathbb{R}^{n-d}$ be a smooth map and let $Z = f^{-1}(0)$. If $df(z)$ is onto for all $z \in Z$, then Z is a submanifold of dimension d.*

Proof. Fix $z \in Z$. The kernel of $df(z)$, denoted by K_z, is a d-dimensional linear subspace of $\mathbb{R}^n$. Choose a basis of K_z as $v_1, \cdots, v_d$, with respect to this basis, write the orthogonal projection $\pi : \mathbb{R}^n \to K_z$ as

$$\pi(x) = \sum_{i=1}^{d} \pi^i(x) v_i.$$

Now we define a map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ by $\phi(x) = (\pi^1(x), \cdots, \pi^d(x), f(x))$. Then the differential of $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$d\phi(z)(v) = \big(\pi^1(v), \cdots, \pi^d(v), df(z)(v)\big).$$

One can check that $d\phi(z)$ is a linear isomorphism. Apply the inverse function theorem, there is an open set U in $\mathbb{R}^n$ with $z \in U$ such that $\phi(U)$ is open and

$$\phi|_U \qquad U \to \phi(U)$$

is a diffeomorphism. This means that (U, ϕ) is a coordinate chart of $\mathbb{R}^n$ and

$$U \cap Z = \{x \in U | f(x) = (\phi^{d+1}(x), \cdots, \phi^n(x)) = 0\},$$

therefore, Z is a submanifold of $\mathbb{R}^n$,in particular, Z is a manifold. $\square$

Exercise 1.17. *Show that the unit sphere S^d in $\mathbb{R}^{d+1}$ is a smooth submanifold.*

Definition 1.18. *Let Z be a subset of a smooth manifold M of dimension n. We call Z a submanifold of dimension d if for every point $z \in Z$, there is a coordinate chart (U, ϕ) on M such that $z \in U$ and*

$$U \cap Z = \{y \in U : |\psi^{d+1}(y) = \cdots = \psi^n(y) = 0\}.$$

Find an atlas of Z which makes Z a smooth manifold and show that the inclusion map i_Z is smooth.

Theorem 1.19. *Let f be a smooth map between two smooth manifolds M and N of dimension m and n respectively, with $m > n$. Let $n \in N$ and $Z = f^{-1}(n)$. If $df(z)$ is onto for all z in Z, then Z is a submanifold of M and the image of $d(i_Z)$ at $z \in Z$ is precise the kernel of $df(z)$.*

1.5 Vector fields and Lie bracket

Definition 1.20. *A vector field is an assignment of a tangent vector $X(x) \in T_xM$ to every point $x \in M$. A vector field $\{X(x)\}_{x\in M}$ is smooth if, for an atlas $\{U_\alpha, \phi_\alpha\}$*

$$X(x) = \sum_{i=1}^{d} X^i_\alpha(x) \frac{\partial}{\partial \phi^i_\alpha}(x),$$

then each component $X^i_\alpha(x)$ is a smooth function on U_α.

Proposition 1.21. *A vector field is smooth if and only if, for any smooth function f on an open set $V \subset M$,*

$$df(x)(X(x)) = < df(x), X(x) >$$

is a smooth function on V. For a smooth vector field X and a smooth function f on M,

$$(fX)(x) = f(x)X(x)$$

is also a smooth vector field.

Proof. Note that $X^i_\alpha(x) = d\phi^i_\alpha(x)(X(x))$ for any coordinate functions $\phi_\alpha = (\phi^1_\alpha, \cdots, \phi^d_\alpha)$. This proposition follows from various definitions. □

Let g be a smooth function on M and X be a smooth vector field, there is a smooth function on M denoted by $X(f)$, which is given by

$$X(f)(x) = df(x)(X(x)),$$

which is really the contraction between the tangent vector defined by X and the cotangent vector defined by the differential df of f.

We have two different operations involving a smooth vector field and a smooth function: one is the multiplication of a smooth function to a vector field to get a new vector field, and the other is the contraction of a vector field and a differential by their duality to get a new smooth function. This latter contraction can be seen as an operation of X on the space of smooth functions. Hence a smooth vector field X define a map on the space of all smooth functions denoted by $C^\infty(M)$.

Lemma 1.22. *For a smooth vector field X on M, the contraction defined above is a $\mathbb{R}$-linear derivation on $C^\infty(M)$, that is, for two smooth functions f and g, and a real number a:*

$$X(f + ag) = X(f) + aX(g),$$
$$X(fg) = X(f)g + fX(g).$$

This Lemma follows from the the Leibnitz rule for partial derivatives.

There is another important concept on the space of all smooth vector fields, which measures the non-commutativity of two vector fields viewed as two derivations on $C^\infty(M)$. This is the Lie bracket.

Let X and Y be two smooth vector fields on M, we can define a new vector field

$$[X, Y](f) = X(Y(f)) - Y(X(f)),$$

for any smooth function f on M.

Proposition 1.23.

1. *$[X, Y]$ is a smooth vector field.*
2. *If f and g are smooth functions on M, then $[fX, gY] = fg[X, Y] + fX(g)Y$.*
3. *$[X, Y] = -[Y, X]$.*

4. $[[X,Y],Z]+[[Y,X],Z]+[[Z,X],Y]=0$ *for smooth vector fields* X*,* Y *and* Z *on* M*.*

Proof. We prove these properties by using local coordinate charts. For any coordinate chart (U,ϕ) with $\phi=(\phi^1,\cdots,\phi^d)$, we can write

$$X|_U=\sum_{i=1}^{d}X^i\frac{\partial}{\partial\phi^i},Y|_U=\sum_{j=1}^{d}Y^j\frac{\partial}{\partial\phi^j},$$

where $\{X^i\}$ and $\{Y^j\}$ are smooth functions on U, Applying $[X,Y]$ to the coordinate functions, we get

$$\begin{aligned}[X,Y](\phi^i) &= X(Y(\phi^i))-Y(X(\phi^i))\\ &= X(Y^i)-Y(X^i)\\ &= \textstyle\sum_{j=1}^{d}\big(X^j\frac{\partial Y^i}{\partial\phi^j}-Y^j\frac{\partial X^i}{\partial\phi^j}\big),\end{aligned}$$

which is a smooth function on U. Hence,

$$[X,Y]|_U=\sum_{i=1}^{d}\Big(\sum_{j=1}^{d}(X^j\frac{\partial Y^i}{\partial\phi^j}-Y^j\frac{\partial X^i}{\partial\phi^j})\Big)\frac{\partial}{\partial\phi^i}$$

is a smooth vector field on M. One can show (b)-(d) by using this local expression. □

Remark 1.24. *A vector space with a bilinear form*

$$[\cdot,\cdot]:\qquad V\times V\to V$$

satisfying properties (c) and (d) of Proposition 1.23 is called a Lie algebra.

1.6 Tangent bundle and cotangent bundle

Let M be a d-dimensional smooth manifold. For x in M, T_xM and T_x^*M, the tangent and cotangent spaces at x are two d-dimensional linear vector spaces.

Denote

$$TM=\bigcup_{x\in M}T_xM.$$

We can now list a few properties about this space.

1. There is a projection map $\pi : TM \to M$ which sends (x, v_x) to x, where v_x is a tangent vector at x.

2. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M, then for any $x \in U_\alpha$, the differential $d\phi_\alpha(x) : T_xM \to \mathbb{R}^d$ is a linear isomorphism. From the standard basis of $\mathbb{R}^d$, we have a basis
$$\{\frac{\partial}{\partial \phi_\alpha^1}(x), \cdots, \frac{\partial}{\partial \phi_\alpha^d}(x)\}$$
of T_xM for any x in U_α. We can define
$$\begin{array}{ccc} \tilde{\phi_\alpha} & \pi^{-1}(U_\alpha) & \cong & U_\alpha \times \mathbb{R}^d \\ \sum_{i=1}^d a^i(x)\frac{\partial}{\partial \phi_\alpha^i}(x) & \mapsto & (x, a^1, \cdots a^d). \end{array}$$
This is called the local trivialization of TM over U_α. Note that $TM = \bigcup_\alpha \pi^{-1}(U_\alpha)$ and $\{(\pi^{-1}(U_\alpha), \tilde{\phi_\alpha})\}$ is an atlas on TM, hence, TM is a smooth manifold.

This is called the tangent bundle of M. Similarly,
$$T^*M = \bigcup_{x \in M} T_x^*M$$
has a smooth manifold structure, T^*M is called the cotangent bundle of M.

The main purpose of this course is to study the following object.

Definition 1.25. *A rank n vector bundle E over M is a quadruple $(E, M, \pi, \mathbb{R}^n)$ such that*

1. *$\pi : E \to M$ is a smooth map.*

2. *For any x in M, $\pi^{-1}(x) \cong \mathbb{R}^n$ is called the fiber over x.*

3. *There is an atlas $\{9U_\alpha, \phi_\alpha)\}$ on M, such that*
$$\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^n,$$
in a fiber preserving way.

A smooth map $s : M \to E$ such that $\pi \circ s = identity$ is called a smooth section of the vector bundle $(E, M, \pi, \mathbb{R}^n)$.

The tangent bundle TM is a rank d vector bundle over M and a vector field is a section of the tangent bundle. Similarly, the differential of a smooth function defines a section of the cotangent bundle, sections of cotangent bundle are also called differential 1-forms on M.

1.7 The exterior algebra, wedge and contraction operations

Let V be a d-dimensional vector space over $\mathbb{R}$, a k-linear map

$$\omega : \qquad \underbrace{V \times V \times \cdots \times V}_{k} \to \mathbb{R}$$

which is linear in each factor of k-copies of V. We call that ω is totally anti-symmetric if

$$\omega(v_1, \cdots, v_i, v_{i+1}, \cdots, v_k) = -\omega(v_1, \cdots, v_{i+1}, v_i \cdots, v_k) \tag{3}$$

for all $v_1, \cdots, v_k$ and any $1 \leq i \leq k-1$.

Lemma 1.26. *Let ω be a totally, anti-symmetric, k-multilinear map on V, then, for $v_1, \cdots, v_k \in V$, we have*

1. $\omega(v_1, \cdots, v, v, \cdots, v_k) = 0$.
2. *If $\sigma \in S_k$ is a permutation of k letters, then*

$$\omega(v_1, \cdots, v_i, \cdots, v_k) = sgn(\sigma)\omega(v_{\sigma(1)}, \cdots, v_{\sigma(i)}, \cdots, v_{\sigma(k)})$$

where $sgn(\sigma)$ is the sign of the permutation σ.

Denote the vector space of all k-linear, totally anti-symmetric maps by $\Lambda^k(V^*)$, whose element is called a k-form. As a convention, we take $\Lambda^0(V^*) = \mathbb{R}$. Obviously, $\Lambda^k(V^*) = 0$ for all $k > d$.

Definition 1.27. *The wedge product $\wedge : \Lambda^p(V^*) \times \Lambda^q(V^*) \to \Lambda^{p+q}(V^*)$ is defined as follows: if $\omega \in \Lambda^p(V^*)$ and $\rho \in \Lambda^q(V^*)$, then*

$$\begin{aligned} & \omega \wedge \rho(v_1, \cdots, v_{p+q}) \\ = \; & \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} sgn(\sigma)\omega(v_{\sigma(1)}, \cdots, v_{\sigma(p)})\rho(v_{\sigma(p+1)}, \cdots, v_{\sigma(p+q)}). \end{aligned}$$

for $p+q$ elements $v_1, \cdots, v_{p+q}$ in V.

Exercise 1.28. *1. Show that $\big(\Lambda(V^*) = \bigoplus_{k=1}^{d} \Lambda^k(V^*), \wedge\big)$ is an associative algebra, which is called the exterior algebra of V^*.*

2. Let $\alpha^1, \cdots, \alpha^k$ be elements of V^*. Show that

$$\begin{aligned} & \big(\alpha^1 \wedge \cdots \wedge \alpha^k\big)(v_1, \cdots, v_k) \\ = \ & \textstyle\sum_{\sigma \in S_k} sgn(\sigma) \alpha^1(v_{\sigma(1)}) \cdots \alpha^k(v_{\sigma(k)}) \end{aligned}$$

3. Choose a basis $\{v_1, \cdots, v_d\}$ for V, whose dual basis is denoted by $\{\alpha^1, \cdots, \alpha^d\}$. Show that

$$\{\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} | 1 \leq i_1 < \cdots < i_k \leq d.\}$$

is a basis for $\Lambda^k(V^*)$. Hence, $\Lambda(V^*)$ is a 2^d-dimensional linear space.

4. If $\omega \in \Lambda^p(V^*)$ and $\rho \in \Lambda^q(V^*)$, then

$$\omega \wedge \rho = (-1)^{pq} \rho \wedge \omega.$$

Definition 1.29. *Given $\omega \in \Lambda^k(V^*)$, and $v \in V$, the contraction of ω and v is the $(k-1)$-form, defined by*

$$\imath_v(\omega)(v_1, \cdots, v_{k-1}) = \omega(v, v_1, \cdots, v_{k-1}).$$

The following formula relates the contraction and the wedge product.

Lemma 1.30. *For $\omega \in \Lambda^p(V^*)$, $\rho \in \Lambda^q(V^*)$ and $v \in V$, then*

$$\imath_v(\omega \wedge \rho) = \imath_v(\omega) \wedge \rho + (-1)^p \omega \wedge \imath_v(\rho).$$

1.8 Differential form and the exterior derivative

Definition 1.31. *Let M be a smooth manifold of dimension n. A smooth differential form k-form is an element of $\Lambda^k T_x^* M$ for any $x \in M$, which depends on x smoothly, that is, ω is a smooth section of the exterior bundle $\Lambda^k(T^*M)$ of power k, where*

$$\Lambda^k(T^*M) = \bigcup_{x \in M} \Lambda^k(T_x^* M).$$

Note that for a coordinate chart (U_α, ϕ_α),

$$\Lambda^k(T*M)|_{U_\alpha} = \bigcup_{x\in U_\alpha} \Lambda^k(T_x^*M)$$

has a trivialization given by the basis

$$\{d\phi_\alpha^{i_1}\wedge\cdots\wedge d\phi_\alpha^{i_k}|1\le i_1<\cdots<i_k\le n.\}$$

In terms of this basis, any differential k-form, when restricted to U_α*, can be written as*

$$\omega = \sum_{1\le i_1<\cdots<i_k\le n} \omega^\alpha_{i_1\cdots i_k} d\phi_\alpha^{i_1}\wedge\cdots d\phi_\alpha^{i_k},$$

where $\omega^\alpha_{i_1\cdots i_k}$ *is a smooth function on* U_α.

From the definition, we see that a differential form defines a k-linear map on the space of sections of the tangent bundle TM. For k smooth vector fields $X_1,\cdots,X_k$, a smooth differential k-form ω gives rise to a map

$$(X_1,\cdots,X_k)\mapsto \omega(X_1,\cdots,X_k),$$

with $\omega(X_1,\cdots,X_k)$ is a smooth function on M. The proof of this' claim can be obtained by using the local expressions.

For example, on the Euclidean space $\mathbb{R}^3$ with the usual coordinate system $\{x,y,z\}$, a smooth differential 0-form is just a smooth function of x,y, and z; a smooth 1-form, 2-form and 3-form can be written respectively as

$$f_1(x,y,z)dx+f_2(x,y,z)dy+f_3(x,y,z)dz;$$

$$g_1(x,y,z)dx\wedge dy+g_2(x,y,z)dx\wedge dz+g_3(x,y,z)dy\wedge dz;$$

$$h(x,y,z)dx\wedge dy\wedge dz.$$

Here f_i, g_i and h are smooth functions.

Exercise 1.32. *1. In the local expression of a differential k-form, the coefficient* $\omega^\alpha_{i_1\cdots i_k}$ *is totally anti-symmetric in their indices.*

2. *On the overlap of two coordinate charts (U_α, ϕ_α) and (U_β, ϕ_α), establish the relationship between the coefficients of local expression of ω under the two coordinate charts.*

Denote $\Omega^k(M)$ the set of all smooth differentiable $k-$forms. Note that $\Omega^0(M) = C^\infty(M)$, and for any $f \in \Omega^0(M)$, $df \in \Omega^1(M)$. Hence, we have a linear differential operator $d : \Omega^0(M) \to \Omega^1(M)$ satisfying the Leibnitz rule:

$$d(fg) = (df)g + f(dg)$$

for any two smooth functions f and g. This differential operator can be generalized to act on any k-forms, which is one of the most important operators in differential geometry.

Theorem 1.33. *There exists a unique linear map*

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

for all $k = 0, \cdots, n-1$ satisfying

1. *For $k = 0$, d is the usual differential operator acting on smooth functions.*
2. $d^2 = 0$.
3. $d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^p \omega \wedge (d\rho)$ *for any p-form ω and q-form ρ.*

This operator is called the exterior derivative on M.

Proof. Define d recursively. Suppose that such linear operator d exists for $p \leq k$. Let (U, ϕ) with $\phi = (x^1, \cdots, x^n)$ be a coordinate chart. For $\omega \in \Omega^{k+1}(M)$, we can express ω as

$$\omega = \sum_{i=1}^{n} \omega_i \wedge dx^i$$

with $\omega_i = (-1)^k \imath_{\frac{\partial}{\partial x^i}} \omega$ is a k-form, obtained from the contraction of ω and $\dfrac{\partial}{\partial x^i}$. We now define

$$d\omega = \sum_{i=1}^{n} d\omega_i \wedge dx^i.$$

We need to check that this definition is independent of the choice of coordinate chart, that is, on the overlap of two charts (U_α, ϕ_α) and (U_β, ϕ_β), we will show that

$$\sum_{i=1}^{n} d\omega_i^\alpha \wedge dx_\alpha^i = \sum_{i=1}^{n} d\omega_i^\beta \wedge dx_\beta^i. \tag{4}$$

Note that on $U_\alpha \cap U_\beta$, we have

$$dx_\beta^i = \sum_{j=1}^{n} \frac{\partial x_\beta^i}{\partial x_\alpha^j} dx_\alpha^j, \tag{5}$$

$$\frac{\partial}{\partial x_\alpha^i} = \sum_{j=1}^{n} \frac{\partial x_\beta^j}{\partial x_\alpha^i} \frac{\partial}{\partial x_\beta^j},$$

which imply that

$$\omega_i^\alpha = \sum_{j=1}^{n} \frac{\partial x_\beta^j}{\partial x_\alpha^i} \omega_j^\beta.$$

From this, we get

$$d\omega_i^\alpha = \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{\partial^2 x_\beta^j}{\partial x_\alpha^i dx_\alpha^l} dx_\alpha^l \omega_j^\beta + \sum_{j=1}^{n} \frac{\partial x_\beta^j}{\partial x_\alpha^i} d\omega_j^\beta.$$

One can check (4) by direct calculation:

$$\begin{aligned} & \textstyle\sum_{i=1}^{n} d\omega_i^\alpha \wedge dx_\alpha^i \\ = & \textstyle\sum_{i,j,l=1}^{n} \frac{\partial^2 x_\beta^j}{\partial x_\alpha^i dx_\alpha^l} dx_\alpha^l \wedge \omega_j^\beta \wedge dx_\alpha^i + \displaystyle\sum_{i,j=1}^{n} \frac{\partial x_\beta^j}{\partial x_\alpha^i} d\omega_j^\beta \wedge dx_\alpha^i. \end{aligned}$$

Notet that $\frac{\partial^2 x_\beta^j}{\partial x_\alpha^i dx_\alpha^l}$ is symmetric for the indices i and l, while $dx_\alpha^l \wedge \omega_j^\beta \wedge dx_\alpha^i$ is anti-symmetric for the indices i and j, hence the first term vanishes, together with (5), we obtain (4).

The claims 2 and 3 are left as an exercise. □

Another local expression of d is also useful in some cases. For a smooth differential k-form ω on M, let (U, ϕ) with $\phi = (x^1, \cdots, x^n)$ be a coordinate chart on M, we can write ω as

$$\omega = \sum_{i_1, \cdots, i_k} \frac{1}{k!} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $\omega_{i_1\cdots i_k}$ is a smooth function on U, then the exterior differential d on ω, in the chart (U, ϕ) can be written as

$$d\omega = \sum_{i_1,\cdots,i_k} \sum_{i=1}^{n} \frac{1}{k!} \frac{\partial \omega_{i_1\cdots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{6}$$

Remark 1.34. *Using the Lie derivatives on differential forms, which will be introduced later in the course, one can have a global expression of d by evaluating on vector fields. For a k-form ω, $d\omega$ can be determined by evaluating on $k+1$ vector fields $X_1, \cdots, X_{k+1}$ as follows:*

$$\begin{aligned} & d\omega(X_1, \cdots, X_{k+1}) \\ = \; & \frac{1}{k+1}\Big(\sum_{i=1}^{k+1}(-1)^{i+1} X_i(\omega(X_1, \cdots, \hat{X}_i, \cdots, X_{k+1})) \\ & + \textstyle\sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{k+1})\Big). \end{aligned}$$

Here the circumflex over a term means that it is to be omitted.

1.9 De Rham cohomology

We first introduce a concept from homological algebra, which a complex and its differentials. A cochain complex is a sequence of linear maps

$$\cdots \to C_{i-1} \xrightarrow{d_{i-1}} C_i \xrightarrow{d_i} C_{i+1} \xrightarrow{d_{i+1}} \to \cdots$$

such that $d_i \circ d_{i-1} = 0$ for all i. From any cochain complex (C_i, d_i), we know that the image of map d_{i-1} is contained in the kernel of map d_{i+1}, the quotient space of the kernel of d_i by the subspace of the image of map d_{i-1} is called the i-th cohomology group of the cochain complex (C_i, d_i).

From Theorem 1.33, we know that we have a cochain complex:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \to \Omega^{d-1}(M) \xrightarrow{d} \Omega^d(M) \to 0.$$

Definition 1.35. *A smooth differential k-form ω on a smooth manifold M is called closed if $d\omega = 0$, and called exact if there is a $(k-1)$ form η such that $\omega = d\eta$. The* *quotient space of the real vector space of closed k-forms on M modulo the subspace of all exact k-forms is called the k-th de Rham cohomology group of M:*

$$H^k(M,\mathbb{R}) = \frac{\{\textit{closed } k\textit{-forms}\}}{\{\textit{exact } k\textit{-forms}\}}.$$

From Theorem 1.33, the wedge product induces a product structure on the cohomology group, which is called the cup product:

$$\cup : \qquad H^i(M,\mathbb{R}) \times H^j(M,\mathbb{R}) \to H^{i+j}(M,\mathbb{R}). \tag{7}$$

This endows the cohomology $H^*(M,\mathbb{R}) = \oplus_{k=0}^d H^k(M,\mathbb{R})$ with a ring structure, which is called the cohomology ring of M.

Let $f : M \to N$ be a smooth map, the dull of the differential map df defines a map

$$df^*(x) : \qquad T^*_{f(x)}N \to T^*_x M.$$

Here $df^*(x)(\omega)(X) = \omega(df(x)(X))$ for $\omega \in T^*_{f(x)}N$ and $X \in T_x M$. This induces an algebra homomorphism on the exterior algebras, denoted by f^*:

$$f^* : \qquad \Lambda(T^*_{f(x)}N) \to \Lambda(T^*_x M).$$

For any differential form ω on N, we can apply f^* to pull ω back to get a differential form $f^*(\omega)$ on M. From this definition, we have

$$f^*(\omega)\big(X_1, \cdots, X_k\big) = \omega\big(df(X_1), \cdots, df(X_k)\big)$$

for $\omega \in \Omega^k(N)$ and for vector fields $X_1, \cdots, X_k$ on M.

Proposition 1.36. *Let $f : M \to N$ be a smooth map. Then*

1. *$d(f^*(\omega)) = f^*(d\omega)$ for a differential form ω on N.*
2. *$f^*(\omega \wedge \rho) = f^*(\omega) \wedge f^*(\rho)$ for two differential forms ω and ρ on N.*

Proof. The proof can be obtained by local calculations. To prove that d commutes with f^*, we choose a coordinate chart (U, ϕ) with $\phi = (x^1, \cdots, x^d)$ around $f(x) \in N$ and a neighborhood V of x such that $f(V) \subset U$. Write a k-form ω as

$$\omega = \sum_{i_1,\cdots,i_k} \frac{1}{k!} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Here $\omega_{i_1\cdots i_k}$ is a smooth function on U. Then

$$f^*(\omega) = \sum_{i_1,\cdots,i_k} \frac{1}{k!}\omega_{i_1\cdots i_k} \circ f d(x^{i_1} \circ f) \wedge \cdots \wedge d(x^{i_k} \circ f).$$

Applying the exterior derivative, we obtain over V

$$\begin{aligned} & d\big(f^*(\omega)\big) \\ = \; & \textstyle\sum_{i_1,\cdots,i_k} \frac{1}{k!} d\big(\omega_{i_1\cdots i_k} \circ f\big) d(x^{i_1} \circ f) \wedge \cdots \wedge d(x^{i_k} \circ f) \\ = \; & \textstyle\sum_{i_1,\cdots,i_k} \frac{1}{k!} f^*\big(\omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}\big) \\ = \; & f^*(d\omega). \end{aligned}$$

This calculation also shows that the pull-back map f^* is an algebra homomorphism on the spaces of differential forms with the wedge product as the product structure. $\square$

Corollary 1.37. *Let $f : M \to N$ be a smooth map. Then the pull-back by f induces a ring homomorphism on the de Rham cohomology rings of N and M.*

2 Lie Groups and Lie Algebras

2.1 Lie groups and examples

Definition 2.1. *A* **Lie Group** G *is a smooth manifold with a group structure such that the map* $G \times G \to G$ *defined by* $(g_1, g_1) \mapsto g_1 g_2^{-1}$ *is a smooth map. A Lie subgroup* H *of a Lie group* G *is a Lie group and a submanifold of* G *such that the inclusion* $i : H \to G$ *is a group homomorphism, if* H *is a closed subset of* G*, then* H *is called a closed Lie subgroup.*

By the definition and the properties of smooth maps, for a Lie group, the group multiplication $\phi : G \times G \to G$ where $\phi(g, h) = gh$ and the inverse $i : G \to G$ $(i(g) = g^{-1})$, are smooth maps.

Example 2.2.

1. *The Euclidean space* $\mathbb{R}^n$ *is a Lie group under vector addition.*
2. *The non-zero complex numbers* $\mathbb{C}^*$ *forms a Lie group under multiplication. The unit circle* $S^1 \subset \mathbb{C}^*$ *is a Lie subgroup of* $\mathbb{C}^*$*.*
3. *The product* $G \times H$ *of two Lie groups is a Lie group with the product smooth manifold structure and the direct group product structure.*
4. *The* **General Linear Group** $GL(n, \mathbb{R})$ *of all* $n \times n$ *matrices over* $\mathbb{R}$ *of non-zero determinant with matrix multiplication is a Lie group of dimension* n^2*, the map in the Definition 2.1, which is given by rational functions of the natural coordinates, is smooth. Similarly, the* **complex general linear group** $GL(n\mathbb{C})$ *of all* $n \times n$ *matrices over* $\mathbb{C}$ *of non-zero determinant is a Lie group of dimension* $2n^2$*.*
5. *The group of* **affine motions** *of* $\mathbb{R}^n$ *is* $GL(n, \mathbb{R}) \times \mathbb{R}^n$ *with the group structure*

   $$(A_1, v_1)(A_2, v_2) = (A_1 A_2, A_1 v_2 + v_1)$$

   is a Lie group.
6. *The* **Orthogonal Group** $O(n, \mathbb{R})$ *is the group of all linear transformations of* $\mathbb{R}^n$ *which preserve the usual inner product on* $\mathbb{R}^n$*. In other words,*

   $$O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) | AA^t = I.\}$$

is a closed subgroup of $GL(n,\mathbb{R})$*, here we denote by* A^t *the transpose of* A*, and* I *is the identity matrix. We can identify* $GL(n,\mathbb{R})$ *with an open subset of* $\mathbb{R}^{n^2}$ *by writing down the rows one after the other. Define a smooth map* $f:\mathbb{R}^{n^2}\to\mathbb{R}^{n^2}$ *by* $f(A)=AA^t-I$*, then* $O(n,\mathbb{R})=f^{-1}(0)$*. Note that* AA^t-I *is a symmetric matrix. Let* S *be the linear subspace of* $\mathbb{R}^{n^2}$ *corresponding to the set of all symmetric matrices, which can be identified with a Euclidean space* $\mathbb{R}^d$ *where* $d=\dfrac{n(n+1)}{2}$*. Then* f *is a smooth map from* $\mathbb{R}^{n^2}$ *to* $\mathbb{R}^d$*. Look at the derivative of* f *at a matrix* A*, we have*

$$df(A)(B)=BA^t+AB^t.$$

For any symmetric matrix B*,* $df(A)(\frac{1}{2}BA)=B$ *for* $A\in O(n,\mathbb{R})$*. This implies that* $df(A):\mathbb{R}^{n^2}\to\mathbb{R}^d$ *is onto for any* $A\in O(n,\mathbb{R})$*. Therefore* $O(n,\mathbb{R})$ *is a Lie subgroup of* $GL(n,\mathbb{R})$ *of dimension* $n^2-d=\dfrac{n(n-1)}{2}$ *by Theorem 1.19.*

7. *Another important example is the* **Unitary Group** $U(n)$*, which consists of matrix* $A\in GL(n,\mathbb{C})$ *with* $A^*A=\bar{A}^tA=I$*. Here the overline indicates complex conjugation in each entry of the matrix. Using the similar function as in the above example, one can show that* $U(n)$ *is a Lie subgroup of* $GL(n,\mathbb{C})$ *of dimension* n^2*. Note that The determinant function on* $U(n)$ *is a map* $det:U(n)\to S^1=U(1)$*. The* **special unitary group**

$$SU(n)=\{A\in U(n)|det(A)=1.\}$$

is a Lie subgroup of $U(n)$ *of dimension* n^2-1*. Similarly, we have the special linear groups* $SL(n,\mathbb{R})$ *and* $SL(n,\mathbb{C})$*, the special orthogonal group* $SO(n)$*.*

8. *Note that* $S^1\cong SO(2)$ *given by* $e^{i\theta}\mapsto\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$*.*

2.2 Left invariant vector fields and Lie algebras

Lie group has many nice properties, which arise from the group multiplication by a fixed element.

Definition 2.3. *Fix an element g in a Lie group G. The left translation L_g by g and right translation R_g by g are the diffeomorphism maps $G \to G$ defined by*

$$L_g(h) = gh, \qquad R_g(h) = hg$$

for all $h \in G$.

The left translation by $g \in G$, being a diffeomorphism, moves a neighborhood U_h around h to $L_g(U_h)$, which is a neighborhood of gh. Obviously, L_g moves the tangent space $T_h(G)$ to the tangent space $T_{gh}(G)$, this is just the differential of L_g:

$$dL_g(h): \qquad T_h(G) \longrightarrow T_{gh}(G).$$

Let e denote the identity element. Then $dL_g(e)$ defines a linear isomorphism between $T_e(G)$ and $T_g(G)$.

Definition 2.4. *A vector field X on G is called* **left invariant** *if $X(g) = dL_g(e)(X(e))$. A left invariant vector field is determined by its value at e.*

The set of all left invariant vector fields on G will be denoted by the lowercase German letter $\mathfrak{g}$, which is a real vector space isomorphic to the tangent space of G at e.

Proposition 2.5. *Let G be a Lie group, and $\mathfrak{g}$ the sets of left invariant vector fields.*

1. *Left invariant vector fields are smooth vector fields.*

2. *Let X, Y be two left invariant vector fields on G, then their Lie bracket $[X, Y]$ is also a left invariant vector field, this gives $\mathfrak{g}$ a Lie algebra structure.*

Recall that a Lie algebra $\mathfrak{g}$ is a real vector space with a bilinear form $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the bracket) such that for all $X, Y, Z \in \mathfrak{g}$,

1. $[X, Y] = -[Y, X]$, that is, $[\ ,\ ]$ is anti-commutative.

2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$, this is the Jacobi identity for the bracket.

Proof. Note that for a left invariant vector field X and any smooth function f on G

$$\begin{aligned}
(Xf)(g) &= < X(g), df(g) > \\
&= < dL_g(X(e)), df(g) > \\
&= < X(e), dL_g^*\big(df(g)\big) > \\
&= < X(e), d(f \circ L_g)(e) >,
\end{aligned}$$

we only need to show that the function $g \mapsto < X(e), d(f \circ L_g)(e) >$ is a smooth function.

The above calculation tells us that for a left invariant vector field X and a smooth function f, then

$$(Xf) \circ L_g = X(f \circ L_g).$$

Denote by ϕ the group multiplication, i_e^1 and i_g^2 the maps of $G \to G \times G$ defined by

$$i_e^1(g) = (g, e), \qquad i_g^2(h) = (g, h).$$

Let Y be any smooth vector field on G such that $Y(e) = X(e)$, then $\big((0, Y)(f \circ \phi)\big)$ is a smooth function on $G \times G$, hence, $\big((0, Y)(f \circ \phi)\big) \circ i_e^1$ is a smooth function on G. Now we can deduce that

$$\begin{aligned}
g &\mapsto < X(e), d(f \circ L_g)(e) > \\
&= X(e)\big(f \circ \phi \circ i_g^2\big) \\
&= Y(e)\big(f \circ \phi \circ i_g^2\big) \\
&= < (0, Y)(g, e), d(f \circ \phi) > \\
&= [(0, Y)(f \circ \phi)] \circ i_e^1(g)
\end{aligned}$$

is a smooth function. Let X and Y be two left invariant vector fields on G, we need to show that

$$dL_g([X, Y](e)) = [X, Y](g).$$

By evaluating on any smooth function f on G, we have

$$\begin{aligned} dL_g([X,Y](e))(f) &= [X,Y](e)(f \circ L_g) \\ &= X(e)(Y(f \circ L_g)) - Y(e)(X(f \circ L_g)) \\ &= X(e)(Y(f) \circ L_g) - Y(e)(X(f) \circ L_g) \\ &= dL_g(X(e))(Yf) - dL_g Y(e)(Xf) \\ &= X(g)(Y(f)) - Y(g)(Xf) \\ &= \big([X,Y](g)\big)(f). \end{aligned}$$

This shows that $[X,Y]$ is a left invariant vector field on G. □

Definition 2.6. *We define the Lie algebra of a Lie group G to the Lie algebra $\mathfrak{g}$ of the left invariant vector fields on G.*

The Lie algebra $Lie(G)$ of the Lie group G can be taken as the tangent space of G at the identity e, as a real vector space, with the Lie bracket given by the Lie bracket of left invariant vector fields. If H is a Lie subgroup of G, then $Lie(H) \subset Lie(G)$ is a Lie subalgebra.

Example 2.7. *1. $Lie(\mathbb{R}^n)$: as a vector space, $Lie(\mathbb{R}^n) \cong \mathbb{R}^n$. The left invariant vector field determined by $v \in \mathbb{R}^n$ is the constant vector field. The Lie bracket of two such constant vector fields is 0 by direct calculation.*

2. $Lie(GL(n,\mathbb{R}))$: Denote by $\mathfrak{gl}(n,\mathbb{R})$ be the set of all $n \times n$ real matrices, which is a real vector space of dimension n^2 with the addition and scalar multiplications by components. Note that $\mathfrak{gl}(n,\mathbb{R})$ is a Lie algebra with the Lie bracket given by $[A,B] = AB - BA$.

Then $GL(n,\mathbb{R})$ is an open set of $\mathfrak{gl}(n,\mathbb{R})$. Hence, there is a linear isomorphism

$$Lie(GL(n,\mathbb{R})) \xrightarrow{\alpha} \mathfrak{gl}(n,\mathbb{R}).$$

We now show that α is a Lie algebra homomorphism, that is,

$$\alpha([X,Y]) = [\alpha(X), \alpha(Y)]$$

for $X, Y \in Lie(GL(n, \mathbb{R}))$.

Let x_{ij} *be the coordinate function on* $\mathfrak{gl}(n, \mathbb{R})$ *assigning to each matrix its* ij*-th entry. This also gives a coordinate system on* $GL(n, \mathbb{R}^n)$, *and*

$$(x_{ij} \circ L_g)(h) = x_{ij}(gh) = \sum_k x_{ik}(g)x_{kj}(h).$$

For a left invariant vector field X, *the smooth function* $X(x_{ij})$ *can be written as*

$$\begin{aligned} X(x_{ij})(g) &= X(e)(x_{ij} \circ L_g) \\ &= \textstyle\sum_k x_{ik}(g)X(e)\big(x_{kj}\big) \\ &= \textstyle\sum_k x_{ik}(g)x_{kj}\big(\alpha(X)\big). \end{aligned}$$

In particular, $X(x_{ij})(e) = x_{ij}(\alpha(X))$, *and*

$$\begin{aligned} x_{ij}\big(\alpha([X, Y])\big) &= [X, Y](e)(x_{ij}) \\ &= X(e)\big(Y(x_{ij})\big) - Y(e)\big(X(x_{ij})\big) \\ &= \textstyle\sum_k \{x_{ik}(\alpha(X))x_{kj}(\alpha(Y)) - x_{ik}(\alpha(Y))x_{kj}(\alpha(X))\} \\ &= x_{ij}([\alpha(X), \alpha(Y)]). \end{aligned}$$

Therefore, we obtain the Lie algebra isomorphism $Lie(GL(n, \mathbb{R})) \cong \mathfrak{gl}(n, \mathbb{R})$.

Similarly, we have $Lie(GL(n, \mathbb{C})) \cong \mathfrak{gl}(n, \mathbb{C})$.

3. $Lie(O(n, \mathbb{R}))$ *is a Lie subalgebra of* $\mathfrak{gl}(n, \mathbb{R})$, *consists of skew-symmetric matrices*

$$\{B \in \mathfrak{gl}(n, \mathbb{R}) | B + B^t = 0.\}$$

To prove this claim, we only to study the tangent space of $O(n, \mathbb{R})$ *at the identity matrix. Let* A_t *be a smooth path in* $O(n, \mathbb{R})$ *through the identity matrix at* $t = 0$, *thought as a smooth path in Euclidean space* $\mathfrak{gl}(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$, *we can calculate the tangent vector of* A_t *by the limit definition, write* $\frac{dA}{dt}(0) = B$, *then differentiating* $A_t A_t^t = I$, *we get*

$$\frac{dA}{dt}(0) + \frac{dA^t}{dt}(0) = 0.$$

That is, B *is a skew-symmetric matrix. By the same argument, we have*

(a) $Lie(U(n)) = \{B \in \mathfrak{g}l(n,\mathbb{C}) | B + \bar{B}^t = 0.\}$ *consists of all skew-hermitian matrices.*

(b) $Lie(SL(n,\mathbb{R})) = \{B \in \mathfrak{g}l(n,\mathbb{R}) | trace(B) = 0.\}$ *consists of all traceless real matrices.*

(c) $Lie(SL(n,\mathbb{C}))$ *consists of all traceless complex matrices.*

4. *Let V be a n-dimensional real vector space. The set of all linear operators on V is denoted by $End(V)$ which becomes a Lie algebra if we set the Lie bracket by*

$$[L_1, L_2] = L_1 \circ L_2 - L_2 \circ L_1,$$

for two linear operators L_1 and L_2. The **Automorphism group** *$Aut(V) \subset End(V)$ is the subset of all invertible linear operators on V. Then $Aut(V)$ is a Lie group with its Lie algebra $End(V)$.*

For d-dimensional Lie algebra $\mathfrak{g}$, if we choose a basis of $\mathfrak{g}$ to be $\{X_1, \cdots, X_d\}$, then the Lie bracket of $\mathfrak{g}$ is defined by constant $\{c_{ij}^k | 0 \le i, j, k \le d\}$ such that

$$[X_i, X_j] = c_{ij}^k X_k \tag{8}$$

These constants $\{c_{ij}^k\}$ are called the structure constant of $\mathfrak{g}$ with respect to the basis $\{X_i\}$. By the anti-commutativity and Jacobi identity, the structure constants satisfy

$$c_{ij}^k + c_{ji}^k = 0,$$
$$\sum_l (c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m) = 0.$$

2.3 Exponential map and the Adjoint representation

Vector fields are also intimately related to flows on manifolds. By local theory of ordinary differential equations, there exists (at least locally) a unique solution to an ordinary differential equation with initial condition. Therefore, a smooth vector field generates a local flow on M.

Definition 2.8. *A local flow Φ on M is a family of smooth maps*

$$\{\Phi^\alpha : (-\epsilon_\alpha, \epsilon_\alpha) \times V_\alpha \to U_\alpha | \epsilon_\alpha > 0\}_{\alpha \in I}$$

such that

1. $\{V_\alpha\}$ *and* $\{U_\alpha\}$ *are open covers of* M*, and* $V_\alpha \subset U_\alpha$*;*

2. Φ_α *and* Φ_β *agree on their common domain;*

3. $\Phi^\alpha(0,\cdot) : V_\alpha \to U_\alpha$ *is the inclusion map;*

4. $\Phi^\alpha(t_1+t_2,\cdot) = \Phi^\alpha(t_1,\cdot) \circ \Phi^\alpha(t_2,\cdot)$ *whenever both sides are well-defined.*

We call that a local flow Φ *is generated by* X*, if the smooth path* $\Phi^\alpha(t,x)$ *(where* $x \in V_\alpha$ *and* $-\epsilon_\alpha < t < \epsilon_\alpha$*) represents the tangent vector* $X(x) \in T_xM$*,* X *is called the infinitesimal generator of* Φ*.*

The inclusion map on the domains of local flow defines a partial order on the set of local flows on M, given a local flow, there is a unique maximal local flow on M containing Φ. There is a one to one correspondence between maximal local flow on M and vector fields on M giving by the infinitesimal generator of flow.

Definition 2.9. *A global flow* Φ *on* M *is a smooth map* $\Phi : \mathbb{R} \times M \to M$ *such that*

1. $\Phi(0,\cdot) = Id_M$*;*

2. $\Phi(t_1+t_2,\cdot) = \Phi(t_1,\cdot) \circ \Phi(t_2,\cdot)$ *for any* t_1, t_2 *in* $\mathbb{R}$*.*

A global flow defines a 1-parameter diffeomorphism groups of M*.*

Proposition 2.10. *For a left invariant vector field* $X \in \mathfrak{g}$ *on a Lie group, the maximal local flow generated by* X *always admits a global flow* Φ^X*.*

Proof. Choose a local flow $\Phi^e : (-\epsilon,\epsilon) \times V_e \to U_e$ around $e \in G$ generated by X, where $V_e \subset U_e$ are open neighborhood of e. As X is left invariant, for any $g \in G$,

$$\Phi^g(t,h) = L_g\big(\Phi^e(t, L_{g^{-1}}(h)\big), \text{for any } h \in L_g(V_e)$$

defines a local flow around g generated by X. Direct calculation shows that these local flow can be fit together to get a local flow of form

$$\Phi : (-\epsilon,\epsilon) \times G \to G.$$

Now write $\Phi(t,\cdot)$ as $\Phi_t : G \to G$. Using the multiplicativity of local flows, we can extend it to a global flow as follows. For $t \in \mathbb{R}$, choose $k \in \mathbb{Z}$ and $s \in (-\epsilon/2, \epsilon/2)$ such that $t = s + k \cdot \frac{\epsilon}{2}$. Then

$$\Phi(t,g) = \begin{cases} \Phi_{\epsilon/2} \circ \cdots \circ \Phi_{\epsilon/2} \circ \Phi_s(g), & k > 0 \\ \Phi_s(g), & k = 0 \\ \Phi_{-\epsilon/2} \circ \cdots \circ \Phi_{-\epsilon/2} \circ \Phi_s(g), & k < 0 \end{cases}$$

is a well-defined global flow on G. $\square$

In the above proof, we also established that the global flow Φ^X generated by $X \in \mathfrak{g}$ satisfies

$$L_g \circ \Phi^X(t,\cdot) = \Phi^X(t,\cdot) \circ L_g,$$

from which we deduce

$$\Phi^X_{t_1+t_2}(e) = \Phi^X_{t_2} \circ L_{\Phi^X_{t_1}(e)}(e) = \Phi^X_{t_1}(e) \cdot \Phi^X_{t_2}(e).$$

Definition 2.11. *Let $\Phi^X : \mathbb{R} \times G \to G$ be the global flow generated by a left invariant vector field $X \in \mathfrak{g}$. Then $\exp(tX) = \Phi^X(t,e)$ is a 1-parameter subgroup of G. The map $\exp(X) : \mathfrak{g} \to G$ is called the* **exponential map**.

Example 2.12. *The exponential map for $GL(n,\mathbb{C})$ is given by the exponentiation of matrices, that is, for $A \in \mathfrak{g}l(n,\mathbb{C})$,*

$$exp(A) = I + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots. \tag{9}$$

We need to show that the right hand side of (9), denoted by e^A, converges and $\{t \to e^{tA}\}$ is the unique 1-parameter subgroup of $GL(n,\mathbb{C})$ whose tangent vector at 0 is A. For any bounded domain in $\mathfrak{g}l(n,\mathbb{C})$, the norm of A, given by the maximum of entries of A, is bounded. This implies that e^A converges uniformly for A in this bounded domain.

One can establish the following properties for e^A:

1. $det(e^A) = e^{traceA}$.

2. $e^{A+B} = e^A e^B$.

Similarly, for a real or complex vector space V, the exponential map

$$exp: \qquad End(V) \longrightarrow Aut(V)$$

is given by

$$exp(L) = 1 + L + \frac{L^2}{2!} + \cdots + \frac{L^n}{n!} + \cdots \tag{10}$$

where L^n means the n-times of composition of L with itself.

Note that the differential of the exponential map at $0 \in \mathfrak{g}$, $d(\exp)(0) : \mathfrak{g} \to T_e G \cong \mathfrak{g}$ is the identity map, so exp gives a diffeomorphism of a neighborhood of 0 in $\mathfrak{g}$ onto a neighborhood of e in G. It is convenient to give an atlas on a Lie group by this exponential map and the left translations of Lie group elements.

The homomorphism between two Lie groups and its differential at identity is related by the exponential map as in the following proposition.

Proposition 2.13. *Let $\phi : H \to G$ be a homomorphism of Lie groups. Then we have the following commutative diagram:*

$$\begin{array}{ccc} H & \xrightarrow{\phi} & G \\ {\scriptstyle exp}\uparrow & & \uparrow{\scriptstyle exp} \\ \mathfrak{h} & \xrightarrow{d\phi} & \mathfrak{g} \end{array}$$

Proof. For any $X \in \mathfrak{h}$, then $\exp(tX)$ is the 1-parameter subgroup of H generated by X. As ϕ is a group homomorphism, $\phi(exp(tX))$ is also a 1-parameter subgroup of G, the tangent vector at $t = 0$ is given by $d\phi(e)([\exp(tX)])$ at $t = 0$, the tangent vector of $[\exp(tX)]$ at $t = 0$ is $X(e) \in \mathfrak{h} \cong T_e H$, hence, $\phi(exp(tX))$ is the 1-parameter subgroup of G generated by $d\phi(X(e))$.

We know that $\exp\big(td\phi(X(e))\big)$ is the unique 1-parameter subgroup of G generated by $d\phi(X(e))$. This implies that

$$\phi(\exp(tX)) = \exp\big(t(d\phi(X))\big),$$

in particular, $\phi(\exp(X)) = \exp\big((d\phi(X))\big)$. $\square$

Remark 2.14. *The 1-parameter group of diffeomorphism of G associated to left invariant vector field $X \in \mathfrak{g}$ is given by $R_{exp(tX)}$, the right multiplication by $exp(tX)$, as $g\exp(tX)$ is the local flow generated by X and taking the value g at $t = 0$.*

2.3.1 The Lie derivative

Let Φ_t^X be the local flow of a smooth vector field X on M, we can define the Lie derivative on vector fields and differential forms as follows.

Let Y be another smooth vector field Y on M. The Lie derivative of Y with respect to X, denoted by $L_X Y$, is defined as follows, for each $x \in M$,

$$L_X Y(x) = \frac{d}{dt}|_{t=0} d\Phi_{-t}^X\big(Y(\Phi_t^X(x))\big),$$

it is easy to see that $L_X Y$ is also a vector field on M.

Let ω be a differential form on M. The Lie derivative of ω with respect to X is define to be

$$L_X\omega(x) = \frac{d}{dt}|_{t=0}(\Phi_t^X)^*\big(\omega(\Phi_t^X(x))\big) = \lim_{t\to 0}\frac{(\Phi_t^X)^*\big(\omega(\Phi_t^X(x))\big) - \omega(x)}{t}.$$

Here we list a few useful properties about the Lie derivative.

Proposition 2.15. *Let X be a smooth vector field on M. Then*

1. *$L_X f = Xf$ for a smooth function f.*
2. *$L_X Y = [X, Y]$ for a smooth vector field Y on M.*
3. *Acting on differential forms, L_X is a derivative which commutes with the exterior derivative d.*
4. *On differential forms, $L_X = \imath(X)\circ d + d\circ\imath(X)$.*
5. *Let $\omega \in \Omega^k(M)$, and let $X_0, X_1, \cdots, X_k$ be smooth vector fields on M. Then*

$$\begin{aligned} L_{X_0}(X_1, \cdots, X_k) &= (L_{X_0}\omega)(Y_1, \cdots, Y_p) \\ &\quad + \textstyle\sum_{i=1}^k \omega(Y_1, \cdots, X_{i-1}, L_{X_0}X_i, X_{i+1}, \cdots, X_k). \end{aligned}$$

and

$$\begin{aligned} d\omega(X_0, \cdots, X_k) &= \textstyle\sum_{i=0}^k X_i\big(\omega(X_0, \cdots, \hat{X}_i, \cdots, Y_p)\big) \\ &\quad + \textstyle\sum_{i<j} (-1)^{i+j}\omega([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k). \end{aligned}$$

We first recall the relation between a tangent vector field and local gradient flow. For a vector field X, the local gradient flow Φ_t^X has the following property: by varying t in a small interval $(-\epsilon, \epsilon)$, $\Phi_t^X(x)$ represents the tangent vector $X(x)$.

Proof. $L_X f = Xf$ follows from the definition and the above mentioned relation. To show that $L_X Y = [X, Y]$, we only need to show that $L_X Y(f) = [X, Y](f)$ for each $f \in C^\infty(M)$. Let $x \in M$. Then

$$\begin{aligned} (L_X Y)(f)(x) &= \Big(\frac{d}{dt}|_{t=0} d\Phi_{-t}^X\big(Y(\Phi_t^X(x))\big)\Big)(f) \\ &= \frac{d}{dt}\,|_{t=0}\big(Y(\Phi_t^X(x))\big)(f \circ \Phi_{-t}^X) \\ &= \frac{d}{dt}\,|_{t=0}\frac{d}{ds}\,|_{s=0}(f \circ \Phi_{-t}^X) \circ (\Phi_s^Y \circ \Phi_t^X(x)) \\ &= \frac{d}{dt}\,|_{t=0}\frac{d}{ds}\,|_{s=0} f\big(\Phi_{-t}^X \circ \Phi_s^Y \circ \Phi_t^X(x)\big) \\ &= \frac{d}{dt}\,|_{t=0}\frac{d}{ds}\,|_{s=0} f\big(\Phi_s^Y \circ \Phi_t^X(x)\big) + \frac{d}{dt}\,|_{t=0}\frac{d}{ds}\,|_{s=0} f\big(\Phi_{-t}^X \circ \Phi_s^Y(x)\big) \\ &= X(Y(f))(x) - Y(X(f))(x) = [X, Y](f)(x). \end{aligned}$$

The derivative property of L_X

$$L_X(\omega \wedge \eta) = L_X\omega \wedge \eta + \omega \wedge L_X(\eta)$$

follows from adding and subtracting suitable terms in the definition before taking the limit in the definition.

We now check that L_X commutes with d on functions: $L_X(df)(x) = d(L_X f)(x)$, for each smooth function f and each point x.

Note that both are cotangent vectors, so we can pair them with an arbitrary tangent vector field Y. Suppose that the local flow Φ_t^X is well-defined for $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$, then we can view $f \circ \Phi_t^X$ as a smooth function over $(-\epsilon, \epsilon) \times U_x$ for some neighborhood U_x of x.

$$d(L_X f)(x)(Y(x)) = Y(L_X f)(x) = Y\Big(\frac{d}{dt}\,|_{t=0}(f \circ \Phi^X)\Big)(x),$$

and

$$\begin{aligned} L_X(df)(x)(Y_x) &= \langle \frac{d}{dt} |_{t=0} (d\Phi_t^X)^* \big(df(\Phi_t^X(x))\big), Y(x)\rangle \\ &= \langle \frac{d}{dt} |_{t=0} df(\Phi_t^X(x)), \circ d\Phi_t^X(Y(x))\rangle \\ &= \frac{d}{dt} |_{t=0} \Big(Y(f \circ \Phi^X)\Big)(x) \end{aligned}$$

imply that $L_X(df) = d(L_X f)$, since $\frac{d}{dt}$ and Y satisfy $[\frac{d}{dt}, Y] = 0$ as vector fields on $(-\epsilon, \epsilon) \times U_x$. Write any differential form in local coordinates, we can check that L_X commutes with d.

By using local coordinates, we can verify that $L_X = \imath(X) \circ d + d \circ \imath(X)$. The proof of claim (5) is also left as an exercise. □

A Lie group acts on itself on the left $a : G \times G \to G$ by the conjugations:

$$a(g, h) = Ad_g(h) = ghg^{-1}.$$

For each $g \in G$, $Ad_g : G \to G$ is a diffeomorphism with $Ad_g(e) = e$. Then the differential of Ad_g at e defines a linear isomorphism:

$$d(Ad_g)(e) : T_eG \longrightarrow T_eG.$$

Under the identification of the Lie algebra $\mathfrak{g}$ of G with T_eG, we have the representation of G on $\mathfrak{g}$, still denoted by Ad:

$$\begin{array}{rccl} Ad: & G & \longrightarrow & Aut(\mathfrak{g}) \\ & g & \to & Ad_g = d(Ad_g)(e) \end{array}.$$

Note that $d(Ad_{g_1 g_2})(e) = d(Ad_{g_1} \circ Ad_{g_2})(e) = d(Ad_{g_1})(e) \circ d(Ad_{g_2})(e)$. This is called the adjoint representation of G.

The differential of the adjoint representation Ad of G at the identity is denoted by ad, which is the adjoint representation of the Lie algebra $\mathfrak{g}$.

$$ad : T_eG \cong \mathfrak{g} \to End(\mathfrak{g}).$$

Proposition 2.16. *1. Let $X, Y \in \mathfrak{g}$, then $ad_X(Y) = [X, Y]$.*

2. $exp\big(Ad_g(X)\big) = Ad_g(expX)$ *for any* $g \in G$ *and* $X \in \mathfrak{g}$.

3. $Ad_{exp(X)}(Y) = exp(ad_X)(Y)$.

Proof. The first property can be proved by direct calculation,

$$\begin{aligned} ad_X Y(e) &= \frac{d}{dt}|_{t=0} d(Ad_{exp(tX)})(Y(e)) \\ &= \frac{d}{dt}|_{t=0} d(R_{exp(-tX)}) \circ d(L_{exp(tx)})(Y(e)) \\ &= \frac{d}{dt}|_{t=0} d(R_{exp(-tX)})\big(Y(exp(tX))\big) \\ &= L_X(Y) = [X, Y]. \end{aligned}$$

Property 2 and property 3 follow from Proposition 2.13 for the following two commutative diagrams.

$$\begin{array}{ccc} G & \xrightarrow{Ad_g} & G \\ {\scriptstyle exp}\uparrow & & {\scriptstyle exp}\uparrow \\ \mathfrak{g} & \xrightarrow{Ad_g} & \mathfrak{g} \end{array} \qquad\qquad \begin{array}{ccc} G & \xrightarrow{Ad} & Aut(\mathfrak{g}) \\ {\scriptstyle exp}\uparrow & & {\scriptstyle exp}\uparrow \\ \mathfrak{g} & \xrightarrow{ad} & End(\mathfrak{g}) \end{array}$$

□

Note that the representation theory of Lie groups and Lie algebra is a very powerful theory to understand Lie groups and Lie algebras.

2.4 Maurer-Cartan forms and Maurer-Cartan equations

Dual to left invariant vector fields, we also have left invariant differential 1-forms on a Lie group G.

Definition 2.17. *A differential 1-form* ω *on a Lie group is left invariant if* $\omega(g) = dL_g^*\omega(e)$ *for any* $g \in G$. *Left invariant 1-forms are also called as Maurer-Cantan forms on* G.

As for left invariant vector fields, left invariant forms are smooth, and are uniquely determined by their value at the identity. Dual to the Lie algebra $\mathfrak{g}$ of G, the set of left invariant 1-forms on G is a $dimG$ dimensional real vector space, denoted by $\mathfrak{g}^*$, which is dual to $\mathfrak{g}$. Using the global expression of the exterior derivative d on G, one can verify that

$$d\omega(X,Y) = -\omega([X,Y]), \tag{11}$$

for a left invariant 1-form ω and $X, Y \in \mathfrak{g}$.

Proposition 2.18. *Let $\{X_1, \cdots, X_d\}$ be a basis of the Lie algebra $\mathfrak{g}$ of a Lie group G with structures constants $\{c_{ij}^k\}$. Denote by $\{\omega_1, \cdots, \omega_d\}$ be the dual basis of $\mathfrak{g}^*$. Then the exterior derivatives of the ω_i are given by the Maurer-Cantan equations:*

$$d\omega_i = -\sum_{j<k} c_{jk}^i \omega_j \wedge \omega_k.$$

Proof.

$$d\omega_i(X_j, X_k) = -\omega_i([X_j, X_k]) = -\omega_i(\sum_l c_{jk}^l X_l) = -c_{jk}^i.$$

□

2.5 Group actions on manifolds

Definition 2.19. *Let G be a Lie group, and let M be a smooth manifold. A smooth map $\mu : G \times M \to M$, write $\mu(g,x)$ as $g \cdot x$, is called a left action of G on M if $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and $e \cdot x = x$ for all $g_1, g_2 \in G$ and $x \in M$. The right action of G on M can be defined similarly.*

Sometimes we say that M is a G-manifold. One can check that G acts on M by diffeomorphisms. For each $x \in M$, the map $\mu_x : G \to M$ defined by

$$g \mapsto g \cdot x$$

is a smooth map, which is called the **orbit map**. The image of μ_x is called the orbit of G-action on M through x.

Definition 2.20. *The action of G on M is called* **transitive** *if M itself is an orbit, in this case, M is called a* **homogeneous space** *of G.* **The stabilizer** *of x of the G-action on M is defined to be*

$$G_x = \{g \in G | g \cdot x = x.\}$$

The differential of μ_x at e, $d\mu_x(e)$, is a linear map:

$$d\mu_x(e): \qquad T_eG \cong \mathfrak{g} \longrightarrow T_xM.$$

With each $X \in \mathfrak{g}$, the associated tangent vector $d\mu_x(e)(X)$ is denoted by $\underline{X}_x \in T_xM$. As x varies in M, the vector field obtained, denoted by $\underline{X}$, is called the fundamental vector field associated with $X \in \mathfrak{g}$. The local flow on M generated by $\underline{X}$, by definition, is given by

$$\Phi_t^{\underline{X}}(x) = exp(tX) \cdot x.$$

Hence $\Phi_t^{\underline{X}}(x)$ is a global flow.

Using Proposition (2.16), one can show that G_x is a closed Lie subgroup of G, and the Lie algebra of G_x is given by

$$\mathfrak{g}_x = Lie(G_x) = \{X \in \mathfrak{g} | \underline{X}_x = 0.\}.$$

Definition 2.21. *An action of G on a manifold M is free if for each $x \in M$, G_x is $\{e\}$. A smooth map $f : M \to N$ between two G-manifolds M and N, we say that f is a G-equivariant map if for each $x \in M$ and each $g \in G$,*

$$f(g \cdot x) = g \cdot f(x).$$

Example 2.22. 1. *$GL(n, \mathbb{R})$ acts naturally on $\mathbb{R}^n$. If we think of elements of $\mathbb{R}^n$ as $n \times 1$ matrices, then $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication. Let $< \cdot, \cdot >$ be the standard inner product. Elements of $GL(n, \mathbb{R})$ which preserve lengths of vectors are exactly those elements in $O(n, R)$. Hence, we get the standard action of $O(n, \mathbb{R})$ on the unit sphere S^{n-1}:*

$$O(n, \mathbb{R}) \times S^{n-1} \to S^{n-1}.$$

2. *$GL(n,\mathbb{C})$ acts naturally on $\mathbb{C}^n$. Denote the standard Hermitian metric on $\mathbb{C}^n$ as $\langle,\cdot,\cdot\rangle$, that is, under the canonical basis $\{e_i : i = 1, \cdots, n\}$, then*

$$\langle\sum_i^n x^i e_i, \sum_i^n y^i e_i\rangle = \sum_{i=1}^n x^i \overline{y^i}.$$

Then elements of the unitary group $U(n)$ are exactly those linear transformations in $GL(n,\mathbb{C})$ preserving lengths of vectors in $\mathbb{C}^n$. Note that the unit sphere in $\mathbb{C}^n$ is diffeomorphic to the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. Hence, we obtain the left action of $U(n)$ on S^{2n-1}.

3. *The unit circle S^1 in $\mathbb{C}$ acts on $\mathbb{C}^n$ by complex multiplication*

$$t \cdot (z_1, \cdots, z_n) = (tz_1, \cdots, tz_n).$$

$0 \in \mathbb{C}^n$ *is the fixed point of this action and S^1 acts on $\mathbb{C}^n - 0$ freely.*

4. *For $p, q \in \mathbb{Z}$, S^1 acts on the unit sphere $S^3 \subset \mathbb{C}^2$ in the following way*

$$t \cdot (z_1, z_2) = (t^p z_1, t^q z_2).$$

This action is free if and only if $(p,q) = 1$.

5. *S^1 acts on the unit sphere $S^2 \subset \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ by rotating around a fixed axis, that is, S^1 acts on $\mathbb{C}$ by complex multiplication and on R trivially. There are two fixed points $(0,0,1)$ and $(0,0,-1)$.*

Another important classes of G-manifolds are provided by the transitive actions, those are called homogeneous manifolds.

Theorem 2.23. *Let H be a closed Lie subgroup of a Lie group G, and let G/H be the set $\{gH | g \in G\}$ of left cosets modulo H. Let $\pi : G \to G/H$ be the natural projection $\pi(g) = gH$. Then G/H has a unique smooth structure such that*

1. *π is smooth, each fiber $\pi^{-1}(gH)$ admits a transitive and free right H-action.*

2. *π is locally trivialized, that is, for each $gH \in G/H$, there is a neighborhood U of gH such that $\pi^{-1}(U) \cong U \times H$ in a fiber preserving way.*

Proof. First we show the existence of such manifold structure on G/H.

We topologize G/H by requiring $U \subset G/H$ to be open if and only if $\pi^{-1}(U)$ is open in G. With this topology on G/H, G/H is a second countable Hausdorff topological space.

Let $\mathfrak{m}$ be a fixed complementary subspace of $\mathfrak{g}$ to $\mathfrak{h}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Suppose that G is of dimension n and that H is of dimension $n-d$. We construct a coordinate chart (U, ϕ) of G at e as follows. Define a map $\psi : \mathfrak{m} \oplus \mathfrak{h} \to G$ by

$$\psi(A, B) = exp(A)exp(B).$$

Choose open neighborhoods W and V of 0 in $\mathfrak{m}$ and $\mathfrak{h}$ respectively. Let $U = \psi(W \times V) \subset G$ and $\phi = \psi^{-1}$. Then (U, ϕ) is a coordinate chart of G at e, and $H \cap U = \psi(\{0\} \times V)$.

Choose an open neighborhood $C \subset W$ such that $-C = C$ and $exp(C)exp(C) \subset U$. Define local slices of G around e by

$$\psi(\{c\} \times V)$$

for each $c \in C$.

Let $c_1, c_2 \in C$, such that $\exp(c_1)exp(V)$ and $exp(c_2)exp(V)$ lie in a common coset modulo H, then

$$exp(-c_1)exp(c_2) \in H \cap U = \psi(\{0\} \times V),$$

from which there exists $v \in V$ such that

$$\psi(c_1, v) = exp(c_1)exp(v) = exp(c_2) = \psi(c_2, 0).$$

This implies that $c_1 = c_2$ and $v = 0$. Hence, each coset gH meets with $\psi(C \times V)$ in at most one slice, and the map $\psi_C : C \times H \to \psi(C \times V)H$ given by

$$\psi_C(c, h) = exp(c)h$$

is a diffeomorphism. Let $U_{eH} = \pi(\psi(C \times V))$. Denote by π_C the projection

$$C \times H \to C \subset \mathfrak{m} \cong \mathbb{R}^d.$$

Then, $(U_{eH}, \phi_e = \pi_C \circ \psi_C^{-1})$ is a coordinate chart on G/H.

As G can be covered by the open set of the form $g\psi(C \times V)H$ for $g \in G$, then

$$\{U_{gH} = \pi(g\psi(C \times V)H)\}_{g \in G}$$

covers G/H.

For each open set U_{gH}, it has a coordinate function of the form $\phi_g = \phi_e \circ L_{g^{-1}} : U_{gH} \to C \subset \mathbb{R}^d$. On the overlap $U_{g_1 H} \cap U_{g_2 H}$, the coordinate functions ϕ_{g_1} and ϕ_{g_2} are related by $\phi_{g_2} = \phi_{g_1} \circ L_{g_1 g_2^{-1}}$, which is smooth. Hence, $\{(U_{gH}, \phi_g)\}$ is an atlas on G/H, which endows G/H with a smooth structure, and π is a smooth map.

On each open set U_{gH}, define $s_g = L_g \circ \phi^{-1}|_C \circ \phi_g$:

$$U_{gH} \to C \to U \to L_g(U) \subset G.$$

Then s_g is a smooth map satisfying $\pi \circ s_g = Id$. s_g is called the **local section** of π. The local trivialization over U_{gH} is obtained by the diffeomorphism

$$U_{gH} \times H \to g\psi(C \times V)H \subset G$$

which maps $(g_1 H, h)$ to $s_g(g_1 H)h$.

To prove the uniqueness of such smooth structure on G/H. For any other smooth structure on G/H satisfying two properties in Theorem, we can check that the identity map on G/H is a diffeomorphism with the help of local trivializations. □

Theorem 2.24. *Let $\mu : G \times M \to M$ be a transitive left action of G on M. Let $x \in M$ and G_x is the stabilizer of G-action at x. Then the map $\theta : G/G_x \to M$ defined by*

$$\theta(gG_x) = g \cdot x$$

is a G-equivariant diffeomorphism.

Proof. It is easy to see that θ is bijective. We only need to show that θ is smooth and its differential is non-degenerate (by inverse function theorem). From the construction of smooth structure on G/H, we know that θ is smooth if and only if $\theta \circ \pi : G \to M$ is smooth, which is obvious as $\theta \circ \pi = \mu_x$ is the orbit map.

Now it is suffice to show that $d\theta(eG_x) : T_{eG_x}(G/G_x) \to T_x M$ is an isomorphism. Note that $T_{eG_x}(G/G_x) \cong \mathfrak{m}$ from the proof the above theorem. For any $X \in \mathfrak{m}$, the representing curve on G/G_x can be written as $exp(tX)H$, under the differential $d\theta$, it is mapped to $exp(tX) \cdot x$, which is zero if and only if $X \in \mathfrak{m} \cap \mathfrak{h} = \{0\}$, hence $d\theta(eG_x)$ is injective. On the other hand, any smooth path on M through x can be written $g_t \cdot x$ for a smooth path in G through e. Then $d\theta(eG_x)$ maps the tangent vector represented by $g_t H$ to the tangent vector represented by $g_t \cdot x$, hence $d\theta(eG_x)$ is surjective. □

Example 2.25. *1. The natural action of orthogonal group $O(n) = O(n, \mathbb{R})$ on S^{n-1} is transitive. Denote e_1 the point of S^{n-1} with n-tuples of all zeros except 1 at the first position. If v is any point in S^{n-1}, then we can construct an orthonormal basis of $\mathbb{R}^n$, $\{v_1, \cdots, v_n\}$, containing v_1 as the first element. Write*

$$v_i = \sum_{j=1}^{n} g_{ij} e_j.$$

Then $g = (g_{ij})$ is an element of $O(n)$, and $g \cdot e_1 = v_1$. Hence, the left action of $O(n)$ on S^{n-1} is transitive. Now we like to understand the stabilizer group G_{e_1}: Direct calculation shows that G_{e_1} consists of elements of form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{g}_{ij} & \\ 0 & & & \end{pmatrix}$$

where $\hat{g} = (\hat{g}_{ij})$ is an element of $O(n-1)$. Theorem 2.24 implies that there is a natural diffeomorphism between S^{n-1} and the homogeneous manifold $O(n)/O(n-1)$. Actually, we have

$$S^{n-1} \cong O(n)/O(n-1) \cong SO(n)/SO(n-1).$$

2. *Similar arguments show that*

$$S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1).$$

Note that $SU(1) = \{e\}$, *we have a Lie group structure on* $S^3 \cong SU(2)$.

3. $\mathbb{RP}^n \cong SO(n+1)/O(n)$ *where* $O(n)$ *is a closed subgroup of* $SO(n+1)$ *under the following identification for each* $\hat{g} \in O(n)$*:*

$$\begin{pmatrix} det(\hat{g}) & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{g}_{ij} & \\ 0 & & & \end{pmatrix}.$$

4. $\mathbb{CP}^n \cong SU(n+1)/U(n)$ *where* $U(n)$ *is a closed subgroup of* $SU(n+1)$ *under the following identification:*

$$\begin{pmatrix} 1/det(\hat{g}) & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{g}_{ij} & \\ 0 & & & \end{pmatrix}.$$

5. **The Grassmann manifold** *of* k*-planes in a real* d*-dimensional vector space,* $\mathbf{Gr}_k(V)$*, is the set of all* k*-dimensional subspaces (*k*-planes) of* V*. Choose a basis* $v_1, \cdots, v_n$ *for* V*, then* $O(n) \subset GL(n, \mathbb{R})$ *acts linearly on* V*, maps* k*-planes to* k*-planes. One can check that* $O(n)$ *acts transitively on* $\mathbf{Gr_k(V)}$*. Note that* $\mathbf{Gr_1(\mathbb{C}^{n+1})} \cong \mathbb{CP}^\mathbf{n}$.

   Let P_0 *be the* k*-plane spanned by the first* k*-elements of the basis, then the stabilizer of* $O(n)$*-action at* P_0 *is given by subset of* $O(n)$ *of form:*

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

where $g_1 \in O(k)$ *and* $g_2 \in O(n-k)$. *We can endow* $\mathbf{Gr}_k(V)$ *with a smooth structure such that* $\mathbf{Gr}_k(V)$ *is diffeomorphic to the homogeneous manifold* $O(n)/(O(k) \times O(n-k))$.

3 Principal Bundles

3.1 Definition, examples and the transition functions

Definition 3.1. *A principal bundle is a quadruple $(P, M; G, \pi)$, where P and M are smooth manifolds and G is a Lie group acting freely on P on the right, $\pi : P \to M$ is a smooth map, called the projection, such that*

1. *For every $x \in M$, the fiber $\pi^{-1}(x) \cong G$ as smooth manifolds, which admits a transitive and free right G-action.*
2. *The right G action on P preserves the fibers of π.*
3. *For each point $x \in M$, there exists an open neighborhood $U_x \subset M$ of x such that $\psi : \pi^{-1}(U_x) \cong U_x \times G$ in a fiber preserving way.*

P is called the total space of the principal bundle, M is called the base.

Note that ψ is equivariant with respect to G-actions where G acts $U_x \times G$ trivially on U_x and by right translation on G

Homogeneous manifolds, see Example 2.25, provide many interesting examples of principal bundles, as we can think the natural projection $\pi : G \to G/H$ as a principal H-bundle over the base G/H, as provided by Theorem 2.23.

Example 3.2. 1. *(**The Hopf bundle**) The unit sphere $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ is a $U(1)$-principal bundle over $SU(2)/U(1) \cong \mathbb{CP}^1$. We like to discuss more about this principal bundle as it is one of important examples which we will meet again. Note that $S^3 \cong SU(2)$ and $U(1)$ is the circle group in $\mathbb{C} - \{0\}$, and*

$$\mathbb{CP}^1 = \{[z_1, z_2] | 0 \neq (z_1, z_2) \in \mathbb{C}^2\}$$

where $[z_1, z_2]$ represents the equivalence class of (z_1, z_2) under the relation $(z_1, z_2) \sim (z_1', z_2')$ if and only if there is a non-zero $t \in \mathbb{C}$ such that $(z_1, z_2) = t(z_1', z_2')$. There is an identification of $\mathbb{CP}^1$ with S^2 using the stereographic projection from the north pole $n \in S^2$, $p_n : S^2 - \{n\} \to \mathbb{R}^2 \cong \mathbb{C}$. On *the other hand, we can identify* $\mathbb{CP}^1 - \{[0,1]\}$ *with* $\mathbb{C}$ *by the map* $[z_1, z_2] \mapsto z_2/z_1$. *By assigning* $[0,1]$ *to* n *and* $[z_1, z_2]$ *to* $p_n^{-1}(z_2/z_1)$, *we can check that this defines a smooth diffeomorphism between* S^2 *with* $\mathbb{CP}^1$.

Define $\pi : S^3 \cong SU(2) \to S^2$ *by mapping* $(z_1, z_2) \to [z_1, z_2]$, *which is a smooth projection. The fiber over each point* $[z_1, z_2] \in \mathbb{CP}^1$ *is*

$$\pi^{-1}([z_1, z_2]) = \{t(z_1, z_2) | t \in \mathbb{C}, |t| = 1\} \cong S^1.$$

To see that π *is locally trivialized without resorting to Theorem 2.23, we use the standard cover from the stereographic projection:* U_1 *and* U_2, *where* $U_i = \{[z_1, z_2] | z_i \neq 0\}$. *Then the local section over* U_i *given by*

$$(z_1, z_2) \mapsto ([z_1, z_2], z_i/|z_i|)$$

defines a local trivialization of $(S^3, S^2; S^1, \pi)$.

2. **The Frame bundle** *of a smooth manifold* M *of dimension* d: $\mathcal{F}r(M)$. *A point in* $\mathcal{F}r(M)$ *consists of a point* $x \in M$ *and a basis for the tangent space* T_xM. *The projection* $\pi : \mathcal{F}r(M) \to M$ *is obvious. There is a right transitive action of* $GL(d, \mathbb{R})$ *on* $\mathcal{F}r(M)$: *for* $A = (a_{ij}) \in GL(d, \mathbb{R})$, *the action is given by*

$$(x, \{v_1, \cdots, v_d\}) \mapsto (x, \{w_1, \cdots, w_d\}) = (x, \{v_1, \cdots, v_d\} \cdot A),$$

where $w_j = \sum_{i=1}^d a_{ij} v_i$. *As the tangent bundle, the smooth structure and the local trivialization are given by any atlas of* M. *Let* $(U, \phi = (\phi^1, \cdots, \phi^d))$ *be a coordinate chart of* M. *Then we can define the local trivialization of* π *over* U *by*

$$(x, A) \mapsto \big(x, \{\frac{\partial}{\partial \phi^1}(x), \cdots, \frac{\partial}{\partial \phi^d}(x)\} \cdot A\big).$$

3. *Let* $\mathbb{Z}_2$ *be the group with two elements* $\{+1, -1\}$. *There is a natural action of* $\mathbb{Z}_2$ *on the unit sphere* S^n *via*

$$x \cdot (\pm 1) = \pm x$$

for $x \in S^n \subset \mathbb{R}^{n+1}$. *It is easy to see that* $S^n/\mathbb{Z}_2 \cong \mathbb{RP}^n$. *Hence* $(S^n, \mathbb{RP}^n; \pi, \mathbb{Z}_2)$ *is a principal* $\mathbb{Z}_2$*-bundle over the real projective space* $\mathbb{RP}^n$. *Note that the fiber of this bundle is discrete and that* $\pi : S^n \to \mathbb{RP}^n$ *is locally diffeomorphic.*

4. *We know that $O(n)$ is a principal $O(n) \times O(k)$-bundle over the Grassmannian manifold $\mathbf{Gr_k}(\mathbb{R}^\mathbf{n})$. Here we give another interesting principal bundle over $\mathbf{Gr_k}(\mathbb{R}^\mathbf{n})$,* **the Stiefel manifold**.

   The Stiefel manifold of k-frames in $\mathbb{R}^n$, denoted by $F(k, \mathbb{R}^n)$, is the space of all k-frames in $\mathbb{R}^n$, which can be identified with the space of all $n \times k$ matrices of rank k. Note that there is a canonical projection $\pi : F(k, \mathbb{R}^n) \to \mathbf{Gr_k}(\mathbb{R}^\mathbf{n})$ where the k-frame generates a unique k-plane. Suppose that A is a $n \times k$ matrix of rank k, then the k-columns of A are independent k-vectors in $\mathbb{R}^n$ which generate the k-plane $\pi(A)$. From this obeservation, we can get coordinate charts for $\mathbf{Gr_k}(\mathbb{R}^\mathbf{n})$: any k-frame in $\mathbb{R}^n$ is given by a $n \times k$ matrix

$$\overrightarrow{v} = \{v_1, \cdots, v_k\} = \begin{pmatrix} v_{11} & \cdots & v_{1k} \\ \vdots & \cdots & \vdots \\ v_{n1} & \cdots & v_{nk} \end{pmatrix}$$

   with some $k \times k$-minor matrix of non-zero determinant, say

$$A = \begin{pmatrix} v_{i_1 1} & \cdots & v_{i_1 k} \\ \vdots & \cdots & \vdots \\ v_{i_k 1} & \cdots & v_{i_k k} \end{pmatrix}.$$

   Then $\overrightarrow{v} \cdot A^{-1} \in U_{i_1 \cdots i_k} \cong \mathbb{R}^{k \times (n-k)}$ where $U_{i_1 \cdots i_k}$ is the subset of the space of $n \times k$-matrix with $k \times k$ minor matrix I determined by $\{i_1, \cdots, i_k\}$-rows. It is easy to see that $\{U_{i_1 \cdots i_k}\}$ covers $\mathbf{Gr_k}(\mathbb{R}^\mathbf{n})$

   There is a natural $GL(k, \mathbb{R})$-action from the right on $F(k, \mathbb{R}^n)$. We can show that this is a free action and $\mathbf{Gr_k}(\mathbb{R}^\mathbf{n}) \cong \mathbf{F}(\mathbf{k}, \mathbf{n})/\mathbf{GL}(\mathbf{k}, \mathbb{R})$. Over each $U_{i_1 \cdots i_k}$, $\pi : F(k, \mathbb{R}^n) \to \mathbf{Gr_k}(\mathbb{R}^\mathbf{n})$ is trivial. Hence, $F(k, \mathbb{R}^n)$ is a principal $GL(k, \mathbb{R})$-bundle over $\mathbf{Gr_k}(\mathbb{R}^\mathbf{n})$.

Definition 3.3. *Two principal G-bundles with the same base are isomorphic if there is a diffeomorphism between their total space, which is G-equivariant and commutes with the projections to the base. Let $\pi : P \to M$ be a principal G-bundle. The* **gauge**

group *consists of the bundle isomorphisms between* $(P, M; \pi, G)$ *to itself which cover the identity map on the base.*

There is another description of principal bundles, which in turn also yields explicit constructions of principal bundles. This involves the transition functions which tell us how two local trivializations are glued together along their overlap.

Let $(P, M; \pi, G)$ be a principal bundle with trivialization given by

$$\psi_\alpha : P_\alpha = P|_{U_\alpha} = \pi^{-1}(U_\alpha) \to U_\alpha \times G$$

for a cover $\{U_\alpha\}$ of M. Note that ψ_α is G-equivariant where G acts on $U_\alpha \times G$ trivially on U_α and by group multiplication on G on the right. For each pair U_α and U_β with $U_\alpha \cap U_\beta \neq \emptyset$, then for each $x \in U_\alpha \cap U_\beta$, we have the following commutative diagram:

$$\begin{array}{ccc} P_\alpha|_{U_\alpha \cap U_\beta} & \xrightarrow{\;=\;} & P_\beta|_{U_\alpha \cap U_\beta} \\ \downarrow{\scriptstyle \psi_\beta} & & \downarrow{\scriptstyle \psi_\alpha} \\ (U_\alpha \cap U_\beta) \times G & \xrightarrow{\psi_\alpha \circ \psi_\beta^{-1}} & (U_\alpha \cap U_\beta) \times G \end{array}$$

which implies that $\psi_\alpha \circ \psi_\beta^{-1}(x, g) = \big(\psi_\alpha \circ \psi_\beta^{-1}(x, e)\big)g$. Define a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ determined by

$$\psi_\alpha \circ \psi_\beta^{-1}(x, e) = (x, g_{\alpha\beta}(x)).$$

Then $\{g_{\alpha\beta}\}$ are smooth maps, called the transition functions of $(P, M; \pi, G)$ associated to a trivializations $\{\psi_\alpha\}$. If $(U_\alpha \cap U_\beta \cap U_\gamma) \times G$ is non-empty, then

$$\big(\psi_\alpha \circ \psi_\beta^{-1}\big) \circ \big(\psi_\beta \circ \psi_\gamma^{-1}\big) \circ \big(\psi_\gamma \circ \psi_\alpha^{-1}\big)(x, e) = (x, e)$$

implies that the transition functions satisfy the following cocycle condition:

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = e \in G,$$

on $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. This cocycle condition also implies that $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$.

Assume that $\{\psi'_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is another trivialization, that is, ψ'_α is another G-equivariant map, with transition functions $\{g'_{\alpha\beta}\}$, then there exist smooths maps

$$h_\alpha : U_\alpha \longrightarrow G$$

such that the following diagram of G-equivaraint maps commutes

$$\begin{array}{ccccc}
\pi^{-1}(U_\alpha) & \xrightarrow{\psi_\alpha} & U_\alpha \times G & & \ni (x,g) \\
& \searrow^{\psi'_\alpha} & \downarrow h_\alpha & & \downarrow \\
& & U_\alpha \times G & & \ni (x, h_\alpha(x)g)
\end{array}$$

which implies that $\psi'_\alpha \circ \psi_\alpha^{-1}(x,g) = (x, h_\alpha(x)g)$, and $\psi_\beta \circ (\psi'_\beta)^{-1}(x,g) = (x, h_\beta^{-1}(x)g)$ by the same argument. Apply

$$\psi'_\alpha \circ (\psi'_\beta)^{-1} = \left(\psi'_\alpha \circ \psi_\alpha^{-1}\right) \circ \left(\psi_\alpha \circ \psi_\beta^{-1}\right) \circ \left(\psi_\beta \circ (\psi'_\beta)^{-1}\right)$$

to $(x,e) \in (U_\alpha \cap U_\beta) \times G$ to get

$$g'_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1}. \tag{12}$$

We say that two transition functions differ by a coboundary if they satisfy (12), and that these two transition functions are equivalent cocycles modulo a coboundary element.

If a principal G-bundle P admits a trivialization with transition function a coboundary element $\{h_\alpha h_\beta^{-1}\}$, then P is a trivial bundle $M \times G$.

Theorem 3.4. *Up to a bundle isomorphism, the principal G-bundle over a smooth manifold M is uniquely determined by the equivalence class of the transition functions.*

Proof. Given a cover $\{U_\alpha\}$ of M, and smooth transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$$

satisfying $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = e$ on $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Define

$$P = \left(\coprod U_\alpha \times G\right) / \sim_{\{g_{\alpha\beta}\}}$$

where the equivalence relation is defined as follows

$$(x,g) \in U_\alpha \times G \sim_{\{g_{\alpha\beta}\}} (x', g') \in U_\beta \times G$$

if and only if $x = x'$ and $g = g_{\alpha\beta}(x) g'$. Put the quotient topology on P. With the smooth structures on M and G, and the smoothness of transition functions, we can give a unique smooth structure on P such that the projection $\pi : P \to M$ is smooth. From the construction, we see that $(P, M; \pi, G)$ is a principal bundle.

From the above discussion, we can show that two principal bundles constructed are isomorphic if the corresponding transition functions differ by a coboundary. □

From this theorem, we can deduce that the isomorphism classes of principal G-bundles are classified by $H^1(M, G)$, the first Cech cohomology group of M with coefficients in G.

For a smooth map $f : M \to N$, we can form a principal G-bundle over M by the pull-back of a principal bundle $(P, N; \pi, G)$. The total space is the fiber product of $\pi : P \to N$ and $f : M \to N$, denoted by f^*P:

$$f^*P = \{(x, p) \in M \times P | f(x) = \pi(p)\}.$$

One can check that f^*P is a principal G-bundle over M. Suppose that $\{g_{\alpha\beta}\}$ is the transition function subordinate to a cover $\{U_\alpha\}$ of N, then $\{g_{\alpha\beta} \circ f\}$ is the transition function for the cover $\{f^{-1}(U_\alpha)\}$ of M. f^*P is called the pull-back principal bundle by f.

Remark 3.5.

1. *For a principal bundle $(P, M; \pi, G)$, G is also the* **structure group**, *by definition, the structure group is the group where the transition functions take values.*

2. *For a Lie group homomorphism $\phi : G \to H$, we can get a principal H-bundle over M from a principal G-bundle over M, in terms of transition functions, which can be described as follows. Let $\{g_{\alpha\beta}\}$ be the transition functions of (P_G, M, π, G) for a cover $\{U_\alpha\}$ of M, then $\{\phi \circ g_{\alpha\beta}\}$ is a H-valued cocycle. Using $\{\phi \circ g_{\alpha\beta}\}$ as transition functions, we obtain a principal bundle (P_H, M, π, H), where*

$$P_H = \left(\coprod U_\alpha \times H\right) / \sim_{\{\phi(g_{\alpha\beta})\}}$$

with the equivalence relation given by

$$(x, h) \in U_\alpha \times H \sim_{\{\phi(g_{\alpha\beta})\}} (x', h') \in U_\beta \times H$$

if and only if $x = x'$ and $h = \phi(g_{\alpha\beta}(x))h'$.

3.2 Associated bundles

Definition 3.6. *Suppose that* (P, M, π, G) *is a principal* G*-bundle over* M*, and a left action of* G *on a manifold* F*, the associated fiber bundle with fiber* F *is a locally trivial bundle constructed as*

$$\pi_F : \quad P \times_G F \longrightarrow M$$

where $P \times_G F = P \times F/\sim$ *with* $(p_1, v_1) \sim (p_2, v_2)$ *if and only if* $p_1 = p_2 \cdot g$*, and* $v_1 = g^{-1} \cdot v_2$*.*

Using the local trivialization of $\pi : P \to M$, we can write

$$P = \bigcup_\alpha (U_\alpha \times G)$$

where over $U_\alpha \cap U_\beta$, the transition function is given by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Then

$$P \times_G F \cong \big(\coprod_\alpha U_\alpha \times F\big)/\sim_{\{\hat{g}_{\alpha\beta}\}}$$

where $\hat{g}_{\alpha\beta}$ is the composition of $g_{\alpha\beta}$ with the left action of G on F:

$$\hat{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to G \to Diff(F),$$

here $Diff(F)$ is the diffeomorphism group of F. As locally, we can identify

$$\begin{array}{ccc} (U_\alpha \times G) \times_G F & \longrightarrow & U_\alpha \times F \\ [(x, g)_\alpha, v] & \mapsto & (x, g \cdot v) \end{array}$$

With this local trivialization for P_F, we can endow $P \times_G F$ with a smooth manifold structure such that π_F is smooth and locally trivial.

If we have a representation of G on a linear vector space V,

$$\rho : G \longrightarrow GL(V)$$

then we have an associated vector bundle $P_V = P \times_G V$ with fiber V. In particular, if $V = \mathbb{R}^n$, then we have a rank n real vector bundle $(E, M; \pi, \mathbb{R}^n)$ associated with $(P, M; \pi, G)$ and a representation $\rho : G \to GL(n, \mathbb{R})$. Using local construction,

$$E = \big(\coprod_\alpha \times F\big)/\sim_{\{\rho(g_{\alpha\beta})\}}$$

with the equivalence relation given by

$$(x, v) \in U_\alpha \times \mathbb{R}^n \sim_{\{\rho(g_{\alpha\beta})\}} (x', v') \in U_\beta \times \mathbb{R}^n$$

if and only if $x = x'$ and $v' = \rho(g_{\alpha\beta}(x))v$. From this local trivialization, we can give a unique manifold structure on E such that $\pi : E \to M$ is smooth.

Similarly, we can get an associated complex vector bundle with fiber $\mathbb{C}^n$ from a representation $\rho : G \to GL(n, \mathbb{C})$.

Example 3.7.

1. *The tangent TM over a smooth manifold M is an associated real vector bundle of the frame bundle $\mathcal{F}r(M)$ with the natural action of $GL(d, \mathbb{R})$ on $\mathbb{R}^n$. The cotangent bundle T^*M of M is an associated real vector bundle of $\mathcal{F}r(M)$ with the representation of $GL(d, \mathbb{R})$ on $\mathbb{R}^n$ given by the transpose-inverse of the natural action.*

2. *Let P be a circle bundle over M, (for example, the Hopf bundle over $\mathbb{CP}^1 \cong S^2$), using the natural embedding $U(1) \subset C^* = GL(1, \mathbb{C})$, we get an associated complex line bundle $L = P \times_{U(1)} \mathbb{C}$.*

3. *Let $(E, M; \pi, R^n)$ be a rank n vector bundle. Using the local trivialization of E for a cover $\{U_\alpha\}$ of M, we get the transition functions $\{g_{\alpha\beta}\}$ so that*

$$E \cong \left(\coprod U_\alpha \times \mathbb{R}^n\right) / \sim_{\{g_{\alpha\beta}\}} .$$

Then $P_E = E \times_{GL(n,\mathbb{R}^n)} GL(n, \mathbb{R})$ is a $GL(n, \mathbb{R})$-principal bundle over M, and E is an associated vector bundle over M of P_E with the natural action of $GL(n, \mathbb{R})$ on R^n. Taking the transpose-inverse of this natural action, we get another associated real vector bundle of rank n, this is called the dual bundle E^*, as we can identify the fiber of E^* as the dual space of the fiber of E.

Remark 3.8.

1. *For a rank n vector bundle $(E, M; \pi, \mathbb{R}^n)$, its transition functions can be applied to define a principal $GL(n, \mathbb{R})$-bundle over. This principal bundle is isomorphic to the principal bundle defined by the frames of fibers.*

2. *Using the transition functions, we can construct many news bundles from the known ones. To list just a few, we suppose that E_1 and E_2 are two vector bundles of rank n_1 and n_2 with transition functions $\{g^1_{\alpha\beta}\}$ and $\{g^2_{\alpha\beta}\}$ for a cover $\{U_\alpha\}$ of M respectively. Then*

   (a) *we can define a rank $n_1 + n_2$ real vector bundle*

   $$E_1 \oplus E_2 = \left(\coprod U_\alpha \times \mathbb{R}^{n_1+n_2}\right) / \sim_{\{g^1_{\alpha\beta} \oplus g^2_{\alpha\beta}\}}$$

   whose fiber is the direct sum of the fibers of E_1 and E_2.

   (b) *we can define a rank $n_1 n_2$ vector bundle*

   $$E_1 \otimes E_2 = \left(\coprod U_\alpha \times \mathbb{R}^{n_1 n_2}\right) / \sim_{\{g^1_{\alpha\beta} \otimes g^2_{\alpha\beta}\}} .$$

   We also have a rank $n_1 n_2$ vector bundle $E_1^* \otimes E_2$, also denoted by $Hom(E_1, E_2)$, whose fiber at x is $(E_1)^*_x \otimes (E_2)_x \cong Hom((E_1)_x, (E_2)_x)$, in particular, $End(E) = Hom(E, E) = E^* \otimes E$.

Definition 3.9. *(***Sections of a fiber bundle***) Let $(P, M; \pi, F)$ be a fiber bundle with fiber F, a smooth section of P is a smooth map*

$$s : M \longrightarrow P$$

such that $\pi \circ s = Id_M$.

Under a local trivialization of P with transition functions $\{g_{\alpha\beta}\}$, a section s can be written as local sections

$$s_\alpha = s|_{U_\alpha} : U_\alpha \to \pi^{-1}(U_\alpha) \cong U_\alpha \times F$$

we can identify the local section s_α as a function $U_\alpha \to F$, still denoted by s_α, then from the following diagram

$$\begin{array}{ccc} \pi^1(U_\alpha \cap U_\beta) & \xrightarrow{\psi_\alpha} & (U_\alpha \cap U_\beta) \times G \\ {\scriptstyle s}\uparrow\downarrow & \searrow^{\psi_\beta} & \uparrow{\scriptstyle g_{\alpha\beta}} \\ U_\alpha \cap U_\beta & & (U_\alpha \cap U_\beta) \times G, \end{array}$$

these local sections satisfy

$$s_\alpha = g_{\alpha\beta} s_\beta.$$

A section s of a fiber bundle $(P, F; \pi, F)$ can also be described as a G-equivariant map $s : P \to F$, that is,

$$s(p \cdot g) = g^{-1} \cdot s(p). \tag{13}$$

We can see that a principal bundle $(P, M; \pi, G)$ admits a section if and only if P is trivial, which means that there exist a G-equivariant diffeomorphism $P \cong M \times G$. On the other hand, the space of section on a vector bundle is infinite dimensional vector space.

Definition 3.10. *(***Gauge group***) Let $(P, M; \pi, G)$ be a principal bundle, the adjoint action of G on itself*

$$Ad : \qquad G \longrightarrow Diff(G)$$

defines an associated bundle $Aut(P) = P \times_{Ad} G$, which is a bundle of groups. The space of sections of Aut(P) is a group under the fiberwise group multiplication. This group is called the gauge group of P. For a rank n vector bundle $(E, M; \pi, \mathbb{R}^n)$, the gauge group for its principal bundle of frames $(P_E, M; \pi, GL(n, \mathbb{R}))$ is also called the gauge group of E, which is the space of sections of

$$Aut(P_E) = P_E \times_{Ad} GL(n, \mathbb{R}).$$

One can check that the gauge group of P consists of all bundle isomorphisms from $(P, M; \pi, G)$ to itself covering the identity map on M. Sometimes, we also denote $Aut(P_E)$ just by $Aut(E)$, then $Aut(E) \subset End(E)$.

Locally, an element u in the gauge group of a principal bundle consists of a family of functions $u_\alpha : U_\alpha \to G$ satisfying

$$u_\alpha = g_{\alpha\beta} u_\beta g_{\alpha\beta}^{-1}.$$

Lemma 3.11. *For a principal bundle $(P, M; \pi, G)$ with abelian group G, then $Aut(P)$ is a trivial bundle, the smooth gauge group is isomorphic to $C^\infty(M, G)$.*

Proof. It is obvious, as the transition function of $Aut(P)$ is trivial. □

3.3 Universal bundles

Now we discuss the universal bundle for real vector bundles, that means, all the real vector bundle can be obtained by the pull-back bundle from this universal bundle. The universal bundle for general principal bundle involves the concept of classifying space of Lie groups, which we will discuss very briefly at the end of this section.

Recall the Grassmannian $\mathbf{Gr_k}(\mathbb{R}^\mathbf{n})$, a homogeneous manifold of dimension $k(n-k)$, which is the set of all k-planes in $\mathbb{R}^n$. Consider the subset of $\mathbf{Gr_k}(\mathbb{R}^\mathbf{n}) \times \mathbb{R}^\mathbf{n}$ consisting of

$$\xi(k,n) = \{(V,v)|V \in \mathbf{Gr_k}(\mathbb{R}^\mathbf{n}), \mathbf{v} \in \mathbf{V}\}$$

with the natural projection $\pi : \xi(k,n) \to \mathbf{Gr_k}(\mathbb{R}^\mathbf{n})$. The fiber of π over $V \in \mathbf{Gr_k}(\mathbb{R}^\mathbf{n})$ is V itself. We can see that $\xi(k,n)$ is the associated bundle of the principal $GL(k,\mathbb{R})$-bunlde $F(k,n)$ with the natural representation of $GL(k,\mathbb{R})$ on $\mathbb{R}^k$.

We have the obvious inclusion $\mathbf{Gr_k}(\mathbb{R}^\mathbf{n}) \subset \mathbf{Gr_k}(\mathbb{R}^\mathbf{n+1})$. Let $\mathbf{Gr_k}(\mathbb{R}^\infty)$ be the union of all these Grassmannian spaces for all $n > k$, which can be viewed as the Grassmannian of k-planes in $\mathbb{R}^\infty$. Then we have a principal $GL(k,\mathbb{R})$-bundle $P(\xi_k)$ over $\mathbf{Gr_k}(\mathbb{R}^\infty)$ and an associated rank k vector bundle ξ_k.

$P(\xi_k)$ and ξ_k are the universal bundle in following sense. Suppose that E is a rank k vector bundle over a manifold M such that the local trivialization is given by

$$\psi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$$

for an locally finite open covering $\{U_\alpha\}$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to this covering. We define

$$\mu_\alpha = \rho_\alpha \cdot (\pi_2 \circ \psi_\alpha) : E \to \mathbb{R}^k_\alpha \cong \mathbb{R}^k$$

where π_2 is the obvious projection $U_\alpha \times \mathbb{R}^k \to \mathbb{R}^k$. Now

$$\mu = \oplus_\alpha \mu_\alpha : E \to \bigoplus_\alpha \mathbb{R}^k_\alpha \subset \mathbb{R}^\infty$$

which embeds each fiber of E linearly into the k-dimensional linear subspace of $\mathbb{R}^\infty$, therefore μ defines a map

$$f: \qquad M \longrightarrow \mathbf{Gr_k}(\mathbb{R}^\infty)$$

and an bundle isomorphism between E and $f^*(\xi_k)$.

Actually, pulling back the universal bundle $P(\xi_k)$ induces a bijective function from the set of homotopy classes of maps $[M, \mathbf{Gr_k}(\mathbb{R}^\infty)]$ to the set of isomorphism classes of principal $GL(k, \mathbb{R})$ bundle over M.

Definition 3.12. *A principal G-bundle $(\mathcal{P}, \mathcal{B}; \pi, G)$ is universal if*

1. *For any principal G-bundle (P, M, π, G), there exists a map $f : M \to \mathcal{B}$ such that $P \cong f^*\mathcal{P}$.*

2. *Two maps f_1 and f_2: $M \to \mathcal{B}$ induce isomorphic bundles if and only if they are homotopic.*

For a Lie group G, there is construction for the classifying space BG and a universal bundle EG over BG. EG is obtained by the so called Milnor construction (an infinite join of G):

$$\begin{aligned} & G * G * \cdots * G * \cdots \\ = \ & \{(t_0 g_0, t_1 g_1, t_2 g_2, \cdots) | t_i \in [0,1], \textstyle\sum_i t_i = 1. \text{ only finitely many non-zero } t_i.\}/\sim \end{aligned}$$

where $(t_0 g_0, t_1 g_1, t_2 g_2, \cdots) \sim (t'_0 g'_0, t'_1 g'_1, t'_2 g'_2, \cdots)$ if and only if

$$\begin{cases} t_i = t'_i & \forall i \\ t_i = t'_i \neq 0 & \Rightarrow g_i = g'_i. \end{cases}$$

The G-action on EG is given by the right group multiplication of G on each component. This action is free, and the quotient space is denoted by BG. One can give a smooth manifold structure on EG such that (EG, BG, π, G) is a principal bundle.

Proposition 3.13. *For any principal bundle $(P, M; \pi_1, G)$, there exists a smooth map $f : M \to BG$ such that $f^*EG \cong P$.*

Proof. (Sketch) The proof is constructive: to construct two maps f and F making the following diagram commute:

$$\begin{array}{ccc} P & \xrightarrow{F} & EG \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & BG. \end{array}$$

Consider a partition of unity $\{(\rho_n) : n \geq 0\}$ on M such that $P|_{\rho^{-1}\left((0,1]\right)}$ is trivialized, let $U_n = \rho^{-1}\left((0,1]\right)$, then the trivialization is given by

$$\begin{array}{ccc} P_{U_n} & \xrightarrow{\psi_n} & U_n \times G \\ & & \big\downarrow \pi_2 \\ & & G \end{array}$$

Define

$$F(p) =< \rho_0(\pi_1(p))\pi_2(\psi_0(p)), \rho_1(\pi_1(p))\pi_2(\psi_1(p)), \cdots, \rho_n(\pi_1(p))\pi_2(\psi_n(p)), \cdots > .$$

Then F is a well-defined G-equivariant map, hence induces a map $f : M \to BG$. It is not hard to check that $f^*EG \cong P$. □

To show that (EG, BG, π, G) is indeed universal, we have to establish the homotopy property for (EG, BG, π, G). We omit the details here. Also we admit without proof that EG is contractible and that any principal G-bundle with contractible total space is homotopic to EG.

Example 3.14. *1. For $G = \mathbb{Z}_2$, $EG = S^\infty$ and $BG = \mathbb{RP}^\infty$.*

2. For $G = S^1$, $EG = S^\infty = \lim_{n\to\infty} S^{2n+1}$ and $BG = \mathbb{CP}^\infty$.

3. For $G = GL(k, \mathbb{R})$, EG is the Stiefel manifold $F(k, \mathbb{R}^\infty) = \lim_{n\to\infty} F(k, \mathbb{R}^n)$ and BG is the infinite Grassmann manifold $\mathbf{Gr}_k(\mathbb{R}^\infty)$.

4. For $G = GL(k, \mathbb{C})$, EG is the complex Stiefel manifold (the space of k-complex frames in $\mathbb{C}^\infty$) $F(k, \mathbb{C}^\infty) = \lim_{n\to\infty} F(k, \mathbb{C}^n)$ and $BG = \mathbf{Gr}_k(\mathbb{C}^\infty)$.

4 Connections and Curvatures

4.1 Connections

To order to study principal bundles and their associated bundles, we need some tools to do differentials on bundles. First concept we need to establish is connection. There are three points of view to say "what is a connection?".

1. A connection is a device to computing derivatives of sections of a vector bundle.
2. A connection is a device to comparing fibers at different points by "parallel transport along a curve".
3. A connection is a device to decomposing the tangent spaces to points in the total space of a bundle into the vertical subspaces and the horizontal subspaces in an equivariant way.

4.1.1 Connections on vector bundles

Start with a trivial vector bundle $\underline{\mathbb{R}^n} = M \times \mathbb{R}^n$. Then the space of smooth sections can be identified with the space of smooth functions from M to $\mathbb{R}^n$.

$$\Omega^0(M, \underline{\mathbb{R}^n}) \cong C^\infty(M, \mathbb{R}^n).$$

We know that the usual exterior derivative defines a map:

$$\begin{array}{rccc} d: & \Omega^0(M, \underline{\mathbb{R}^n}) & \longrightarrow & \Omega^0(M, T^*M \otimes \underline{\mathbb{R}^n}) = \Omega^1(M, \underline{\mathbb{R}^n}) \\ & f & \mapsto & df \end{array}$$

satisfying

1. d is linear.
2. For any $u \in C^\infty(M, \mathbb{R})$, $d(uf) = du \otimes f + udf$, this is called the Leibniz rule.

Definition 4.1. *Let $(E, M; \pi, \mathbb{R}^n)$ be a vector bundle. Denote by $\Omega^0(M, E)$ the space of smooth sections of E. A connection on E is a map*

$$\begin{array}{rccc} D: & \Omega^0(M, E) & \longrightarrow & \Omega^0(M, T^*M \otimes E) = \Omega^1(M, E) \\ & s & \mapsto & Ds \end{array}$$

satisfying

1. *D is linear.*

2. *$D(fs) = df \otimes s + fDs$, for all $f \in C^\infty(M, \mathbb{R})$ and $s \in \Omega^0(M, E)$.*

A connection D on E is a local operator, just as the exterior derivative is a local operator. Under a local trivialization: $\psi : E|_U \cong U \times \mathbb{R}^n$, we have the corresponding local frame $\{e_i\}_{i=1}^n$ defined by the standard basis for $\mathbb{R}^n$. Over U, any section s can be written as $s = \sum_i s_i e_i$ with $s_i : U \to \mathbb{R}$ a smooth function. Then

$$Ds = \sum_i ds_i \otimes e_i + s_i(De_i)$$

where De_i is a section of $T^*U \otimes E|U$. There exist 1-forms A_{ji} defined on U such that

$$De_i = \sum_j A_{ji} \otimes e_j$$

Hence,

$$Ds = \sum_{i,j} (ds_i + A_{ij} s_j) \otimes e_i.$$

We identify $s = (s_1, \cdots, s_n)^t$ under the corresponding local frame from a local trivialization, we can write

$$D \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = (d + A) \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix},$$

where $A = (A_{ij})$ is a $n \times n$ matrix of 1-forms on U. So we get a local expression of a connection D over U: $d + A$ with A is a matrix-valued 1-form on U. On the other hand, any local operator of form $d + A$ with A is a matrix-valued 1-form on U defines a local connection on $E|_U \cong U \times \mathbb{R}^n$.

Exercise 4.2. 1. *Let $(E, M; \pi, \mathbb{R}^n)$ be a vector bundle with a local trivialization*

$$\psi_\alpha : \qquad E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^n$$

and a partition of unity $\{\rho_\alpha\}$ subordinate to the cover $\{U_\alpha\}$ on M. Choose any local connection D_α on $E|_{U_\alpha}$. Define

(a) $D_1 = \sum_\alpha \rho_\alpha D_\alpha$;

(b) $D_2 = \sum_\alpha D_\alpha \circ \rho_\alpha$.

Show that D_1 and D_2 are connections on E, and

$$D_1 - D_2 = \sum_\alpha d\rho_\alpha \otimes Id.$$

is a $End(E)$-valued 1-form on M.

2. *Fix a connection D_0 on a vector bundle E. Denote by $\mathcal{A}(E)$ the space of all connections on E, then $\mathcal{A}(E) = D_0 + \Omega^1(M, End(E))$, hence is an infinite dimensional affine space based on $\Omega^1(M, End(E))$.*

Proposition 4.3. *Let D be a connection on E with local expression $\{d + A_\alpha\}_\alpha$ under the local trivialization*

$$E = \left(\coprod U_\alpha \times \mathbb{R}^n\right) / \sim_{\{g_{\alpha\beta}\}}$$

such that A_α is $n \times n$-matrix valued 1-form on U_α. Then

$$A_\alpha = g_{\alpha\beta}^{-1} A_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} \tag{14}$$

on any non-empty $U_\alpha \cap U_\beta$.

Proof. As the local frames $\{e_i^\alpha\}_{i=1}^n$ and $\{e_i^\beta\}_{i=1}^n$ over $U_\alpha \cap U_\beta$ are related by

$$\{e_1^\alpha, \cdots, e_n^\alpha\} = \{e_1^\beta, \cdots, e_n^\beta\} \cdot g_{\alpha\beta}.$$

From this, we see that $e_i^\alpha = g_{\alpha\beta}^{ji} e_j^\beta$. Applying D and the Leibniz rule, the proof can be obtained:

$$\begin{aligned}
(A_\alpha)_{ji} \otimes e_j^\alpha &= De_i^\alpha \\
&= D(g_{\alpha\beta}^{ji} e_j^\beta) \\
&= d(g_{\alpha\beta})_{ji} \otimes e_j^\beta + (g_{\alpha\beta})_{ji} De_j^\beta \\
&= d(g_{\alpha\beta})_{ji} (g_{\beta\alpha})_{kj} \otimes e_k^\alpha + (g_{\alpha\beta})_{ji} A_{kj}^\beta \otimes e_k^\beta \\
&= d(g_{\alpha\beta})_{ji} (g_{\beta\alpha})_{kj} \otimes e_k^\alpha + (g_{\alpha\beta})_{ji} A_{kj}^\beta (g_{\beta\alpha})_{lk} \otimes e_l^\alpha \\
&= \left(g_{\alpha\beta}^{-1} A_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}\right)_{ji} \otimes e_j^\alpha.
\end{aligned}$$

□

Therefore, a connection D on a vector bundle is specified by a family of matrix-valued 1-forms $\{A_\alpha\}$ for a cover $\{U_\alpha\}$ satisfying the relation (14).

Remark 4.4. *If $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$ is locally constant, then $\{A_\alpha = 0\}$ is a solution to (14). We call a bundle with locally constant transition functions a flat bundle.*

As a principal bundle admits a section if and only if the bundle is trivial. So we need to develop the concept of connection in terms of the second and the third points of view at the beginning of this section.

Given a connection D on a vector bundle $(E, M, \pi, \mathbb{R}^n)$, D is a linear map $D : \Omega^0(M, E) \to \Omega^1(M, E)$. Hence, for a vector field X on M, we define the so called covariant derivative along the direction determined by X:

$$\begin{array}{cccc} \nabla_X : & \Omega^0(M,E) & \longrightarrow & \Omega^0(M,E) \\ & s & \mapsto & < Ds, X > \end{array}$$

which is a linear map. ∇ is called the covariant differential corresponding to D. Then ∇ satisfies

1. $\nabla_{\lambda X+Y}(s) = \lambda \nabla_X(s) + \nabla_Y(s)$, for any $s \in \Omega^0(M, E)$ and $\lambda \in \mathbb{R}$.

2. $\nabla_X(fs) = X(f)s + f\nabla_X(s)$.

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth path through $x \in M$ which gives a vector field along γ, denoted by $\dot{\gamma}$. Choose a trivialization of E over U around x: $E|_U \cong U \times \mathbb{R}^n$ so that any section of E over U can be write as

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$$

and the connection $D|_U = d + A$ for a matrix valued 1-form A on U. Then there is a unique solution to the following ordinary differential equation with initial condition:

$$\begin{cases} \dfrac{ds\big(\gamma(t)\big)}{dt} + < A\big(\gamma(t)\big) \cdot s(\gamma(t)), \dot{\gamma}(t) >= 0 \\ s(\gamma(0)) = e \in E_{\gamma(0)}. \end{cases} \tag{15}$$

The solution is a curve $\tilde{\gamma}_e$ in E through e, which is called the horizontal lift of γ. This lift defines an isomorphism, called the parallel transport along γ:

$$\begin{array}{rccc} \tau_\gamma(t): & E_x & \longrightarrow & E_{\gamma(t)} \\ & e & \mapsto & \tilde{\gamma}_e(t). \end{array} \tag{16}$$

Fix a section s and a smooth path $\gamma(t)$ through $x \in M$. As the horizontal lift of $\gamma(t)$ through $s(x)$, denoted by $\tilde{s} = \tau_\gamma(t)(s(x))$, it satisfies the equation

$$< d\tilde{s}, \dot{\gamma}(0) >= - < A\big(\gamma(0)\big) \cdot s(\gamma(0)), \dot{\gamma}(0) > .$$

Hence, we have the following lemma.

Lemma 4.5. *The covariant derivative along $X = [\gamma] \in T_x M$ is given by*

$$\nabla_X(s) = \lim_{t \to 0} \frac{\tau_\gamma^{-1}(t) \circ s(\gamma(t)) - s(x)}{t}.$$

So the parallel transport along a curve, which is uniquely determined by the connection D allows us to identify fibers at different points on this curve.

Let $e \in E_x$, the differentials of the inclusion map of the fiber E_x and the projection $\pi : E \to M$ define the following exact sequence of vector spaces:

$$0 \to \mathbb{R}^n \to T_e E \to T_x M \to 0.$$

The horizontal lift defines a map

$$\begin{array}{rccc} h_x: & T_x M & \longrightarrow & T_e E \\ & X = [\gamma] & \mapsto & X_h = [\tilde{\gamma}] \end{array} \tag{17}$$

where $\tilde{\gamma}$ is the horizontal lift of γ through $e \in E_x$. We see that $d\pi \circ h_x = Id$. Therefore we have a unique decomposition

$$T_e E = T_e E_x \oplus H_x$$

where H_x is the space of horizontal tangent vectors in $T_e E$, and $T_e E_x$ is called the vertical tangent subspace of $T_e E$.

These two points (16) and (17) of view of connections can be generalized to any fiber bundle, in particular, any principal bundle. For a principal bundle, we will require that the decomposition and the parallel transport are equivariant under the group action.

4.1.2 Connections on principal bundles

We now define a connection for a smooth principal bundle to be an infinitesimal version of an equivariant family of cross sections.

Definition 4.6. *Let $\pi : P \to M$ be a principal G-bundle over an n-dimensional manifold M. A connection P is an n-dimensional distribution $\mathcal{H}$, that is, a smooth family of n-dimensional linear subspaces of tangent bundle TP of P, which is horizontal in the sense that $d\pi$ sends each plane in $\mathcal{H}$ isomorphically the corresponding tangent space of M, and which is invariant under G-action.*

For any $p \in P$, a connection, as a equivariant horizontal lift, defines a right inverse to $d\pi$ in the following exact sequence

$$0 \longrightarrow \mathfrak{g} \xrightarrow{\upsilon_p} T_pP \underset{h_p}{\overset{d\pi_p}{\rightleftarrows}} T_{\pi(p)}M \longrightarrow 0$$

where we identify the tangent of the fiber, $T_p(p \cdot G)$, with the Lie algebra of G under the differential of the orbit map through p, which gives a inclusion map

$$\upsilon_p : \qquad \mathfrak{g} \longrightarrow T_pP.$$

Hence a connection induces an isomorphism

$$T_pP \cong \mathfrak{g} \oplus T_{\pi(p)}M.$$

Note that the tangent spaces along the fibers of P is a sub-bundle of TP, and $T_p(p \cdot G) \cong \mathfrak{g}$ is called the vertical tangent space, the image of the horizontal lift (given by the connection) is called the horizontal tangent space.

Proposition 4.7. *Give a smooth path $\gamma : [0,1] \to M$ from x_0 to x_1, a connection on $(P, M; \pi, G)$ defines an isomorphism*

$$\tau_\gamma : \qquad P_{x_0} = \pi^{-1}(x_0) \longrightarrow P_{x_1}$$

which is equivariant with the G-actions on these fibers.

Proof. Such path γ also defines a smooth vector field along γ in M. Given a initial point p_0 in P_{x_0}, a connection on P determines a unique horizontal lift of this vector field, whose integral path $\tilde{\gamma}$ for this lifted vector field with the given initial condition exists and is unique. Then we can define $\tau_\gamma(p_0) = \tilde{\gamma}(1)$. Obviously, the horizontal lift of γ beginning at $p \cdot g$ is $\tilde{\gamma} \cdot g$. □

The splitting of the exact sequence

$$0 \longrightarrow \mathfrak{g} \underset{\theta_p}{\overset{\upsilon_p}{\rightleftarrows}} T_pP \underset{h_p}{\overset{d\pi_p}{\rightleftarrows}} T_{\pi(p)}M \longrightarrow 0$$

determined by a connection is equivalent to a linear map

$$\theta_p : \qquad T_pP \longrightarrow \mathfrak{g}$$

in a G-equivariant fashion, such that θ_p is a left inverse of υ_p

$$\theta_p \circ \upsilon_p = Id.$$

In fact, we have the following proposition which can be used as an equivalent definition of a connection on P.

Proposition 4.8. *A connection on a principal G-bundle $\pi : P \to M$ is equivalent to a $\mathfrak{g}$-valued differential form $\theta \in \Omega^1(P, \mathfrak{g})$ satisfying*

1. *For any $p \in P$, $\theta_p \circ \upsilon_p = Id$*
2. *Under the right action of G, θ transforms via the adjoint representation G on $\mathfrak{g}$:*

$$R_g^*\theta = Ad(g^{-1}) \circ \theta$$

for any $g \in G$.

Proof. A connection determines a G-equivariant decomposition of T_pP into vertical and horizontal subspaces

$$T_pP = T_p^vP \oplus T_p^hP$$

where $T_p^h P = \mathcal{H}_p$ and $T_p^v P = T_p(p{\cdot}G)$. This decomposition induces the linear projection

$$\pi_p^v : \qquad T_pP \longrightarrow T_p^v P.$$

The differential of the orbit map μ_p of G-action through p defines a linear isomorphism:

$$d\mu_p(e) : \qquad T_eG \cong \mathfrak{g} \longrightarrow T_p^v P$$

Note that θ_p is the composition $(d\mu_p(e))^{-1} \circ \pi_p^v$. We only need to verify the second property of θ_p which comes from the following commutative diagram.

$$\begin{array}{ccccc} T_pP & \xrightarrow{\pi_p^v} & T_p^vP & \xleftarrow[d\mu_p(e)]{\cong} & \mathfrak{g} \\ \downarrow{\scriptstyle dR_g(p)} & & \downarrow{\scriptstyle dR_g(p)} & & \downarrow{\scriptstyle Ad_{g^{-1}}} \\ T_{pg}P & \xrightarrow[\pi_{pg}^v]{} & T_{pg}^vP & \xleftarrow[d\mu_{pg}(e)]{\cong} & \mathfrak{g}. \end{array}$$

Here we can see that the first square commutes by the equivariant condition of connection, and the second square commutes via direct calculation:

$$\begin{array}{ccc} [p\exp(tX)] & \xleftarrow[d\mu_p(e)]{} & [\exp(tX)] \\ \downarrow{\scriptstyle dR_g(p)} & & \downarrow{\scriptstyle Ad_{g^{-1}}} \\ [p\exp(tX)g] & \xleftarrow[d\mu_{pg}(e)]{} & [g^{-1}\exp(tX)g]. \end{array}$$

Conversely, the kernel of θ_p defines a G-equivariant horizontal subspace of T_pP. □

Proposition 4.9. *Any principal G-bundle $\pi : P \to M$ has a connection. The space of all connections is an affine space whose underlying vector space can be identified with $\Omega^1(M, adP)$, where adP is the associated bundle of P with the adjoint representation $Ad : G \to Aut(\mathfrak{g})$.*

Proof. First we show that any trivial principal bundle $M \times G$ admits a connection: the obvious one given by the natural horizontal distribution, namely the tangent space to the first factors, this is called the trivial connection.

Cover M by open sets $\{U_\alpha\}$ over which P is trivialized. Choose the trivial connection A_α for $P|_{U_\alpha}$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$. Define

$$A = \sum_\alpha \rho_\alpha A_\alpha.$$

Then A is a connection on P.

Let θ_1 and θ_2 be two connections on $(P, M; \pi, G)$. Then $(\theta_1 - \theta_2)_p$ defines a linear map

$$T_p P \longrightarrow \mathfrak{g}$$

such that $(\theta_1 - \theta_2)_p \circ \upsilon_p = 0$. Hence, $\omega_p = (\theta_1 - \theta_2)_p$ defines a linear map

$$T_{\pi(p)} M \longrightarrow \mathfrak{g}$$

satisfying $\omega_{p \cdot g} = Ad_{g^{-1}} \circ \omega_p$, which implies that ω defines a section of $T^*M \otimes adP$. □

In the proof, we use the fact that a G-equivariant map

$$s: \qquad P \longrightarrow F$$

defines a section of the associated bundle $P \times_G F$ on M.

Proposition 4.10. *Let $(E, M; \pi, \mathbb{R}^n)$ be an associated vector bundle of a principal G-bundle P over M for a linear representation $\rho : G \to GL(n, \mathbb{R})$. Let $\theta \in \Omega^1(P, \mathfrak{g})$ be a connection on P. Then $D = d + \theta$ defines a connection on E:*

$$\begin{array}{rrcl} D: & \Omega^0(M,E) & \longrightarrow & \Omega^1(M,E) \\ & s & \mapsto & ds + \theta \cdot s \end{array}$$

where we identify $s \in \Omega^0(M, E)$ as a G-equivariant function: $s : P \to \mathbb{R}^n$.

Proof. A section $s \in \Omega^0(M, E)$ can be viewed as a function $s : P \to \mathbb{R}^n$ satisfying

$$s(p \cdot g) = g^{-1} \cdot s(p)$$

from which we can show that $(ds + \theta \cdot s)_p$ vanishes on the vertical tangent space of P, the image of ν_p. For $X \in \mathfrak{g}$, the associated tangent vector $\nu_p(X) = \underline{X}_p$ is given by $[p \cdot exp(tX)]$. This implies that

$$ds(p)(\underline{X}_p) + \theta_p(\underline{X}_p) \cdot s(p) = -\underline{X}_p \cdot s(p) + \theta_p(\nu_p(X)) \cdot s(p) = 0.$$

On the other hand, we can prove that $Ds = ds + \theta \cdot s$ is G-equivariant:

$$\begin{aligned}(ds + \theta \cdot s)_{p \cdot g} &= g^{-1} \cdot ds(p) + (Ad_{g^{-1}} \cdot \theta) \cdot g^{-1} \cdot s(p) \\ &= g^{-1}(ds + \theta \cdot s)_p.\end{aligned}$$

Hence, $D = d + \theta$ defines a connection on E. □

Remark 4.11. *1. Suppose that $(f^*P, N; \pi, G)$ is the pull-back principal G-bundle from a principal G-bundle P over M for a smooth map $f : N \to M$. If P has a connection θ, then $f^*\theta$ defines a connection on f^*P.*

*2. (**Induced connections**)*

(a) Let D_1 and D_2 be connections on vector bundles E_1 and E_2 over M respectively, then $D^{\oplus} = D_1 \oplus D_2$ defines a connection on $E_1 \oplus E_2$, and $D^{\otimes} = D_1 \otimes I_2 + I_1 \otimes D_2$ defines a connection on $E_1 \otimes E_2$.

(b) Given a connection D on a vector bundle $\pi : E \to M$. Then

$$d < s^*, s > = < D^* s^*, s > + < s^*, Ds >$$

defines a connection D^* on E^*, where $s \in \Omega^0(M, E)$ and $s^* \in \Omega^0(M, E^*)$. With respect to a local frame $\{e_i\}$ for E, we write $D = d + A$, and with respect to the dual frame $\{e_i^*\}$ for E^*, we write $D^* = d + A^*$. Then $A^* = -A^t$.

3. In particular, a connection D on $E \to M$ induces a connection on $E \otimes E^* \cong End(E)$: locally, $D = d + A$ on E with respect to $\{e_i\}$, then the induced connection on $End(E)$, with respect to $\{e_i \otimes e_j^*\}$, is given

$$D^{End(E)} u = du + Au - uA = du + [A, u]$$

for a section u of $End(E)$.

4. *A bundle metric* $< \cdot, \cdot >$ *on a rank* n *real vector vector bundle* $\pi : E \to M$ *is a nowhere vanishing section on* $E^* \otimes E^* \cong Hom(E, E^*)$. *In terms of local frame* $\{e_i\}$ *of* E, *then a bundle metric can be written as*

$$H = \sum_{i,j} H_{ij} e_i^* \otimes e_j^*$$

where $H_{ij} =< e_i, e_j >$. *We say that the metric* $<,>$ *on* E *is compatible with a connection* D *on* E *if*

$$d < s_1, s_2 >=< Ds_1, s_2 > + < s_1, Ds_2 > .$$

Such connection is called an orthogonal connection, as the connection 1-form A, *under a local orthonormal frame* $\{e_i\}$, *satisfies*

$$A + A^t = 0,$$

hence A *is a* $Lie(O(n))$*-valued 1-form. We can check that* D *is an orthogonal connection on* $(E, <,>)$ *if and only if the induced connection* $D^{E^* \otimes E^*}$ *satisfies*

$$D^{E^* \otimes E^*}(H) = 0$$

under the local frame $\{e_i^* \otimes e_j^*\}$.

Similarly, a Hermitian metric on a rank n *complex vector bundle is compatible with a connection if and only if the connection 1-form under an orthonormal complex local frame is a* $Lie(U(n))$*-valued 1-form, such a connection is sometimes called a unitary connection.*

5. *As an example, a Riemannian metric* g *on a manifold* M *is a bundle metric on the tangent bundle* TM. *The condition for an orthogonal connection* ∇ *on* TM *is given by*

$$\nabla g = 0.$$

Then a path $\gamma(t)$ *on* M *is called geodesic if* $\gamma(t)$ *is parallel with respect to the compatible connection* ∇, *which means,*

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0,$$

where $\dot{\gamma}(t)$ *is the tangent vector at* $\gamma(t)$ *determined by* $\gamma(t)$.

4.2 Curvatures

Corresponding to three points of view for connections, we have three interpretations of curvatures.

First, recall that a connection θ on a principal G-bundle $P \to M$ defines a covariant derivative on the space of sections of the associated vector bundle $E = P \times_G V$ for a representation of $\rho : G \to GL(V)$ on a vector space V:

$$D : \qquad \Omega^0(M, E) \longrightarrow \Omega^1(M, E).$$

This covariant derivative can be extended uniquely to a linear operator for all $k > 0$:

$$D : \qquad \Omega^k(M, E) \longrightarrow \Omega^{k+1}(M, E)$$

satisfying the graded Leibniz rule:

$$D(\eta \wedge \lambda) = d\eta \wedge \lambda + (-1)^p \eta \wedge D\lambda$$

for any $\eta \in \Omega^p(M)$ and $\lambda \in \Omega^k(M, E)$. Locally, write an element of $\Omega^k(M, E)$ as $\sigma = \sum_i \omega_i \otimes s_i$ for a local k-form ω_i on M and a local section s_i of E, then define

$$D\sigma = \sum_i \big(d\omega_i \otimes s_i + (-1)^k \omega_i \wedge Ds_i\big).$$

Definition 4.12. *The curvature of a connection D on a vector bundle $E \to M$ is defined to be $F_D = D \circ D$, which is an element in $\Omega^2(M, End(E))$.*

Proposition 4.13. *F_D is a well-defined global section of $\Lambda^2 T^* M \otimes End(E)$.*

Proof. With respect to a local frame $\{e_i^\alpha\}$ for E over U_α, then D acting on a section $s = \sum_i s_i e_i^\alpha$ over U_α can be written as $Ds = \sum_i (ds_i + \sum_j A_{ij}^\alpha s_j) e_i^\alpha$. Hence,

$$\begin{aligned} D \circ D(s) &= D(\textstyle\sum_i (ds_i + \sum_j A_{ij}^\alpha s_j) e_i^\alpha) \\ &= \textstyle\sum_{i,k} (-ds_i \wedge A_{ki}^\alpha) e_k^\alpha + \sum_{i,j} d(A_{ij}^\alpha s_j) e_i^\alpha - \sum i, j, k A_{ij}^\alpha s_j \wedge A_{ki}^\alpha e_k^\alpha \\ &= \textstyle\sum_{i,j} (dA^\alpha + A^\alpha \wedge A^\alpha)_{ij} s_j e_i^\alpha \end{aligned}$$

That is, over U_α, $(D \circ D)|_{U_\alpha} = dA^\alpha + A^\alpha \wedge A^\alpha$ is a $\mathfrak{g}l(n, \mathbb{R})$-valued 2-form on U_α. Over any non-empty $U_\alpha \cap U_\beta$, we have

$$A^\alpha = g_{\alpha\beta}^{-1} A_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}.$$

Direct calculation shows that

$$dA^\alpha + A^\alpha \wedge A^\alpha = g_{\alpha\beta}^{-1}(dA^\beta + A^\beta \wedge A^\beta) g_{\alpha\beta}.$$

Hence, $F_D = \{F^\alpha = dA^\alpha + A^\alpha \wedge A^\alpha\}$ defines a global section of $\Lambda^2 T^* M \otimes End(E)$. $\square$

Therefore, the curvature of D measures the failure of

$$\Omega^0(M,E) \xrightarrow{D} \Omega^1(M,E) \xrightarrow{D} \Omega^2(M,E) \cdots \xrightarrow{D} \Omega^{n-1}(M,E) \xrightarrow{D} \Omega^n(M,E)$$

to be complex, unlike the de Rham complex.

From another point of view, we know that a connection D on a vector bundle E defines a covariant derivative ∇. For two vector fields X and Y on M, then we can establish the following identity:

$$F_D(X,Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]}. \tag{18}$$

Using local expression $F_D = dA + A \wedge A$, we have

$$dA(X,Y) = X(A(Y)) - Y(A(X)) - A([X,Y])$$

and

$$(A \wedge A)(X,Y) = A(X)A(Y) - A(Y)A(X)$$

as $\mathfrak{g}l(n, \mathbb{R})$-valued functions on M. On the other hand,

$$\begin{aligned}
\nabla_X \circ \nabla_Y s &= \nabla_X(d_Y s + A(Y)s) \\
&= d_X(d_Y s + A(Y)s) + A(X)(d_Y s + A(Y)s) \\
&= X(Y(s)) + X(A(Y))s + A(Y)X(s) + A(X)Y(s) + A(X)A(Y)s,
\end{aligned}$$

$$\begin{aligned}
\nabla_Y \circ \nabla_X s &= \nabla_Y(d_X s + A(X)s) \\
&= d_Y(d_X s + A(X)s) + A(Y)(d_X s + A(X)s) \\
&= Y(X(s)) + Y(A(X))s + A(X)Y(s) + A(Y)X(s) + A(Y)A(X)s,
\end{aligned}$$

and

$$\nabla_{[X,Y]}s = [X,Y]s + A([X,Y])s.$$

Put all these together, we get the formula (18). In particular, if $[X,Y]=0$, then F_D measures the failure of ∇_X and ∇_Y to commute.

For a principal bundle $(P,M;\pi,G)$, a connection is a $\mathfrak{g}$-valued differential form $\theta\in\Omega^1(P,\mathfrak{g})$ satisfying

$$\theta_p(\underline{X}_p) = X \in \mathfrak{g},$$

and

$$\theta_{p\cdot g} = Ad(g^{-1})\theta_p.$$

Note that the Lie bracket of $\mathfrak{g}$ induced a map on

$$\wedge: \qquad \Omega^1(P,\mathfrak{g})\otimes\Omega^1(P,\mathfrak{g})\to\Omega^2(P,\mathfrak{g})$$

satisfying $\theta\wedge\theta(X,Y) = [\theta(X),\theta(Y)]$ for two vector fields X and Y.

Proposition 4.14. *Let $\theta\in\Omega^1(P,\mathfrak{g})$ be a connection on a principal bundle $(P,M;\pi,G)$. Then $d\theta+\theta\wedge\theta$ defines $F_\theta\in\Omega^2(M,adP)$ such that*

$$\pi^*(F_\theta) = d\theta+\theta\wedge\theta.$$

This 2-form is called the curvature of the connection θ, denoted by F_θ.

Proof. It is easy to see that $d\theta+\theta\wedge\theta$ is G-equivaraint. In order to complete the proof, it is enough to show that for any $p\in P$ and for any $X,Y\in T_pP$ with X vertical, we have

$$(d\theta+\theta\wedge\theta)(X,Y) = 0. \tag{19}$$

Extend X to a vertical vector field $\underline{X}$ on P associated to $X\in\mathfrak{g}$, then $\theta(\underline{X}) = X$ is constant. Using the connection θ, we can decompose Y into a vertical tangent vector and a horizontal tangent vector. So we can verify (19) for Y vertical and horizontal respectively. For Y vertical, extend Y to a vertical vector field $\underline{Y}$ on P associated to $Y\in\mathfrak{g}$, then

$$d\theta(\underline{X},\underline{Y}) = -\theta([\underline{X},\underline{Y}]) = -\theta(\underline{[X,Y]}) = -[X,Y],$$

and

$$\theta \wedge \theta(\underline{X}, \underline{Y}) = [\theta(\underline{X}), \theta(\underline{Y})] = [X, Y],$$

which imply (19) for Y vertical.

For Y is horizontal, we can also extend Y to a G-equivariant horizontal vector field on P, then $\theta(Y) = 0$ and $L_X(Y) = [X, Y] = 0$, hence

$$(d\theta + \theta \wedge \theta)(X, Y) = -\theta([X, Y]) = 0.$$

□

Remark 4.15. *Let θ be a connection on a principal G-bundle $\pi : P \to M$. Let s_1 be a local section of P over an open set U in M (so P is trivialized over U). Then $s^*\theta \in \Omega^1(U, \mathfrak{g})$ is called the connection 1-form for (U, s_1), and under the trivialization over U given by s_1:*

$$F_{\theta, s_1} = d(s_1^*\theta) + (s_1^*\theta \wedge s_1^*\theta).$$

For another trivialization s_2 over U with $s_1 = s_2 \cdot g_{12}$, we obtain F_{θ, s_2} and

$$F_{\theta, s_1} = g_{12}^{-1} F_{\theta, s_2} g_{12}.$$

Hence, the curvature F_θ is given by a family of $\mathfrak{g}$-valued 2-form on M, $\{F_\theta^\alpha\}$, for a cover $\{U_\alpha\}$ over which P is trivialized by $\{\psi_\alpha\}$, satisfying

$$F_\theta^\alpha = g_{\alpha\beta}^{-1} F_\theta^\beta g_{\alpha\beta},$$

where $\{g_{\alpha\beta}\}$ are the transition functions of P under the trivialization $\{U_\alpha, \psi_\alpha\}$. This gives another proof that $F_\theta \in \Omega^2(M, adP)$.

Fix a connection θ on a principal bundle $(P, M; \pi, G)$. Then we have a lift map on the space of vector fields on M to the space of horizontal vector fields on P. For any vector field X on M, we denote by $\tilde{X}$ the unique horizontal lift of X. Then the curvature of the connection θ evaluates on a pair of vector fields X and Y on M to give

$$F_\theta(X, Y) = \widetilde{[X, Y]} - [\tilde{X}, \tilde{Y}]. \tag{20}$$

Definition 4.16. *We call a connection θ flat if $F_\theta = 0$.*

This provides the third interpretation of the curvature, that is, the obstruction to integrating the horizontal distribution of the connection over two dimensional submanifolds of the base, this relies on Frobenius's Theorem. We should not elaborate here, but only mention the holonomy of a connection, and to illustrate that the curvature of a connection essentially measure the non-triviality of holonomy along a contractible loop on the base.

4.2.1 The holonomy of a connection

Let θ be a connection on a principal G-bundle $\pi : P \to M$. The holonomy of θ at a point $p \in P$ is a map from the loop space $\Omega(M, \pi(p))$ of M based at $\pi(p)$ to G:

$$h_{\theta,p} : \qquad \Omega(M, \pi(p)) \longrightarrow G.$$

For a loop $\gamma : [0,1] \to M$ with $\pi(p) = \gamma(0) = \gamma(1)$, $h_{\theta,p}(\gamma)$ is determined by taking the horizontal life $\tilde{\gamma}$ starting from p and ending at

$$p \cdot h_{\theta,p}(\gamma).$$

Since the horizontal lift is G-equivariant, we have the following consequences: the holonomy map satisfies:

1. $h_{\theta,p\cdot g} = g^{-1} h_{\theta,p} g$.
2. $h_{\theta,p}(\gamma^{-1}) = h_{\theta,p}(\gamma)^{-1}$.
3. $h_{\theta,p}(\gamma_1\gamma_2) = h_{\theta,p}(\gamma_2) h_{\theta,p}(\gamma_1)$, here $\gamma_1\gamma_2$ denotes the composition of two loops γ_1 and γ_2.

These imply that the image of the holonomy map is a subgroup of G and independent of reference point up to conjugations.

In fact, we have the following holonomy theorem.

Theorem 4.17. *Let $(P, M; \pi, G)$ be a principal bundle with connected manifold M and a connection θ. Let $Hol_p(\theta)$ be the subgroup of G obtained by the image of the honolomy map $h_{\theta,p}$. Then the Lie algebra of $Hol_p(\theta)$ is spanned by the elements of the form*

$$\pi_p^*(F_\theta)(X, Y)$$

for two arbitrary horizontal tangent vectors X and Y at p. In particular, a connection θ is flat if and only if the holonomy map $h_{\theta,p}$ factors through the map $\Omega(M, \pi(p)) \to \pi_1(M, \pi(p))$ by taking homotopy classes:

$$\begin{array}{ccc} \Omega(M, \pi(p)) & \xrightarrow{h_{\theta,p}} & G \\ \downarrow & \nearrow & \\ \pi_1(M, \pi(p)) & & \end{array}$$

Here $\pi_1(M, \pi(p))$ is the fundamental group of M, the homotopy classes of loops based at $\pi(p)$.

Proof. We only give a sketch of the proof here. A detailed proof can be found in *Foundations of Differential Geometry, Vol I, by Kobayashi and Nomizu, page 89.* For any smooth contractible loop γ based at $\pi(p)$, take a smooth retraction γ_t of γ to $\pi(p)$, then $h_{\theta,p}$ provide a smooth path $h_{\theta,p}(\gamma_t)$ in $Hol_p(\theta)$ joining $h_{\theta,p}(\gamma)$ with the identity. Let $\mathfrak{h}_{\theta,p}$ be the set of vectors in $\mathfrak{g}$ tangent to smooth curves in $Hol_p(\theta)$, a theorem of Freudenthal implies that $\mathfrak{h}_{\theta,p}$ is a Lie algebra of $Hol_p(\theta)$.

For any pair of horizontal vector fields X and Y,

$$\pi_p^*(F_\theta)(X, Y) = -\theta([X, Y]).$$

Note that $d\pi_p(X)$ and $d\pi_p(Y)$ define a small rectangle $[0, \epsilon] \times [0, \epsilon]$ in $T_{\pi(p)}M$, the exponential map maps the four sides of this rectangle to a piecewise smooth loop $\gamma_{XY} \in \Omega(M, \pi(p))$, then a local calculation shows that

$$\pi_p^*(F_\theta)(X, Y) = -\lim_{\epsilon \to 0} \frac{log(h_{\theta,p}(\gamma_{XY})}{\epsilon^2} \in \mathfrak{h}_{\theta,p}.$$

□

5 Characteristic Classes

Just as the curvature to measure the obstruction to integrating horizontal distribution of a connection over a two dimensional submanifolds of the base, we can use the curvature to define certain cohomology classes in M to measure the non-triviality of the bundle. These cohomology classes are called the characteristic classes, in the sense that isomorphic bundles have same characteristic classes.

The first key identity we need is the so-called Bianchi identity.

Lemma 5.1. *(**Bianchi Identity**) Let θ be a connection on a principal G-bundle $\pi : P \to M$. The curvature of θ, F_θ is an element of $\Omega^2(M, adP)$. Under the covariant derivative D_θ induced by θ, we have,*

$$D_\theta F_\theta = dF_\theta + [\theta, F_\theta] = 0.$$

Proof. Under the adjoint representation $ad : \mathfrak{g} \to \mathfrak{g}l(\mathfrak{g})$, F_θ is a matrix of 2-form on each open set where P is trivialized. Suppose that the trivialization over U is given by a local section s, then $s^*\theta$ is a $\mathfrak{g}$-valued 1-form on U, still denoted by θ for convenience. On U, we have

$$F_\theta = d\theta + \theta \wedge \theta.$$

We can compute directly

$$\begin{aligned} dF_\theta &= d\theta \wedge \theta - \theta \wedge d\theta \\ &= (F_\theta - \theta \wedge \theta) \wedge \theta - \theta \wedge (F_\theta - \theta \wedge \theta) \\ &= F_\theta \wedge \theta - \theta \wedge F_\theta \\ &= [F_\theta, \theta]. \end{aligned}$$

Hence, $D_\theta F_\theta = 0$. □

For a connection D on a vector bundle $(E, M; \pi, \mathbb{R}^n)$, the curvature F_D is an element of $\Omega^2(M, End(E))$. Then the Bianchi identity reads as

$$D(F_D) = 0.$$

Here $D = D^{End(E)}$ acts on $\Omega^2(M, End(E))$ as the covariant derivative on $End(E)$ induced by D. Locally, $D = d + A$ and $F_D = dA + A \wedge A$ over U where E is trivialized by a local frame, then the Bianchi identity reads as

$$dF_D + [A, F_D] = 0.$$

Given a linear map

$$\Phi : \qquad \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{k} \longrightarrow \mathbb{R}$$

which is symmetric and invariant under the simultaneous adjoint action of G on $\mathfrak{g}$, we can form

$$\Phi(F_\theta, \cdots, F_\theta) \in \Omega^{2k}(M, \mathbb{R}).$$

Now we come to the main object of this course: the Chern-Weil theory, which claims that $\Phi(F_\theta, \cdots, F_\theta)$ is closed, and independent of the choice of connections. $\Phi(F_\theta, \cdots, F_\theta)$ is the characteristic class corresponding to Φ.

5.1 The Chern-Weil homomorphism

Let $\mathcal{I}^k(\mathfrak{g}^*)$ be the G-invariant (under the simultaneous conjugation action of G on $\mathfrak{g}$) part of the k-multilinear symmetric functions on $\mathfrak{g}$. There is a product

$$\mathcal{I}^k(\mathfrak{g}^*) \otimes \mathcal{I}^l(\mathfrak{g}^*) \longrightarrow \mathcal{I}^{k+l}(\mathfrak{g}^*)$$

defined by

$$\Phi * \Psi(v_1, \cdots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \Phi(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \Psi(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}).$$

Let $\mathcal{I}^*(\mathfrak{g}^*) = \oplus_{k \geq 0} \mathcal{I}^k(\mathfrak{g}^*)$ ($\mathcal{I}^0(\mathfrak{g}^*) = \mathbb{R}$). Notice that any homogeneous $\Phi \in \mathcal{I}^*(\mathfrak{g}^*)$ is determined by the polynomial function on $\mathfrak{g}$ given by

$$v \mapsto \Phi(v, \cdots, v),$$

whose inverse is called polarization. Hence, we can call $\mathcal{I}^*(\mathfrak{g}^*)$ the algebra of invariant polynomials on $\mathfrak{g}$.

Consider a principal bundle $(P, M; \pi, G)$ with a connection θ, and its curvature $F_\theta \in \Omega^2(M, adP)$. Suppose that P is defined by transition functions

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$$

over an open cover $\{U_\alpha\}$ of M, then the connection θ is defined by a collection $\{\theta_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}\}$ and its curvature is given by a collection

$$\{F_\alpha = d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha\}$$

satisfying

$$F_\alpha = g_{\alpha\beta}^{-1} F_\beta g_{\alpha\beta}, \quad dF_\alpha + [\theta_\alpha, F_\alpha] = 0.$$

Then for any $k \geq 0$ and $\Phi \in \mathcal{I}^k(\mathfrak{g}^*)$, we have a well-defined $2k$-form on M

$$\Phi(F_\theta) = \Phi(F_\theta, \cdots, F_\theta).$$

This is because that Φ is Ad-invariant and $F_\alpha = g_{\alpha\beta}^{-1} F_\beta g_{\alpha\beta}$. The locally defined forms $\{\Phi(F_\alpha)\}$ can be glued together to get a global $2k$-form on M, denoted by $\Phi(F_\theta)$.

Theorem 5.2. *The form $\Phi(F_\theta)$ is closed, and its cohomology class does not depend on the choice of connection and in particular depends only on the isomorphic class of P. This cohomology class is denoted by $cw_P(\Phi)$. The map (called the Chern-Weil homomorphism)*

$$cw_P : \qquad \mathcal{I}^*(\mathfrak{g}^*) \longrightarrow H^*(M, \mathbb{R})$$

is an algebra homomorphism and behaves naturally under the bundle pull-back, that is, for a smooth map $f : N \to M$, we have

$$cw_{f^*P} = f^* \circ cw_P.$$

For $\Phi \in \mathcal{I}^*(\mathfrak{g}^*)$, $cw_P(\Phi)$ is called the characteristic class of P corresponding to Φ.

Proof. Suppose that Φ is a homogeneous polynomial of degree k. As Φ is symmetric and Ad-invariant, hence, we have

$$\Phi(Ad_{exp(tX)}Y, \cdots, Ad_{exp(tX)}Y) = \Phi(Y, \cdots, Y)$$

by differentiation, which implies that

$$\Phi([X,Y],Y,\cdots,Y)=0$$

for any $X,Y\in\mathfrak{g}$. Now from Bianchi identity, we get

$$\begin{aligned} d\Phi(F_\theta) &= \Phi(dF_\theta,F_\theta,\cdots,F_\theta) \\ &= -k\Phi([\theta,F_\theta],F_\theta,\cdots,F_\theta)=0. \end{aligned}$$

Hence, $\Phi(F_\theta)$ is a closed $2k$-form on M.

Suppose that θ_0 and θ_1 are two connections on P, note that $\theta_i\in\Omega^1(P,\mathfrak{g})$, then

$$\tilde{\theta}_{(p,t)}=(1-t)\theta_0(p)+t\theta_1(p), \qquad for \;\; (p,t)\in P\times[0,1]$$

defines a connection on the principal G-bundle $P\times[0,1]\to M\times[0,1]$. The curvature of $\tilde{\theta}$ is denoted by $\tilde{F}$. Therefore, $\Phi(\tilde{F})$ defines a closed $2k$-form on $M\times[0,1]$. Notice that the de Rham differential d on $M\times[0,1]$ acting on $\Phi(\tilde{F})$ is given by

$$0=d(\Phi(\tilde{F}))=d_M\Phi(\tilde{F})+\frac{\partial}{\partial t}\Phi(\tilde{F})\wedge dt,$$

here d_M is the de Rham differential on M.

The map $e_i: x\to(x,i)$ defines an embedding of M as the boundary of $M\times[0,1]$, then

$$e_0^*\tilde{F}=F_{\theta_0}, \;\; and \;\; e_1^*\tilde{F}=F_{\theta_1}.$$

Now integrating $d(\Phi(\tilde{F}))=0$ over $[0,1]$, we get

$$\begin{aligned} -d_M\int_{t=0}^1\Phi(\tilde{F}) &= \int_{t=0}^1\frac{\partial}{\partial t}\Phi(\tilde{F})\wedge dt \\ &= e_1^*(\Phi(\tilde{F}))-e_0^*(\Phi(\tilde{F})) \\ &= \Phi(F_{\theta_1})-\Phi(F_{\theta_0}). \end{aligned}$$

Hence, $\Phi(F_{\theta_1})$ is cohomologous to $\Phi(F_{\theta_0})$.

It is straight forward to check that for $\Phi\in\mathcal{I}^k(\mathfrak{g}^*)$ and $\Psi\in\mathcal{I}^l(\mathfrak{g}^*)$,

$$(\Phi*\Psi)(F_\theta)=\Phi(F_\theta)\wedge\Psi(F_\theta),$$

which implies that cw_P is an algebra homomorphism.

The naturality of cw_P follows from that if θ is a connection on P with curvature form $F_\theta \in \Omega^2(M, adP)$, then $f^*\theta$ is a connection on f^*P with curvature form $f^*F_\theta \in \Omega^2(N, f^*adP)$, then

$$f^*\Phi(F_\theta) = \Phi(f^*F_\theta) = \Phi(F_{f^*\theta}).$$

□

Similarly, for the algebra of complex valued G-invariant polynomials on $\mathfrak{g}$, we have a complex Chern-Weil homomorphism

$$\mathcal{I}^*_{\mathbb{C}}(\mathfrak{g}^*) \longrightarrow H^*(M, \mathbb{C}).$$

For some classical Lie groups, we have some standard invariant polynomials. With suitable normalizations, these characteristic classes are respectively the Pontrjagin classes, the Euler class, the Chern classes. These characteristic classes are integral classes, which follows from the universal characteristic classes on classifying spaces.

5.2 The Chern classes

Let $(E, M; \pi, \mathbb{C}^n)$ be a complex vector bundle with a connection A and its curvature form

$$F_A \in \Omega^2(M, End(E)).$$

The corresponding principal bundle has structure group $GL(n, \mathbb{C})$ with a connection and its curvature form. Locally, F_A is a $\mathfrak{g}l(n, \mathbb{C})$-valued 2-form on M.

Suppose that $A \in \mathfrak{g}l(n, \mathbb{C})$ has eigenvalues $\lambda_1, \cdots, \lambda_n$, then

$$\begin{aligned} det(I + A) &= det(\begin{pmatrix} 1+\lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & 1+\lambda_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 1+\lambda_n \end{pmatrix} \\ &= \prod_{i=1}^n (1+\lambda_i) \\ &= 1 + \sigma_1 + \cdots + \sigma. \end{aligned}$$

Here

$$
\begin{aligned}
&\sigma_0(\lambda_1, \cdots, \lambda_n) = 1 \\
&\sigma_1(\lambda_1, \cdots, \lambda_n) = \textstyle\sum_{i=1}^n \lambda_i \\
&\sigma_2(\lambda_1, \cdots, \lambda_n) = \textstyle\sum_{i \neq j} \lambda_i \lambda_j \\
&\vdots \\
&\sigma_0(\lambda_1, \cdots, \lambda_n) = \lambda_1 \lambda_2 \cdots \lambda_n.
\end{aligned} \tag{21}
$$

Lemma 5.3. *Any Ad-invariant polynomial Φ on $\mathfrak{g}l(n, \mathbb{C})$ can be expressed as a polynomial function of $\sigma_1, \cdots, \sigma_n$.*

Proof. For any $A \in \mathfrak{g}l(n, \mathbb{C})$, we can choose B so that the conjugation by B, BAB^{-1} is a Jordan canonical form (an upper triangular matrix). Replacing B by $diag(\epsilon, \epsilon^2, \cdots, \epsilon^n)B$, let $\epsilon \to 0$, we can make the off diagonal entries approach to zero. By the continuity of Φ, $\Phi(A)$ depends only on the diagonal entries of BAB^{-1}, that is, the eigenvalues of A. As a symmetric functions of these eigenvalues, Φ must be a polynomial functions of $\sigma_1, \cdots, \sigma_n$. □

Therefore, the algebra of *Ad*-invariant polynomials on $\mathfrak{g}l(n, \mathbb{C})$ is generated by the elementary symmetric functions

$$\{\sigma_0(\lambda_1, \cdots, \lambda_n), \cdots, \sigma_n(\lambda_1, \cdots, \lambda_n)\}$$

where $\{\lambda_1, \cdots, \lambda_n\}$ are considered as the eigenvalue functions of a matrix. With suitable normalization, we consider the complex valued *Ad*-adjoint invariant polynomials C_k which are coefficients to λ^{n-k} in

$$det(\lambda \cdot I - \frac{A}{2\pi i}) = \sum_k C_k(A, \cdots, A)\lambda^{n-k}$$

where $i = \sqrt{-1}$ and $A \in \mathfrak{g}l(n, \mathbb{C})$. The characteristic classes corresponding to $C_k \in \mathcal{I}^k(\mathfrak{g}l(n, \mathbb{C})^*)$ is called **the k-th Chern class** of the principal $GL(n, \mathbb{C})$-bundle. The k-th Chern class lies in $H^{2k}(M, \mathbb{C})$.

In particular, for a complex vector bundle $(E, M; \pi, \mathbb{C}^n)$ with a connection A and curvature form F_A, the k-th Chern class of E has a differential representative given by the coefficient to λ^{n-k} in

$$det(\lambda \cdot I - \frac{F_A}{2\pi i}),$$

which is exactly $\sigma_k(\frac{iF_A}{2\pi})$.

Denote by $c_k(E) \in H^{2k}(M, \mathbb{C})$ the k-th Chern class of E, note that $c_0(E) = 1$ and $c_k(E) = 0$ for $k > n$. Then we have the following theorem, which characterizes the Chern classes uniquely.

Theorem 5.4. *For a complex vector bundle $\pi : E \to M$ of rank n. Define* **the total Chern class** *to be the sum*

$$\begin{aligned} c(E) &= c_0(E) + c_1(E) + \cdots + c_n(E) \\ &= det(I - \frac{F_A}{2\pi i}). \end{aligned}$$

Then

1. $c_k(E) \in H^{2k}(M, \mathbb{C})$ *and* $c_0(E) = 1$.

2. *If $f : N \to M$ is a smooth map, then*

$$c(f^*E) = f^*(c(E)).$$

If $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are two complex vector bundle of rank n_1 and n_2 respectively. Then

$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2).$$

3. *The canonical line bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$ on $\mathbb{CP}^n$, coming from the natural $\mathbb{C}^*$-bundle $\mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$ and the natural action of $\mathbb{C}^* = GL(1, \mathbb{C})$ on $\mathbb{C}$. Then*

$$c(\mathcal{O}_{\mathbb{CP}^n}(-1)) = 1 - h_n$$

where h_n is the canonical generator of $H^2(\mathbb{CP}^n, \mathbb{Z})$, that is, the restriction map $H^2(\mathbb{CP}^n, \mathbb{Z}) \to H^2(\mathbb{CP}^1, \mathbb{Z})$ maps h_n to h_1, and $< h_1, [\mathbb{CP}^1] >= 1$ for $\mathbb{CP}^1$ with the canonical orientation.

Proof. The naturality follows from the definition of characteristic classes and the Chern-Weil homomorphism. The total Chern class is multiplicative under direct sum, this follows from the identity

$$det(I - \frac{F_{A_1} \oplus F_{A_2}}{2\pi i}) = det(I_{n_1 \times n_1} - \frac{F_{A_1}}{2\pi i}) \cdot det(I_{n_2 \times n_2} - \frac{F_{A_2}}{2\pi i}).$$

To show that $c_1(\mathcal{O}_{\mathbb{CP}^n}(-1)) = -h_n$, by the naturality, we only need to show that

$$< c_1(\mathcal{O}_{\mathbb{CP}^1}(-1)), [\mathbb{CP}^1] >= -1.$$

So we consider the principal $\mathbb{C}^*$-bundle

$$\pi : \qquad \mathbb{C}^2 - \{0\} \longrightarrow \mathbb{CP}^1.$$

Let (z_0, z_1) be the coordinate on $\mathbb{C}^2 - \{0\}$. Then the complex valued 1-form

$$\theta = \frac{\overline{z_0} dz_0 + \overline{z_1} dz_1}{|z_0|^2 + |z_1|^2},$$

where the bar denotes the complex conjugation and $|z|^2 = \overline{z} z$, is a connection and the curvature form is determined by $d\theta$.

Let $U_1 = \mathbb{CP}^1 - \{[0, 1]$ with its local coordinate $z = z_1/z_0$. Then we can calculate that on U_1

$$d\theta = \frac{d\overline{z} \wedge dz}{(1 + |z|^2)^2}.$$

Then on U_1, $c_1(\mathcal{O}_{\mathbb{CP}^1}(-1)$ is represented by

$$-\frac{1}{2\pi i} \cdot \frac{d\overline{z} \wedge dz}{(1 + |z|^2)^2}.$$

Therefore,

$$< c_1(\mathcal{O}_{\mathbb{CP}^1}(-1)), [\mathbb{CP}^1] >= -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{d\overline{z} \wedge dz}{(1 + |z|^2)^2}.$$

Using the polar coordinate (r, t) on $\mathbb{C}$, $z = r \cdot e^{2\pi i t}$, then

$$< c_1(\mathcal{O}_{\mathbb{CP}^1}(-1)), [\mathbb{CP}^1] >= -\int_0^\infty \frac{dr}{(1 + r)^2} = -1.$$

□

If a complex vector bundle has a hermitian metric which is compatible to a connection A, then the curvature form F_A is a $Lie(U(n))$-valued 2-from, that is, F_A is skew-hermitian ($F_A = -\overline{F_A}^t$). Therefore,

$$det(I - \frac{F_A}{2\pi i}) = \overline{det(I - \frac{F_A}{2\pi i})}.$$

Hence, any Hermitian vector bundle has its Chern classes in real cohomology groups of M.

The connection A on a complex vector bundle induces a connection $\overline{A}$ on its conjugate bundle $\overline{E}$, then $F_{\overline{A}} = \overline{F_A}$. Hence the Chern classes of the conjugate bundle $\overline{E}$ of E is given by

$$c_k(\overline{E}) = (-1)^k c_k(E). \tag{22}$$

Similar argument shows that the Chern classes of the dual bundle E^* is given by

$$c_k(E^*) = (-1)^k c_k(E). \tag{23}$$

Example 5.5. *The total Chern class of the tangent bundle to* $\mathbb{CP}^n$ *is equal to* $(1+h_n)^n$.

Proof. Let $L = \mathcal{O}(-1)$ be the canonical line bundle over $\mathbb{CP}^n$. Using the standard hermitian metric on trivial complex vector bundle $\underline{\mathbb{C}^{n+1}}$ on $\mathbb{CP}^n$, we get

$$\underline{\mathbb{C}}^{n+1} = L \oplus L^{\perp}$$

where $L^{\perp}$ is the orthogonal complement of L. If l is a complex line through origin in $\mathbb{C}^{n+1}$, and $l^{\perp}$ is its orthogonal complement, then there is a local diffeomorphism between $Hom(l, l^{\perp})$ and the neighborhood of $l \in \mathbb{CP}^n$, via graphs of linear maps from l to $l^{\perp}$. This implies that

$$T(\mathbb{CP}^n) \cong Hom(L, L^{\perp}).$$

Hence, we have

$$\begin{aligned} T(\mathbb{CP}^n) \oplus \underline{\mathbb{C}} &\cong Hom(L, L^{\perp} \oplus L) \\ &\cong Hom(L, \underline{\mathbb{C}}^{n+1}) \\ &\cong \underbrace{L^* \oplus \cdots \oplus L^*}_{n+1}. \end{aligned}$$

Thus the total Chern class of $T(\mathbb{CP}^n)$ is given by

$$c(\underbrace{L^* \oplus \cdots \oplus L^*}_{n+1}) = c(L^*)^{n+1} = (1 + h_n)^{n+1}.$$

□

Remark 5.6. *We can apply the Chern-Weil homomorphism to a Ad-invariant formal power series to define a characteristic class, for example,*

$$Tr(exp(\frac{iA}{2\pi})).$$

Then the image of $Tr(exp(\frac{iA}{2\pi}))$ under the Chern-Weil homomorphism, for a complex vector bundle E, is still well-defined, as $H^k(M, \mathbb{C}) = 0$ for $k > dimM$. This is called the Chern character, denoted by $ch(E)$. Then the Chern character satisfies the following nice formulae, for two complex vecotr bundles E_1 and E_2 over M, then,

1. $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$.

2. $ch(E_1 \otimes E_2) = ch(E_1) \cdot ch(E_2)$.

The proof is left to readers.

5.3 The Pontrjagin Classes

For a real vector bundle E or a principal $GL(n, \mathbb{R})$-bundle P, we can also define its characteristic classes, which is called the Pontrjagin classes.

Notice that every principal $GL(n, \mathbb{R})$-bundle can be reduced to a principal $O(n)$-bundle, as the inclusion $O(n) \subset GL(n, \mathbb{R})$ is a homotopy equivalence. In other word, we can fix a bundle metric on E (a smoothly varying fiberwise metric), and take the orthogonal frame bundle and its orthogonal connection θ, then F_θ is a $Lie(O(n))$-valued 2-form.

For $GL(n, \mathbb{R})$, the Lie algebra $\mathfrak{g}l(n, \mathbb{R})$ is the Lie algebra of all matrices with Lie bracket $[A, B] = AB - BA$. The Lie algebra of $O(n)$, $\mathfrak{o}(n)$, is a Lie subalgebra of

$\mathfrak{g}l(n,\mathbb{R})$, consisting of all skew-symmetric matrices. For any $A \in \mathfrak{o}(n)$, A is conjugate (in $O(n)$) to, if $n = 2k$,

$$\begin{pmatrix} 0 & \lambda_1 & \cdots & 0 & 0 \\ -\lambda_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_k \\ 0 & 0 & \cdots & -\lambda_k & 0 \end{pmatrix},$$

or if $n = 2k+1$,

$$\begin{pmatrix} 0 & \lambda_1 & \cdots & 0 & 0 & 0 \\ -\lambda_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & \lambda_k & 0 \\ 0 & 0 & \cdots & -\lambda_k & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Some elementary algebraic calculation leads to the following lemma.

Lemma 5.7. $\mathcal{I}^*(\mathfrak{o}(n)^*) \sqsupseteq \mathbb{R}[\sigma_1(\lambda_1^2,\cdots,\lambda_{[n/2]}^2),\cdots,\sigma_{[n/2]}(\lambda_1^2,\cdots,\lambda_{[n/2]}^2)]$.

Consider the real-valued invariant polynomial $P_{\frac{k}{2}}$, $k = 0,\cdots,n$, defined by

$$det(\lambda I - \frac{A}{2\pi}) = \sum_k P_{\frac{k}{2}}(A,\cdots,A)\lambda^{n-k}, \text{ for } A \in \mathfrak{o}l(n,\mathbb{R}).$$

$P_{\frac{k}{2}}(A)$ is called the $k/2$-th Pontrjagin polynomial, whose image under the Chern-Weil homomorphism, for a principal $GL(n,\mathbb{R})$-bundle, is called the $k/2$-th Pontrjagin class. For any $A \in \mathfrak{o}(n)$, we have

$$det(\lambda I - \frac{A}{2\pi}) = det(\lambda I + \frac{A}{2\pi}),$$

from which we get the restriction of $P_{\frac{k}{2}}$ to $\mathfrak{o}(n)$, for k odd, is zero. One can check that

$$P_{\frac{k}{2}}(A) = \sigma_k(\lambda_1^2(A/2\pi),\cdots,\lambda_{[n/2]}^2(A/2\pi)).$$

In particular, for a real vector bundle E of rank n, the $k/2$-th Pontrjagin class of E is defined by

$$p_{\frac{k}{2}}(E) = cw(P_{\frac{k}{2}}).$$

By the Chern-Weil theory, we know that the $k/2$-th Pontrjagin class of E is represented by the $2k$-form $P_{\frac{k}{2}}(F_A)$ for a connection A on E.

Define the total Pontrjagin class of a real vector bundle E over M to be

$$p(E) = 1 + p_1(E) + \cdots + p_{[n/2]} \in H^*(M, \mathbb{R}).$$

Then we have the naturality for the Pontrjagin classes similar to the naturality for the Chern classes.

Comparing the definition of the Chern classes with the definition of the Pontrjagin classes,

$$p(E) = [det(I + \frac{F_A}{2\pi})], \quad c(E \otimes_{\mathbb{R}} \mathbb{C}) = [det(I + \frac{iF_A}{2\pi}],$$

we get

$$p_k(E) = (-1)^k c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}), \tag{24}$$

where $E \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of E under the inclusion map $GL(n, \mathbb{R}) \subset GL(n, \mathbb{C})$. This relation also follows from the identity for symmetric polynomials

$$\sigma_k(\lambda_1^2, \cdots, \lambda_m^2) = (-1)^k \sigma_{2k}(\lambda_1, -\lambda_1, \cdots, \lambda_m, -\lambda_m).$$

Note that, as the transition function takes value in $GL(n, \mathbb{R}) \subset GL(n, \mathbb{C})$, we have

$$E_{\mathbb{C}} \cong \bar{E}_{\mathbb{C}},$$

which implies that $c_k(E_{\mathbb{C}}) = 0$ for k odd. Hence, we can also take (24) as another definition of the Pontrjagin classes.

Next we want to study the Pontrjagin classes for the underlying real vector bundle of a complex bundle and the Chern classes of the complex vector bundle.

Recall that, for a real vector space V, $V \otimes_{\mathbb{R}} \mathbb{C}$ can be identified with $V \oplus V$ with complex structure

$$J(x, y) = (-y, x).$$

For a complex vector space F, suppose that the underlying real vector space $V = F_{\mathbb{R}}$, then $V \otimes_{\mathbb{R}} \mathbb{C}$ is canonically isomorphic to $F \oplus \overline{F}$ as a complex vector space. The isomorphism is given by the sum of two maps

$$V \oplus V \ni (x, y) \mapsto (\frac{x+iy}{2}, \frac{-ix+y}{2}) + (\frac{x-iy}{2}, \frac{ix+y}{2})$$

where the first map is complex linear, and the second map is conjugate linear.

Let E be a complex vector bundle of rank n over M. Viewed as a real vector bundle, we write it as $E_{\mathbb{R}}$. Then $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector bundle of rank $2n$, then as complex vector bundles,

$$E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \overline{E}.$$

Hence, from $c(E_{\mathbb{R}} \otimes \mathbb{C}) = c(E)c(\overline{E})$, we have

$$\begin{aligned} & 1 - p_1(E_{\mathbb{R}}) + p_2(E_{\mathbb{R}}) - \cdots + (-1)^n p_n(E_{\mathbb{R}}) \\ = \; & \big(1 + c_1(E) + c_2(E) + \cdots + c_n(E)\big)\big(1 - c_1(E) + c_2(E) - \cdots + (-1)^n c_n(E)\big). \end{aligned}$$

As an application of this formula, we can calculate the Pontrjagin classes of of $\mathbb{CP}^n$, $p_k(\mathbb{CP}^n) = p_k(T\mathbb{CP}^n_{\mathbb{R}})$. The Pontrjagin classes of $\mathbb{CP}^n$ is determined by

$$1 - p_1 + \cdots + (-1)^n p_k = c(T\mathbb{CP}^n)c(\overline{T\mathbb{CP}^n}) = (1 - h_n^2)^{n+1}.$$

Therefore, the total Pontrjagin class of $\mathbb{CP}^n$ is equal to $(1 + h_n^2)^{n+1}$. This implies that for $1 \leq k \leq n/2$,

$$p_k(\mathbb{CP}^n) = \begin{pmatrix} n+1 \\ k \end{pmatrix} h_n^{2k},$$

and $p_k(\mathbb{CP}^n) = 0$ for $k > n/2$ as $H^{4k}(\mathbb{CP}^n) = 0$.

Remark 5.8. *A theorem of Novikove says that rational Pontrjagin classes are topological invariant. It follows that manifolds with different Pontrgajin classes are not homeomorphic.*

5.4 The Euler Class

Recall that an orientation of a real vector space is an equivalence class of ordered bases, where two ordered bases are equivalence if and only if the base change is given by a matrix of positive determinant.

For a real vector bundle E of rank n, an orientation is a function which assigns an orientation to each fiber of E such that the local trivialization can be chosen to be orientation preserving. Hence E is orientable if and only if it admits a trivialization with transition functions of positive determinant metrixs, which implies that the top exterior bundle $\Lambda^n(E)$ is trivial. In other word, an oriented real vector bundle is an associated vector bundle of a principal $SO(n)$-bundle.

For a real vector bundle E, let $g_{\alpha\beta} : U\alpha \cap U_\beta \to GL(n, \mathbb{R})$ be the transition function associated to a trivialization of E over a cover $\{U_\alpha\}$ of M. We can get a sign function

$$\epsilon_{\alpha\beta} : U_\alpha \cap U_\beta \to \{\pm 1\} = \mathbb{Z}_2$$

satisfying the cocycle condition

$$\epsilon_{\alpha\beta}\epsilon_{\beta\gamma}\epsilon_{\gamma\alpha} = 1.$$

Hence, we can define an element $w_1(E) \in H^1(M, \mathbb{Z}_2)$ determined by the equivalent class of $[\epsilon_{\alpha\beta}]$. This is called the first Stiefel-Whitney class of E.

Proposition 5.9. *A real vector bundle E is orientable if and only if its first Stiefel-Whitney class is trivial.*

For an oriented real vector bundle E, the Euler class $e(E)$ is defined in terms of the Thom class and the Thom isomorphism, see Milnor and Stasheff's book on *Characteristic Classes* for details.

Theorem 5.10. *For an oriented real vector bundle E of rank n. Denote E_0 the complement of the zero section in E. There exists a unique cohomology class $U \in H^n(E, E_0; \mathbb{Z})$ whose restriction to each fiber the standard generator given by the orientation. Moreover, there is an isomorphism*

$$\begin{array}{ccc} H^*(E) & \to & H^*(E, E_0; \mathbb{Z}) \\ \omega & \mapsto & \omega \cup U. \end{array} \tag{25}$$

The class U is called the Thom class of E. The isomorphism (25) is called the Thom isomorphism.

Definition 5.11. *Let $(E, M; \pi, \mathbb{R}^n)$ be an oriented real vector bundle. The unique class $e(E) \in H^n(M, \mathbb{Z})$ such that $\pi^* e(E)$ is the image of the Thom class of E under the natural map*

$$\begin{array}{ccccc} H^n(E, E_0; \mathbb{Z}) & \to & H^n(E, \mathbb{Z}) & \cong & H^n(M, \mathbb{Z}) \\ U & \mapsto & \pi^* e(E) & \mapsto & e(E) \end{array}$$

which is called the Euler class of E.

The Euler class of an oriented real vector bundle changes sign when the orientation of the bundle is reversed. For an oriented real vector bundle with odd dimensional fiber, there is an obvious orientation reversing bundle-automorphism. Hence, the Euler class of an oriented vector bundle E with odd dimensional fiber satisfies $2e(E) = 0$.

The Euler class can be applied to give the Gysin exact sequence see Milnor and Stasheff's book.

We will give a brief definition of Euler class of a principal $SO(2n)$-bundle in terms of Chern-Weil theory.

The Lie algebra of $SO(2n)$, $\mathfrak{so}(2n)$, is isomorphic to $\mathfrak{o}(2n)$, the Lie algebra of skew-symmetric matrices. There is a special $SO(2n)$-invariant polynomial of degree n, called the Pfaffian, given by, for $A = (a_{ij}) \in \mathfrak{so}(2n)$,

$$Pf(A) = \frac{1}{2^n (2\pi)^n n!} \sum_{\sigma \in S_{2n}} sgn(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)}.$$

For example, $Pf(\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}) = \frac{a}{2\pi}$.

We need to show that $Pf(A)$ is invariant under the conjugation action of $SO(2n)$. If $g = (g_{ij}) \in SO(2n)$, then

$$gAg^{-1} = gAg^t = (\sum_{k_1, k_2 = 1}^{2n} g_{ik_1} a_{k_1 k_2} g_{jk_2}).$$

So

$$\begin{aligned} Pf(gAg^{-1}) &= \frac{1}{2^n (2\pi)^n n!} \sum_{k_1, k_2, \cdots, k_{2n}} a_{k_1 k_2} \cdots a_{k_{2n-1} k_{2n}} \\ &\quad \cdot \textstyle\sum_{\sigma \in S_{2n}} sgn(\sigma) g_{\sigma(1)k_1} g_{\sigma(2)k_2} \cdots g_{\sigma 2n-1 k_{2n-1}} g_{\sigma(2n)k_{2n}}. \end{aligned}$$

The coefficient of $a_{k_1k_2}\cdots a_{k_{2n-1}k_{2n}}$ is the determinant of the matrix (g_{ik_j}). This determinant is zero unless $\{k_1,\cdots,k_{2n}\}$ is a permutation of $\{1,\cdots,2n\}$. As $det(g)=1$, when $\{k_1,\cdots,k_{2n}\}$ is a permutation of $\{1,\cdots,2n\}$, we get the sign of the permutation $\{k_1,\cdots,k_{2n}\}$. Hence,

$$Pf(gAg^{-1}) = Pf(A).$$

Notice that $Pf(A)$ is not invariant under the action of $O(n)$, as for $g\in O(n)$ with $det(g)=-1$, we get $Pf(gAg^{-1}) = -Pf(A)$.

For a $SO(2n)$-vector bundle E over M, we define the **Euler class** to be

$$e(E) = cw(Pf) \in H^{2n}(M,\mathbb{R}).$$

The Chern-Weil theorem gives the following theorem which uniquely determines the Euler class.

Theorem 5.12. *For a $SO(2n)$-vector bundle E over M, its Euler class $e(E)$ satisfies the following properties:*

1. *(*Natuality*) For a smooth map $f: N\to M$, $f^*(e(E)) = e(f^*(E))$.*
2. *For a $SO(2m)$-vector bundle F and a $SO(2n)$-vector bundle E over M, $e(F\oplus E) = e(E)\cup c(F)$.*
3. *Let $\pi: E\to M$ be a $U(n)$-vector bundle. Under the inclusion $U(n)\subset SO(2n)$ visa the idnetification*

$$\mathbb{C}^n = \mathbb{R}\oplus i\mathbb{R}\oplus\cdots\oplus\mathbb{R}\oplus i\mathbb{R}\cong\mathbb{R}^{2n},$$

we get a $SO(2n)$-vector bundle $E_{\mathbb{R}}$, then $e(E_{\mathbb{R}}) = c_n(E)$.

4. *For a $SO(2n)$-vector bundle, $e(E)^2 = p_n(E)$.*

Proof. The naturality is obvious by Chern-Weil theorem. To show the second property, we only need to prove that

$$Pf\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = Pf(A)\cdot Pf(B).$$

Notice that Pf is invariant under SO-action, it is enough to consider A and B of the form

$$A = \begin{pmatrix} 0 & a_1 & \cdots & 0 & 0 \\ -a_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_m \\ 0 & 0 & \cdots & -a_m & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & b_1 & \cdots & 0 & 0 \\ -b_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b_n \\ 0 & 0 & \cdots & -b_n & 0 \end{pmatrix}$$

Then $Pf(A) = \dfrac{a_1 \cdots a_m}{(2\pi)^m}, Pf(B) = \dfrac{b_1 \cdots b_n}{(2\pi)^n}$, and

$$Pf \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \frac{a_1 \cdots a_m b_1 \cdots b_n}{(2\pi)^{m+n}}.$$

Similarly, we can show that

$$Pf(A)^2 = det(-\frac{A}{2\pi}),$$

for $A \in o(2n)$, which implies Property 4.

The inclusion $U(n) \subset SO(2n)$ induces a map on their Lie algebras: $i_* : \mathfrak{u}(n) \to \mathfrak{o}(2n)$ which sends the skew-hermitian matrix $A = (a_{ij} + ib_{ij})$ to the matrix

$$i_*(A) = \begin{pmatrix} a_{11} & -b_{11} & \cdots & a_{1n} & b_{1n} \\ b_{11} & a_{11} & \cdots & b_{1n} & a_{1n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & -b_{n1} & \cdots & a_{nn} & -b_{nn} \\ b_{n1} & a_{n1} & \cdots & b_{nn} & a_{nn} \end{pmatrix}.$$

We only need to prove that

$$Pf(i_*(A)) = C_n(A, \cdots, A),$$

for any $A \in \mathfrak{u}(n)$. As both sides are invariant polynomial on $\mathfrak{u}(n)$, we can assume that A is diagonalized, that is, $A = diag(ia_1, \cdots, ia_n)$, then

$$Pf(i_*(A)) = (-1)^n \frac{a_1 \cdots a_n}{(2\pi)^{2n}} = det(-\frac{A}{2\pi i}) = C_n(A, \cdots, A).$$

□

5.5 Appendix: Characteristic classes for classifying spaces

A characteristic class can be defined by any cohomology class w on a classifying space of a Lie group G. The isomorphic class of a principal G-bundle over M is determined by the homotopy class of map $f : M \to BG$. The naturality property is automatically satisfied if we define the characteristic class to be $f^*(w) \in H^*(M)$. Thus, characteristic classes of principal G-bundle correspond bijectively to cohomology classes of BG.

As an example, we will discuss those characteristic classes with $\mathbb{Z}_2$ coefficient for real vector bundles. These are the Stiefel-Whitney classes.

We know that the classifying space for real vector bundles of rank n is the Grassmann manifold $\mathbf{Gr}_n(\mathbb{R}^\infty)$: the space of n-dimensional subspaces of $\mathbb{R}^\infty$.

We assume some knowledge from algebraic topology about $H^*(\mathbf{Gr}_n(\mathbb{R}^\infty), \mathbb{Z}_2)$: the cohomology ring $H^*(\mathbf{Gr}_n(\mathbb{R}^\infty), \mathbb{Z}_2)$ is a polynomial algebra over $\mathbb{Z}_2$ freely generated by $w_i \in H^i(\mathbf{Gr}_n(\mathbb{R}^\infty), \mathbb{Z}_2)$ with $1 \le i \le n$. The proof can be obtained by studying the cell decomposition of the Grassmannian manifold.

For each real vector bundle $(E, M; \pi, \mathbb{R}^n)$, we define the i-th Stiefel-Whitney class of E to be

$$w_i(E) = f^*(w_i)$$

for the classifying map $f : M \to \mathbf{Gr}_n(\mathbb{R}^\infty)$. The Stiefel-Whitney classes are uniquely determined the following axioms, whose proof can be found in Milnor-Stasheff's book.

Theorem 5.13. *To each real vector bundle $E \to M$, there is a unique sequence of classes $w_i(E) \in H^i(M, \mathbb{Z}_2)$ (called the Stiefel-Whitney classes of E), depending only on the isomorphism type of E, such that*

1. *$w_i(f^*(E)) = f^*(w_i(E))$ for a pull-back bundle f^*E.*
2. *(Whitney sum formula) $w(E_1 \oplus E_2) = w(E_1) \cup w(E_2)$ for the total Stiefel-Whitney class $w = 1 + w_1 + w_2 + \cdots \in H^*(M, \mathbb{Z}_2)$.*
3. *$w_i(E) = 0$ for $i > rank(E)$.*
4. *For the canonical line bundle $E \to \mathbb{RP}^\infty$, $w_1(E) = \alpha$ is a generator of $H^1(\mathbb{RP}^\infty, \mathbb{Z}_2)$.*

Similar to the calculation of the total Chern class for complex projective spaces, we have the total Stiefel-Whitney class for real projective class:

$$w(\mathbb{RP}^n) = (1+\alpha)^n.$$

From which, we get, $w(\mathbb{RP}^n) = 1$ if and only if $n+1$ is a power of 2.

We conclude this section by stating the following theorem of H. Cartan.

Theorem 5.14. *Let G be a compact Lie group. Then the Chern-Weil homomorphism is an isomorphism:*

$$\mathcal{I}^*(\mathfrak{g}^*) \cong H^*(BG, \mathbb{R}).$$